MICROSCOPIC ENERGY BALANCES WITH THE ENERGY EQUATION

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www.mtu.edu/~fmorriso/cm310/cm310.html

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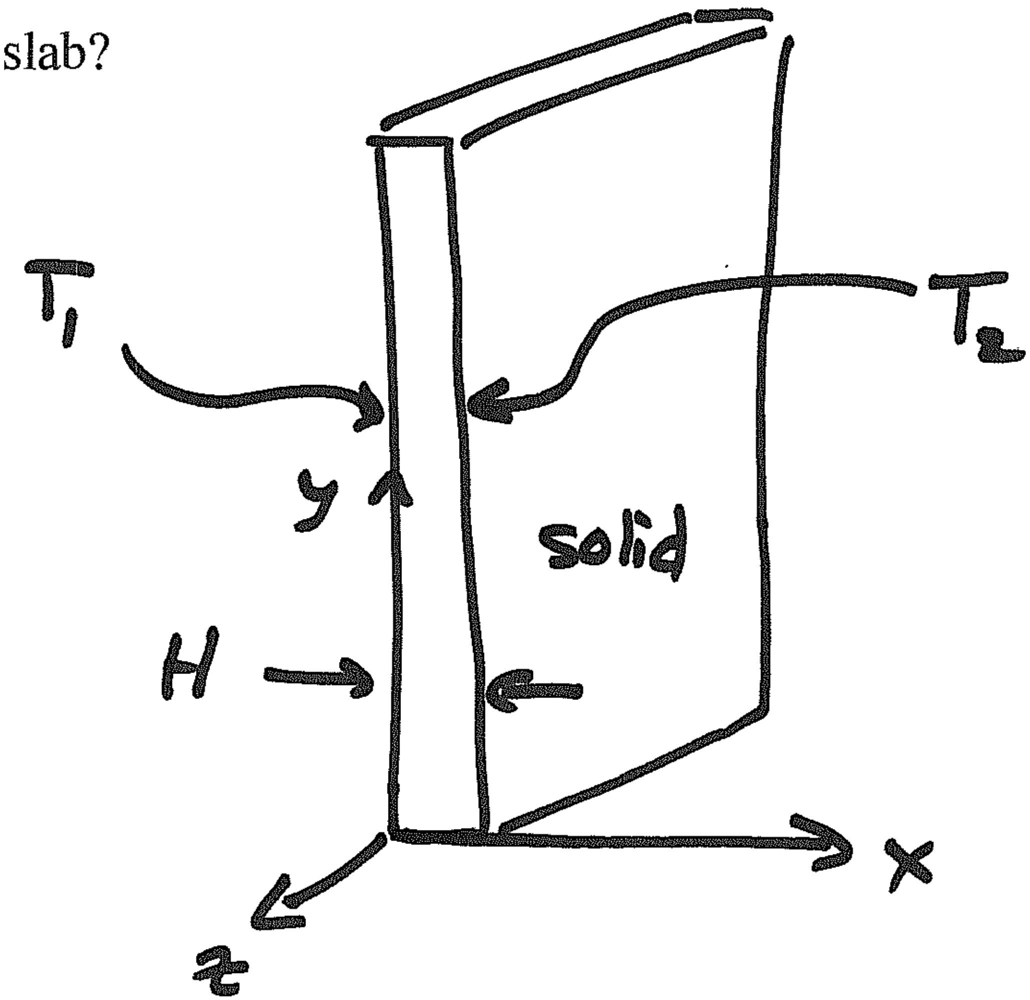
Example: A tall, wide, rectangular slab of steel has a heater on one side that holds the wall at T_1 and a heater on the other side that holds that wall at T_2. What is the steady state temperature profile in the slab? What is the heat flux in the slab?



$T_1 > T_2$

$x = 0 \quad T = T_1$
$x = H \quad T = T_2$

Equation of energy for Newtonian fluids of constant density, ρ, and thermal conductivity, k, with source term (source could be viscous dissipation, electrical energy, chemical energy, etc., with units of energy/(volume time)).

http://www.chem.mtu.edu/~fmorriso/cm310/energy_eqn.pdf
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Gibbs notation (vector notation)

$$\left(\frac{\partial T}{\partial t}+\underline{v}\cdot\nabla T\right)=\frac{k}{\rho\hat{C}_p}\nabla^2 T+\frac{S}{\rho\hat{C}_p}$$

Cartesian (xyz) coordinates:

$$\frac{\partial T}{\partial t}+v_x\frac{\partial T}{\partial x}+v_y\frac{\partial T}{\partial y}+v_z\frac{\partial T}{\partial z}=\frac{k}{\rho\hat{C}_p}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+\frac{S}{\rho\hat{C}_p}$$

Cylindrical (rθz) coordinates:

$$\frac{\partial T}{\partial t}+v_r\frac{\partial T}{\partial r}+\frac{v_\theta}{r}\frac{\partial T}{\partial \theta}+v_z\frac{\partial T}{\partial z}=\frac{k}{\rho\hat{C}_p}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}\right)+\frac{S}{\rho\hat{C}_p}$$

Spherical (rθφ) coordinates:

$$\frac{\partial T}{\partial t}+v_r\frac{\partial T}{\partial r}+\frac{v_\theta}{r}\frac{\partial T}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial T}{\partial \phi}=\frac{k}{\rho\hat{C}_p}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}\right)+\frac{S}{\rho\hat{C}_p}$$

Source: R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Processes*, Wiley, NY, 1960, page 319.

Equation of energy for Newtonian fluids of constant density, ρ, and thermal conductivity, k, with source term
(source could be
1. ~~viscous dissipation~~,
2. ~~electrical energy~~,
3. ~~chemical energy~~, etc., with units of energy/(volume time)).

Cartesian (xyz) coordinates:

$$\underbrace{\cancel{\frac{\partial T}{\partial t}}}_{\text{steady state}} + \underbrace{\cancel{v_x}\frac{\partial T}{\partial x} + \cancel{v_y}\frac{\partial T}{\partial y} + \cancel{v_z}\frac{\partial T}{\partial z}}_{\text{no flow } \underline{v}=0} = \frac{k}{\rho \hat{C}_p}\left(\frac{\partial^2 T}{\partial x^2} + \underbrace{\cancel{\frac{\partial^2 T}{\partial y^2}}}_{\text{tall}} + \underbrace{\cancel{\frac{\partial^2 T}{\partial z^2}}}_{\text{wide}}\right) + \underbrace{\cancel{\frac{S}{\rho \hat{C}_p}}}_{\text{no flow, no current, no rxn}}$$

$$0 = \frac{k}{\rho \hat{C}_p}\frac{\partial^2 T}{\partial x^2}$$

$$\boxed{0 = \frac{\partial^2 T}{\partial x^2}} \Rightarrow \boxed{0 = \frac{d^2 T}{dx^2}}$$

(since $T = T(x)$ only)

$$\frac{d^2T}{dx^2} = 0$$ Now, integrate twice ....

1 integrate: $\frac{dT}{dx} = c_1$

2 integrate $\boxed{T = c_1 x + c_2}$

Apply BC + Solve.

Fourier's Law:

$$\frac{q_x}{A} = -k \frac{dT}{dx}$$

Solution:

$$T = C_1 x + C_2$$

BC: $x = 0 \quad T = T_1$
$x = H \quad T = T_2$

BC1: $T_1 = C_1(0) + C_2$

$$\Rightarrow \boxed{C_2 = T_1}$$

BC2: $T_2 = C_1(H) + C_2$

$$T_2 = C_1 H + T_1$$

$$T_2 - T_1 = C_1 H$$

$$\boxed{C_1 = \frac{T_2 - T_1}{H}}$$

$$\Rightarrow \boxed{T = \left(\frac{T_2 - T_1}{H}\right) x + T_1}$$

Heat flux in slab:

Fourier's Law

$$\frac{q_x}{A} = -k \frac{dT}{dx} \leftarrow \text{calculate from Temp profile solution.}$$

$$= -k \left(\frac{T_2 - T_1}{H} \right)$$

$$\boxed{\frac{q_x}{A} = \frac{k(T_1 - T_2)}{H}}$$

$$//$$