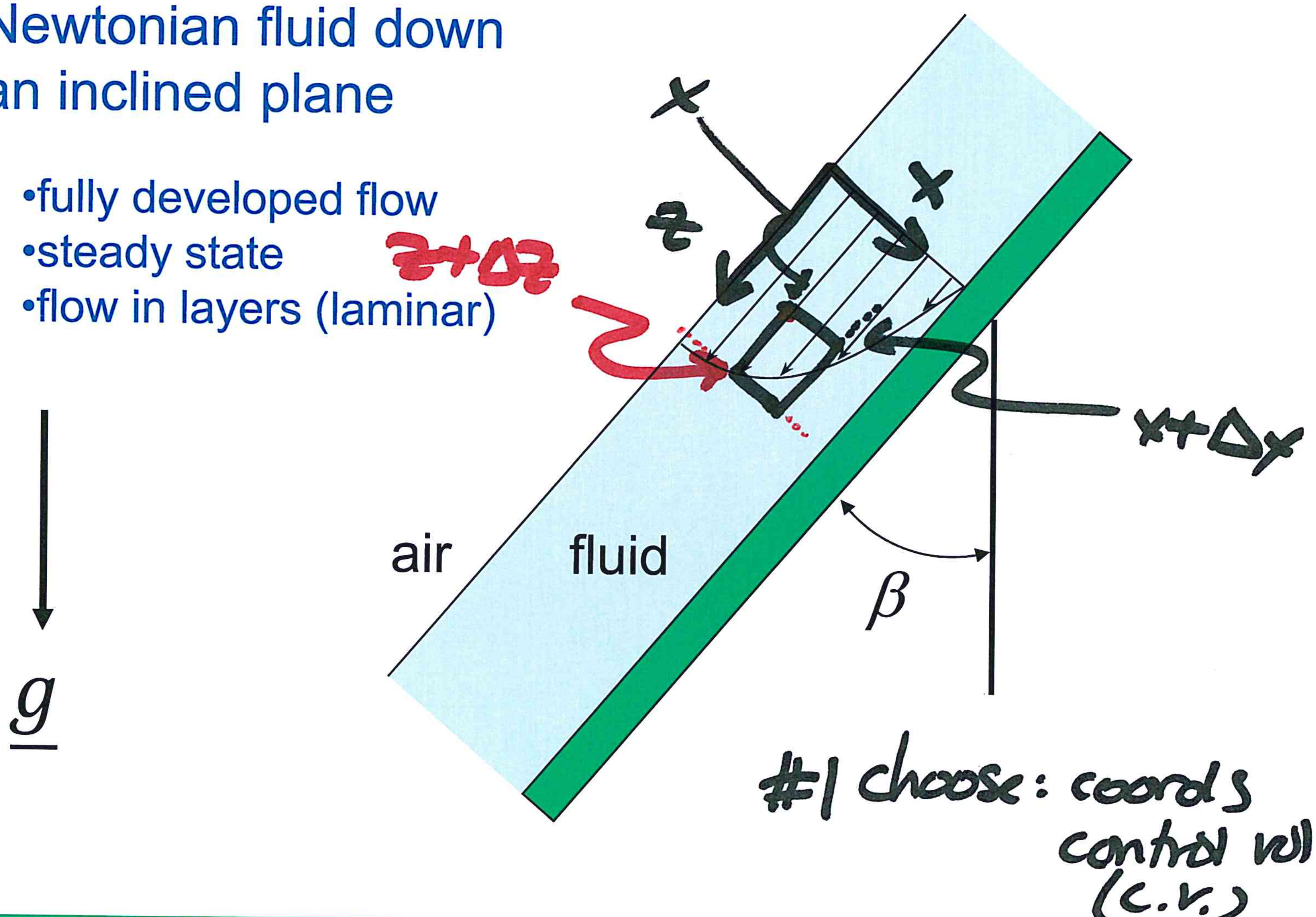
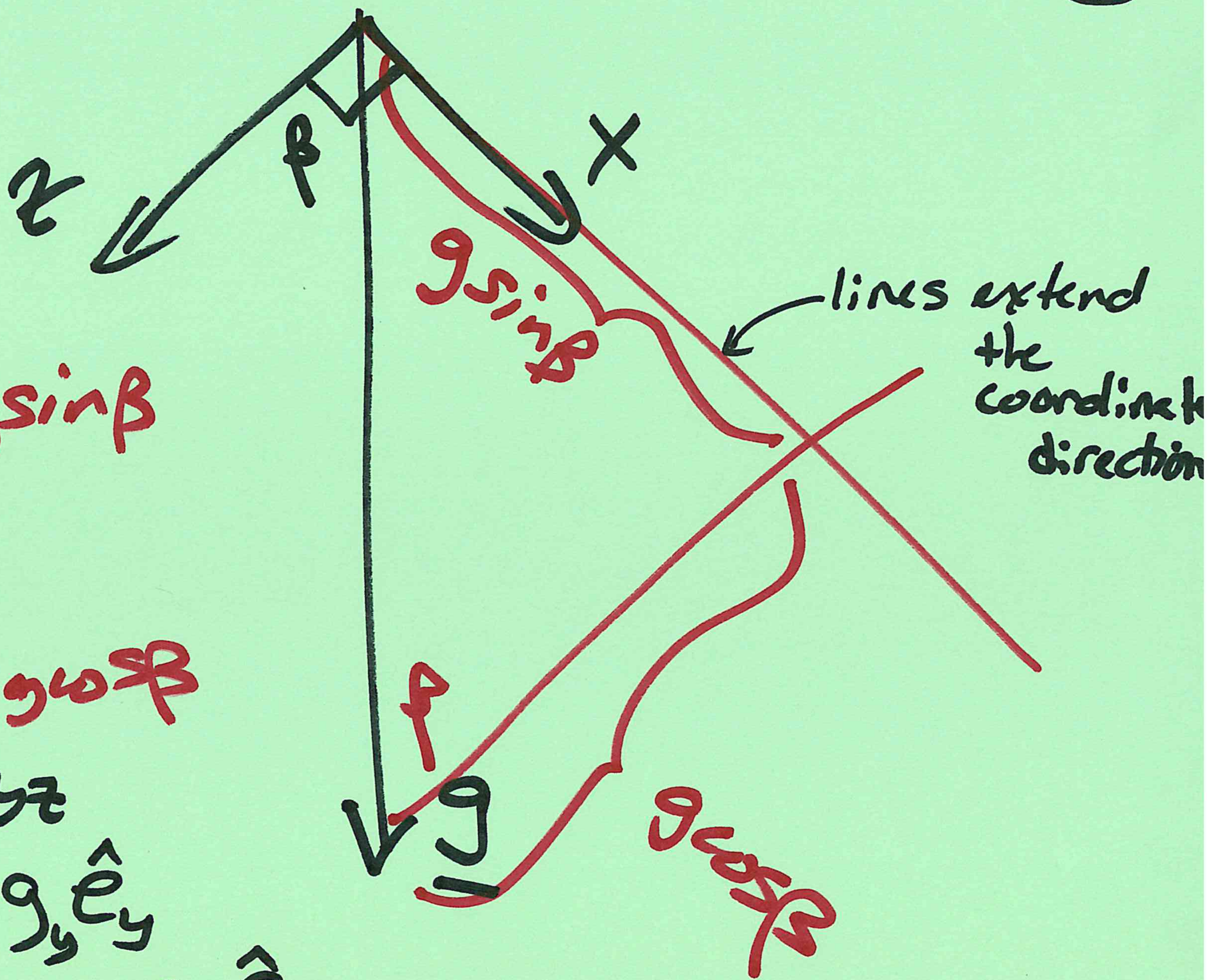
Week 2-3 2016 Morrison ①

EXAMPLE 1: Flow of a Newtonian fluid down an inclined plane

- •fully developed flow
- •steady state
- •flow in layers (laminar)



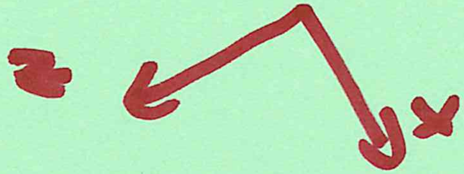
write $\underline{g}$ in chosen coord system:

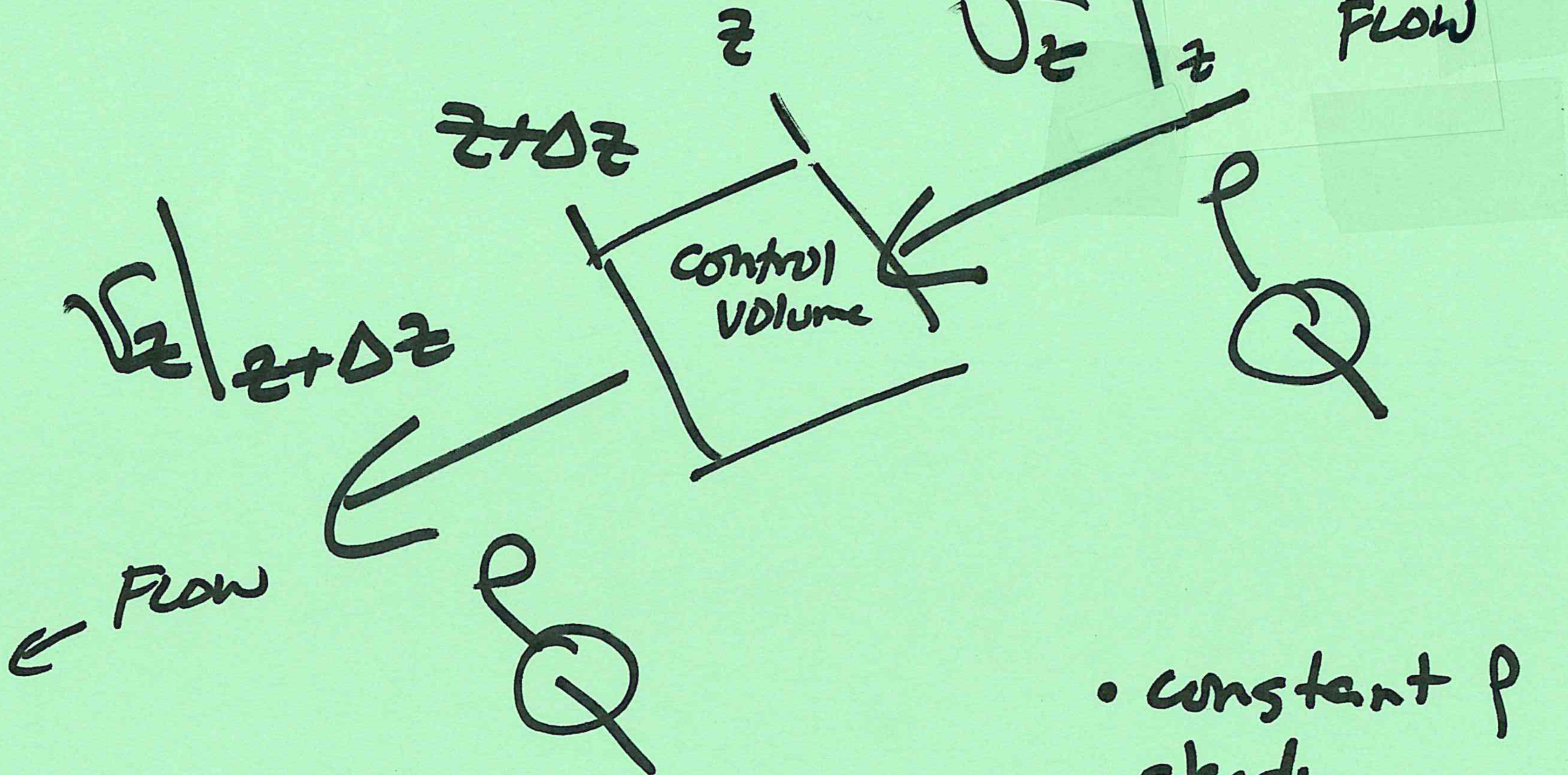


$$\underline{g} = \begin{pmatrix} g_x = g\sin\beta \\ \cancel{g_y}\ 0 \\ g_z = g\cos\beta \end{pmatrix}_{xyz}$$

$$\underline{g} = g_x\hat{e}_x + g_y\hat{e}_y + g_z\hat{e}_z$$

MICRO-MASS BAL?





- constant ρ
- steady

How are flow rate Q and velocity v_z related? (4)

$$\frac{Q}{\text{X Sec AREA}} = \langle V \rangle$$

$$\langle V \rangle (\text{X-Sec AREA}) = Q$$

$$\left(V_z \Big|_z \right) \left(\Delta x \, \Delta y \right)$$

(can also do for out-flow)

MICROSCOPIC MASS BAL

RATE of MASS ~~IN~~: OUT

$$\underbrace{\rho}_{\left(\frac{\text{mass}}{\cancel{\text{volume}}}\right)} \underbrace{\left(v_z\big|_{\cancel{z}\;\, z+\Delta z}\right)(\Delta x \, \Delta y)}_{\left(\frac{\cancel{\text{volume}}}{\text{time}}\right)}$$

Microscopic Mass Bal on C.V.:

$$\cancel{in} - out = \underbrace{0}_{steady}$$

$$\Rightarrow \boxed{in = out}$$

$$\cancel{\rho}\left(v_z\Big|_z\right)(\cancel{\Delta x}\,\cancel{\Delta y}) = \cancel{\rho}\left(v_z\Big|_{z+\Delta z}\right.\cancel{\Delta x}\,\cancel{\Delta y})$$

$$\Rightarrow \boxed{v_z\Big|_z = v_z\Big|_{z+\Delta z}}$$

velocity does not vary in z-dir

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MICROSCOPIC

Momentum balance, flowing system (open system; control volume):

$$\left\{\begin{matrix} \text{sum of forces} \\ \text{acting } \underline{\text{on}} \text{ control vol} \end{matrix}\right\} + \left\{\begin{matrix} \text{net momentum} \\ \text{flowing in} \end{matrix}\right\} = \cancel{\left\{\begin{matrix} \text{rate of} \\ \text{accumulation} \\ \text{of momentum} \end{matrix}\right\}}$$

$$\sum_{in} - \sum_{out}$$

steady state

gravity + viscous

$$\sum_j \underline{F}_{on_i} + \underbrace{\sum_i \left\{\begin{matrix} \text{momentum} \\ \text{flowing } \underline{\text{in}} \\ \text{in the streams} \end{matrix}\right\} - \sum_i \left\{\begin{matrix} \text{momentum} \\ \text{flowing } \underline{\text{out}} \\ \text{in the streams} \end{matrix}\right\}}_{\text{convective}} = 0$$

steady

note that **momentum** is a vector quantity

Convective terms – pattern on micro mass balance

momentum flow ~~into~~ out of control volume:

$$\left(\frac{\text{momentum}}{\text{vol}}\right)\left(\frac{\text{vol}}{\text{time}}\right)$$

$$\cancel{\left(\frac{(\text{mass})(\text{velocity})}{\text{vol}}\right)} \quad v_z\Big|_{\cancel{z}\;z+\Delta z} \quad \Delta x \Delta y$$

$$\rho \quad v_z\Big|_{\cancel{z}\;z+\Delta z}$$

$$= \rho \left(v_z\big|_{\cancel{z}\;z+\Delta z}\right)^2 \Delta x \Delta y$$

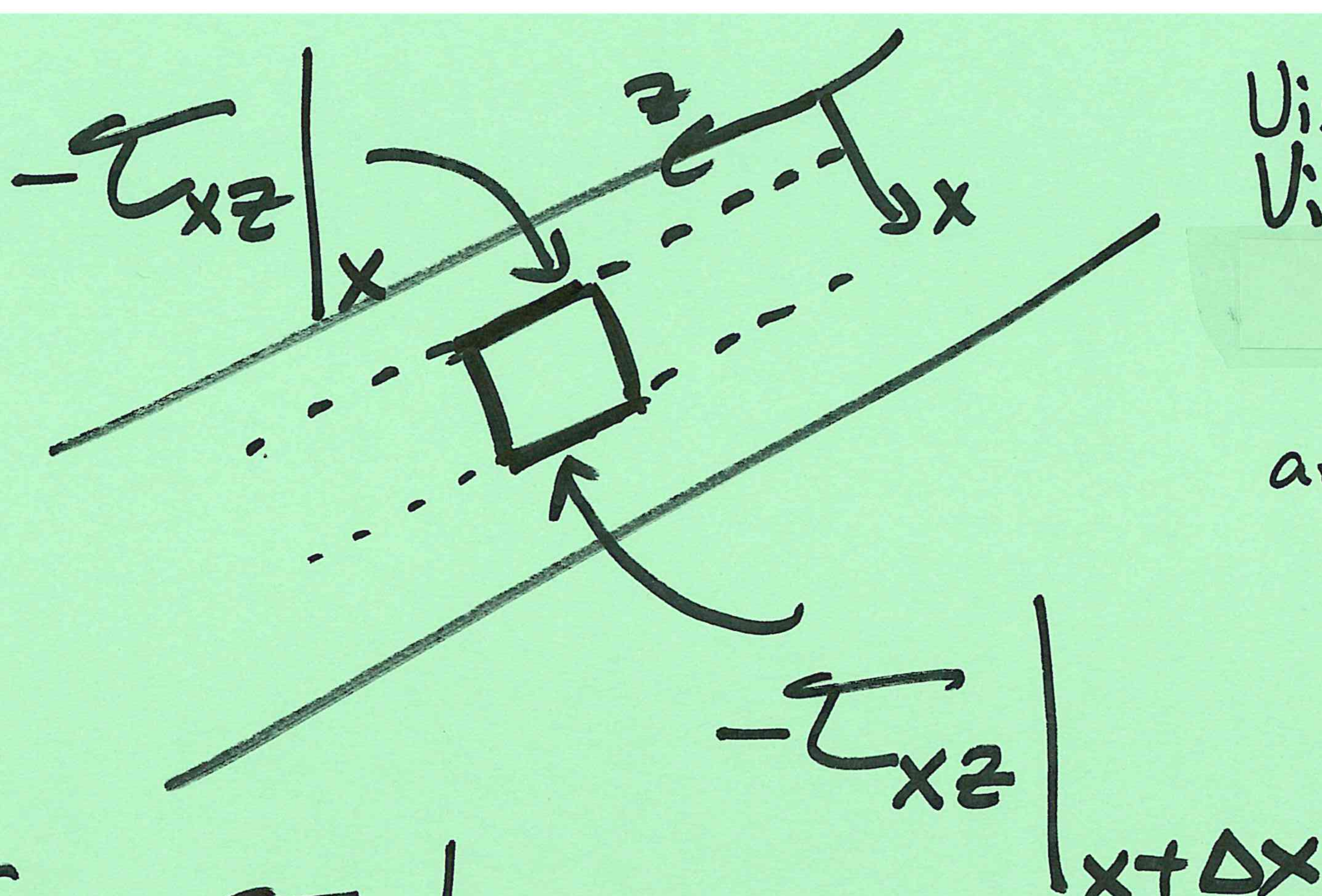
Σ FORCES ON C.V : Gravity

$$\begin{pmatrix} \text{mass} \\ \text{of} \\ \text{C.V.} \end{pmatrix} \left(\underline{g} \right)$$

$$\rho \left(\text{volume} \right) \underline{g}$$

$$\rho \, \Delta x \Delta y \Delta z \begin{pmatrix} g \sin\beta \\ 0 \\ g \cos\beta \end{pmatrix}_{xyz}$$

☆ we'll concentrate on z-momentum



Viscous force or Viscous flux into top of c.v and out of bottom of c.v.

Top $\tau_{xz}|_x \, \Delta y \, \Delta z$

$\left(\frac{\text{force}}{\text{area}}\right)$ (area of top)

Bottom $\tau_{xz}|_{x+\Delta x} \, \Delta y \, \Delta z$

* negative sign is due to stress sign convention

MicroScopic Momentum Bal on a C.V. (z-dir) ASSEMBLE

$$\Sigma\begin{pmatrix}\text{viscous}\\ \text{forces}\end{pmatrix} + \cancel{\begin{pmatrix}\Sigma \overset{\text{(convective)}}{\text{momentum}} \\ \text{flux} \\ \text{in}\end{pmatrix}} + \begin{matrix}\text{gravity}\\ \text{forces}\end{matrix} = \underbrace{0}_{\text{steady}}$$

equal + opposite

z-direction

$$0 = \underbrace{\tau_{xz}\Big|_x \cancel{\Delta y}\cancel{\Delta z}}_{\text{viscous flux into C.V.}} - \underbrace{\tau_{xz}\Big|_{x+\Delta x} \cancel{\Delta y}\cancel{\Delta z}}_{\text{viscous flux out of C.V.}} + \overbrace{\rho g \Delta x \cancel{\Delta y}\cancel{\Delta z}}^{\text{gravity force on C.V.}} (\cos\beta)$$

z-component of $\underline{g}$

$$0 = -\tau_{xz}\big|_x + \tau_{xz}\big|_{x+\Delta x} + \rho g \cos\beta \, \Delta x$$

$$-\tau_{xz}\big|_{x+\Delta x} + \tau_{xz}\big|_x = \rho g \cos\beta \, \Delta x$$

$$\underbrace{\lim_{\Delta x \to 0} \frac{\tau_{xz}\big|_{x+\Delta x} - \tau_{xz}\big|_x}{\Delta x}} = -\rho g \cos\beta$$

$$\frac{d\tau_{xz}}{dx} = -\rho g \cos\beta$$

$$\frac{d\tau_{xz}}{dx} = -\rho g \cos\beta$$

$$\tau_{xz} = -(\rho g \cos\beta)x + C_1$$

(or

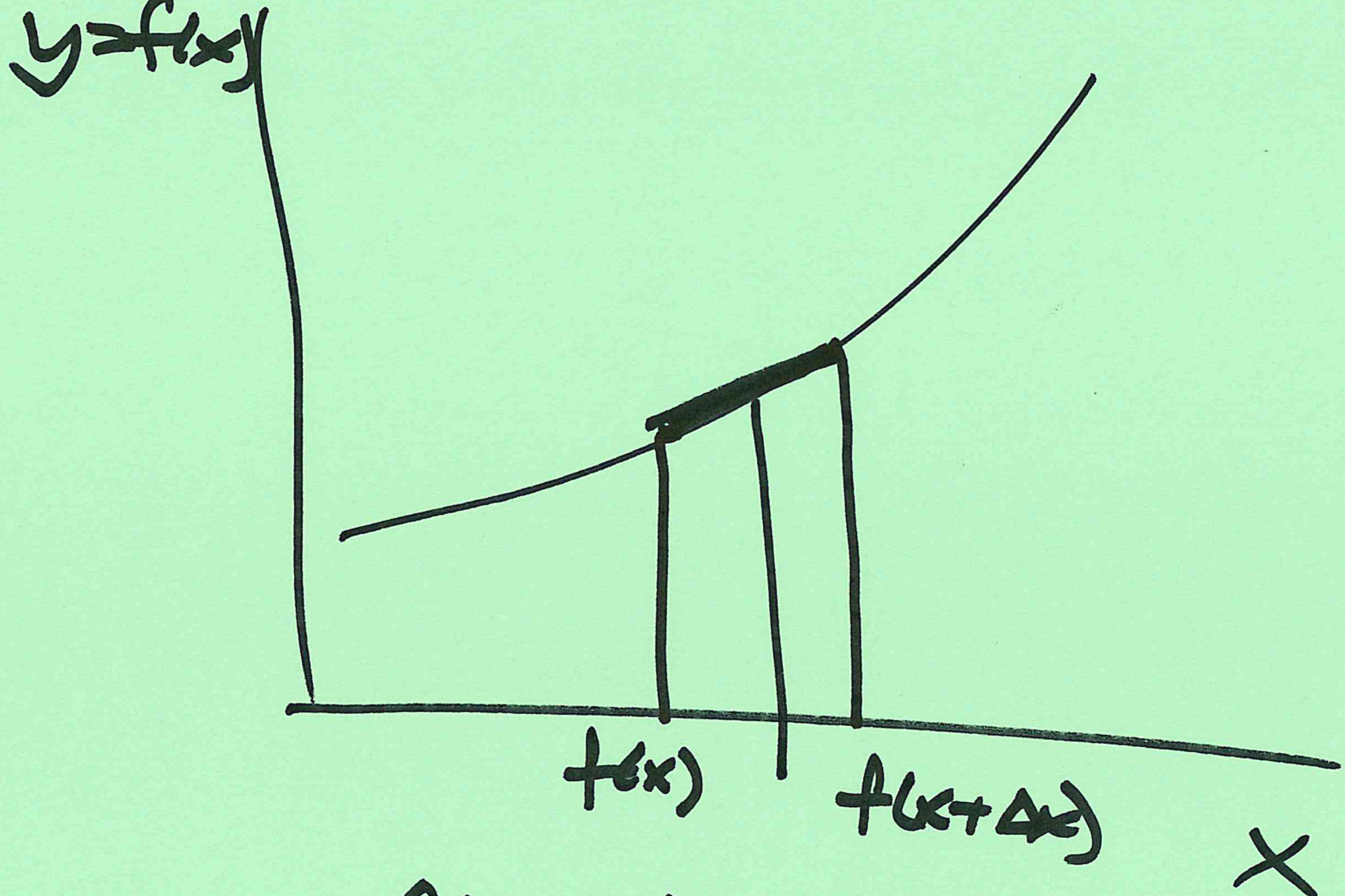
$$\int d\tau_{xz} = -\int(\rho g \cos\beta)\, dx$$

Use Newton's Law of viscosity $\rightarrow$

$$\boxed{\tau_{xz} = -(\rho g \cos\beta)x + C_1}$$

shear stress as a function of position

The definition of derivative:



$$\frac{df}{dx} \equiv \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\tau_{xz} = \mu \frac{dV_z}{dx}$$

$$\mu \frac{dV_z}{dx} = -(\rho g \cos\beta)x + C_1$$

$$\frac{dV_z}{dx} = -\left(\frac{\rho g \cos\beta}{\mu}\right)x + \frac{C_1}{\mu}$$

$$\int dV_z = \int\left[-\left(\frac{\rho g \cos\beta}{\mu}\right)x + \frac{C_1}{\mu}\right]dx$$

$$V_z = -\left(\frac{\rho g \cos\beta}{\mu}\right)\frac{x^2}{2} + \left(\frac{C_1}{\mu}\right)x + C_2$$

* We need two boundary conditions

BC#1 $x = H \quad V_z = 0$

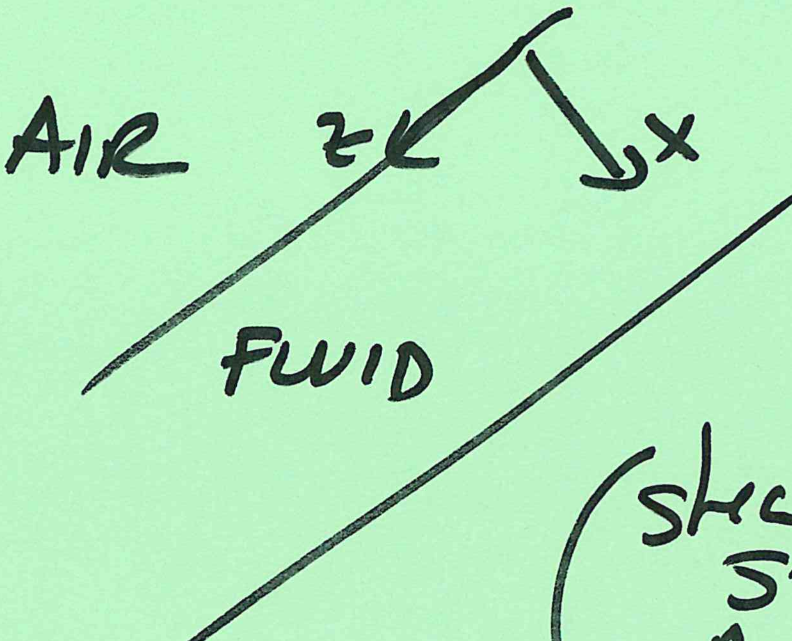
no slip boundary condition

BC#2 $x = 0 \quad \frac{dV_z}{dx} = 0 \Rightarrow C_1 = 0$

BC1 $\Rightarrow \quad 0 = -\left(\frac{\rho g \cos\beta}{2\mu}\right) H^2 + C_2$

$$\boxed{C_2 = +\left(\frac{\rho g \cos\beta}{2\mu}\right) H^2}$$

Where do we get that Second BC?



$$\begin{pmatrix}\text{shear}\\ \text{stress}\\ \text{AIR}\end{pmatrix} = \begin{pmatrix}\text{shear}\\ \text{stress}\\ \text{FLUID}\end{pmatrix}$$

$$0 \cong \mu_{AIR}\left(\frac{dV_z}{dx}\right)_{AIR} = \mu_{FLUID}\left(\frac{dV_z}{dx}\right)_{fluid}$$

$$\mu_{AIR} \ll \mu_{FLUID}$$

AT THE FREE SURFACE
(AIR)

$$0 = \mu_{FLUID}\left(\frac{dV_z}{dx}\right)$$

BC 2.

$$\boxed{\frac{dV_z}{dx} = 0 \quad \text{at} \quad x = 0}$$

$$v_z = -\left(\frac{\rho g \cos\beta}{2\mu}\right)x^2 + \left(\frac{\rho g \cos\beta}{2\mu}\right)H^2$$

$$= -\left(\frac{\rho g \cos\beta}{2\mu}\right)\left(x^2 - H^2\right)$$

check units:

$$\frac{\cancel{kg}}{\cancel{m^3}} \; \frac{m}{s^{\cancel{2}}} \; \frac{\cancel{m\,s}}{\cancel{kg}} \; \cancel{m^2}$$

$$[=] \frac{m}{s} \checkmark$$

(20) MASS BAL FIRST

$v_y = 0$

The Equation of Continuity and the Equation of Motion in Cartesian, cylindrical, and spherical coordinates

steady $\quad v_x = 0$

CM3110 Fall 2011 Faith A. Morrison

Continuity Equation, Cartesian coordinates

$$\frac{\partial \rho}{\partial t} + \left(v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} \right) + \rho \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) = 0$$

Continuity Equation, cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial (\rho r v_r)}{\partial r} + \frac{1}{r}\frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{\partial (\rho v_z)}{\partial z} = 0$$

$\rho = \text{constant}$

Continuity Equation, spherical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial (\rho r^2 v_r)}{\partial r} + \frac{1}{r \sin\theta}\frac{\partial (\rho v_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial (\rho v_\phi)}{\partial \phi} = 0$$

Equation of Motion for an incompressible fluid, 3 components in Cartesian coordinates

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial P}{\partial x} + \left(\frac{\partial \tilde{\tau}_{xx}}{\partial x} + \frac{\partial \tilde{\tau}_{yx}}{\partial y} + \frac{\partial \tilde{\tau}_{zx}}{\partial z} \right) + \rho g_x$$

$$\rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = -\frac{\partial P}{\partial y} + \left(\frac{\partial \tilde{\tau}_{xy}}{\partial x} + \frac{\partial \tilde{\tau}_{yy}}{\partial y} + \frac{\partial \tilde{\tau}_{zy}}{\partial z} \right) + \rho g_y$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial P}{\partial z} + \left(\frac{\partial \tilde{\tau}_{xz}}{\partial x} + \frac{\partial \tilde{\tau}_{yz}}{\partial y} + \frac{\partial \tilde{\tau}_{zz}}{\partial z} \right) + \rho g_z$$

$$\underline{v} = \begin{pmatrix} 0 \\ 0 \\ v_z(x,y) \end{pmatrix}_{xyz}$$

Equation of Motion for an incompressible fluid, 3 components in cylindrical coordinates

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) = -\frac{\partial P}{\partial r} + \left(\frac{1}{r}\frac{\partial (r\tilde{\tau}_{rr})}{\partial r} + \frac{1}{r}\frac{\partial \tilde{\tau}_{\theta r}}{\partial \theta} - \frac{\tilde{\tau}_{\theta\theta}}{r} + \frac{\partial \tilde{\tau}_{zr}}{\partial z} \right) + \rho g_r$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta v_r}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \left(\frac{1}{r^2}\frac{\partial (r^2\tilde{\tau}_{r\theta})}{\partial r} + \frac{1}{r}\frac{\partial \tilde{\tau}_{\theta\theta}}{\partial \theta} + \frac{\partial \tilde{\tau}_{z\theta}}{\partial z} + \frac{\tilde{\tau}_{\theta r} - \tilde{\tau}_{r\theta}}{r} \right) + \rho g_\theta$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial P}{\partial z} + \left(\frac{1}{r}\frac{\partial (r\tilde{\tau}_{rz})}{\partial r} + \frac{1}{r}\frac{\partial \tilde{\tau}_{\theta z}}{\partial \theta} + \frac{\partial \tilde{\tau}_{zz}}{\partial z} \right) + \rho g_z$$

$$\boxed{\frac{\partial v_z}{\partial z} = 0}$$

Equation of Motion for an incompressible fluid, 3 components in spherical coordinates

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r} \right)$$

$$= -\frac{\partial P}{\partial r} + \left(\frac{1}{r^2}\frac{\partial (r^2\tilde{\tau}_{rr})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (\tilde{\tau}_{\theta r}\sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tilde{\tau}_{\phi r}}{\partial \phi} - \frac{\tilde{\tau}_{\theta\theta} + \tilde{\tau}_{\phi\phi}}{r} \right) + \rho g_r$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2 \cot\theta}{r} \right)$$

$$= -\frac{1}{r}\frac{\partial P}{\partial \theta} + \left(\frac{1}{r^3}\frac{\partial (r^3\tilde{\tau}_{r\theta})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (\tilde{\tau}_{\theta\theta}\sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tilde{\tau}_{\phi\theta}}{\partial \phi} + \frac{\tilde{\tau}_{\theta r} - \tilde{\tau}_{r\theta}}{r} - \frac{\tilde{\tau}_{\phi\phi}\cot\theta}{r} \right) + \rho g_\theta$$

$$\rho \left(\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{v_\phi v_\theta \cot\theta}{r} \right)$$

$$= -\frac{1}{r\sin\theta}\frac{\partial P}{\partial \phi} + \left(\frac{1}{r^3}\frac{\partial (r^3\tilde{\tau}_{r\phi})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (\tilde{\tau}_{\theta\phi}\sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tilde{\tau}_{\phi\phi}}{\partial \phi} + \frac{\tilde{\tau}_{\phi r} - \tilde{\tau}_{r\phi}}{r} + \frac{\tilde{\tau}_{\phi\theta}\cot\theta}{r} \right) + \rho g_\phi$$

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Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation) 3 components in Cartesian coordinates

steady

$$\rho\left(\cancel{\frac{\partial v_x}{\partial t}}+\cancel{v_x\frac{\partial v_x}{\partial x}}+\cancel{v_y\frac{\partial v_x}{\partial y}}+\cancel{v_z\frac{\partial v_x}{\partial z}}\right) = -\frac{\partial P}{\partial x}+\mu\left(\cancel{\frac{\partial^2 v_x}{\partial x^2}}+\cancel{\frac{\partial^2 v_x}{\partial y^2}}+\cancel{\frac{\partial^2 v_x}{\partial z^2}}\right)+\rho g_x \quad g_x = g\sin\beta$$

$$\rho\left(\cancel{\frac{\partial v_y}{\partial t}}+\cancel{v_x\frac{\partial v_y}{\partial x}}+\cancel{v_y\frac{\partial v_y}{\partial y}}+\cancel{v_z\frac{\partial v_y}{\partial z}}\right) = -\frac{\partial P}{\partial y}+\mu\left(\cancel{\frac{\partial^2 v_y}{\partial x^2}}+\cancel{\frac{\partial^2 v_y}{\partial y^2}}+\cancel{\frac{\partial^2 v_y}{\partial z^2}}\right)+\cancel{\rho g_y}$$

$$\rho\left(\cancel{\frac{\partial v_z}{\partial t}}+\cancel{v_x\frac{\partial v_z}{\partial x}}+\cancel{v_y\frac{\partial v_z}{\partial y}}+\cancel{v_z\frac{\partial v_z}{\partial z}}\right) = -\frac{\partial P}{\partial z}+\mu\left(\frac{\partial^2 v_z}{\partial x^2}+\cancel{\frac{\partial^2 v_z}{\partial y^2}}+\cancel{\frac{\partial^2 v_z}{\partial z^2}}\right)+\rho g_z \quad g\cos\beta$$

continuity

wide flow

continuity

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation) 3 components in cylindrical coordinates

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial P}{\partial r}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_r)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}\right)+\rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial P}{\partial \theta}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_\theta)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right)+\rho g_\theta$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right)+\rho g_z$$

$$\underline{v} = \begin{pmatrix} 0 \\ 0 \\ v_z \end{pmatrix}_{xyz} \qquad \underline{g} = \begin{pmatrix} g\sin\beta \\ 0 \\ g\cos\beta \end{pmatrix}$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in spherical coordinates

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi}-\frac{v_\theta^2+v_\phi^2}{r}\right)$$
$$= -\frac{\partial P}{\partial r}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right)\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v_r}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial \phi^2}\right.$$
$$\left.-\frac{2}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(v_\theta\sin\theta\right)-\frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right)+\rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}+\frac{v_r v_\theta}{r}-\frac{v_\phi^2\cot\theta}{r}\right)$$
$$= -\frac{1}{r}\frac{\partial P}{\partial \theta}+\mu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(v_\theta\sin\theta\right)\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial \phi^2}\right.$$
$$\left.+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}-\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right)+\rho g_\theta$$

$$\rho\left(\frac{\partial v_\phi}{\partial t}+v_r\frac{\partial v_\phi}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}+\frac{v_r v_\phi}{r}+\frac{v_\phi v_\theta\cot\theta}{r}\right)$$
$$= -\frac{1}{r\sin\theta}\frac{\partial P}{\partial \phi}+\mu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(v_\phi\sin\theta\right)\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial \phi^2}\right.$$
$$\left.+\frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial \phi}+\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right)+\rho g_\phi$$

Note: the r-component of the Navier-Stokes equation in spherical coordinates may be simplified by adding $0 = \frac{2}{r}\nabla\cdot\underline{v}$ to the component shown above. This term is zero due to the continuity equation (mass conservation). See Bird et. al.

References:

1. R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena*, 2nd edition, Wiley: NY, 2002.
2. R. B. Bird, R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric Fluids: Volume 1 Fluid Mechanics*, Wiley: NY, 1987.

x-component of Navier-Stokes:

$$0 = -\frac{\partial P}{\partial x} + \rho g \sin\beta$$

$$\frac{dP}{dx} = \rho g \sin\beta$$

$$P = \rho g \sin\beta\, x + C_3$$

$x = 0$

$P = P_{atm}$

$$\boxed{P = (\rho g \sin\beta)\, x + P_{atm}}$$

y-component of NS:

$$0 = -\frac{\partial P}{\partial y} \implies$$ Pressure is not a function of y

z-component of NS: $v_z(x)$

$$0 = \cancel{-\frac{\partial P}{\partial z}} + \mu \frac{\partial^2 v_z}{\partial x^2} + \rho g \cos\beta$$

not a function of z

$$-\mu \frac{d^2 v_z}{dx^2} = \rho g \cos\beta$$

$\Phi = \frac{dv_z}{dx}$

$$\frac{d^2 v_z}{dx^2} = \left(\frac{\rho g \cos\beta}{-\mu}\right)$$

$$\frac{d}{dx}(\Phi) = \left(\frac{\rho g \cos\beta}{-\mu}\right)$$

$$\int d\Phi = \int \left(\frac{\rho g \cos\beta}{-\mu} \right) dx$$

$$\frac{dV_z}{dx} = \Phi = \left(\frac{\rho g \cos\beta}{-\mu} \right) x + \cancel{C_4}\}_0$$

$$V_z = \left(\frac{\rho g \cos\beta}{-\mu} \right) \frac{x^2}{2} + C_4 x + C_5$$

BC:

$$\boxed{\begin{array}{ll} x = 0 & \dfrac{dV_z}{dx} = 0 \\ x = H & V_z = 0 \end{array}}$$

BC1 $\Rightarrow C_4 = 0$

BC2 $\Rightarrow 0 = -\left(\frac{\rho g \cos\beta}{2\mu}\right) H^2 + C_5$

$$C_5 = \frac{\rho g \cos\beta \, H^2}{2\mu}$$

$$\Rightarrow v_z(x) = \frac{\rho g \cos\beta}{2\mu}\left(H^2 - x^2\right)$$

(matches lecture slides) ✓