# Equations Summary from Inside Cover of Morrison, 2013

Mechanical Energy Balance

$$\frac{\Delta p}{\rho} + \frac{\Delta \langle v \rangle^2}{2\alpha} + g\Delta z + F_{friction} = -\frac{W_{s,by\ fluid}}{m} \qquad \begin{cases} \alpha_{laminar} = 0.5 \\ \alpha_{turbulent} \approx 1 \end{cases}$$

$$F_{friction} = \left[ 4f\frac{L}{D} + \sum_{fittings_i} n_i K_{f,i} \right] \frac{\langle v \rangle^2}{2}$$

Fanning Friction Factor (pipe flow)

$$f = \frac{F_{drag}}{\frac{1}{2}\rho \langle v \rangle^2 \, (2\pi R L)} = \frac{\Delta p D}{2L\rho \langle v \rangle^2}$$

**Note this is correct; there is an error on the inside cover**

Drag Coefficient (sphere drop)

$$C_D = \frac{F_{drag}}{\frac{1}{2}\rho v_\infty^2 \, (\pi R^2)} = \frac{4gD(\rho_{body} - \rho)}{3\rho v_\infty^2}$$

Momentum balance on a CV (Reynolds transport theorem)

$$\frac{d\underline{\mathbf{P}}}{dt} + \iint_{CS} (\hat{n} \cdot \underline{v})\rho \underline{v} \, dS = \sum_{on\ CV} \underline{f}$$

Hydrostatic Pressure

$$p_{bottom} = p_{top} + \rho g h$$

Hagen-Poiseuille Equation (steady, laminar tube flow, incompressible)

$$Q = \frac{\pi(p_0 - p_L)R^4}{8\mu L}$$

Prandtl Equation (steady, turbulent tube flow)

$$\frac{1}{\sqrt{f}} = -4.0 \log\left(\frac{4.67}{\mathrm{Re}\sqrt{f}}\right) + 2.28$$

Stokes-Einstein-Sutherland Equation (steady, slow flow around a sphere)

$$F_{drag} = 6\pi R \mu v_\infty$$

Macroscopic Momentum Balance on a CV

$$\frac{d\mathbf{P}}{dt} + \sum_{i=1}^{\#streams} \left[ \frac{\rho A cos(\theta) \langle v \rangle^2}{\beta} \hat{v} \right] \Bigg|_{A_i} = \sum_{i=1}^{\#streams} [-pA\hat{n}]_{A_i} + \underline{R} + M_{CV}\underline{g} \qquad \begin{cases} \beta_{laminar} = 0.75 \\ \beta_{turbulent} \approx 1 \end{cases}$$

Navier-Stokes equation (microscopic momentum balance, incompressible, Newtonian fluids)

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v}\right) = -\nabla p + \mu \nabla^2 \underline{v} + \rho \underline{g}$$

Continuity equation (microscopic mass balance, incompressible fluids)

$$\nabla \cdot \underline{v} = 0$$

Total stress tensor $\underline{\underline{\tilde{\Pi}}} = -p\underline{\underline{I}} + \underline{\underline{\tilde{\tau}}}$

$$\begin{pmatrix} \tilde{\Pi}_{11} & \tilde{\Pi}_{12} & \tilde{\Pi}_{13} \\ \tilde{\Pi}_{21} & \tilde{\Pi}_{22} & \tilde{\Pi}_{23} \\ \tilde{\Pi}_{31} & \tilde{\Pi}_{32} & \tilde{\Pi}_{33} \end{pmatrix}_{123} = \begin{pmatrix} \tilde{\tau}_{11} - p & \tilde{\tau}_{12} & \tilde{\tau}_{13} \\ \tilde{\tau}_{21} & \tilde{\tau}_{22} - p & \tilde{\tau}_{23} \\ \tilde{\tau}_{31} & \tilde{\tau}_{32} & \tilde{\tau}_{33} - p \end{pmatrix}_{123}$$

Dynamic pressure $\mathcal{P} \equiv p + \rho g h$

Newtonian constitutive equation

$$\begin{aligned} \underline{\underline{\tilde{\tau}}} &= \mu \left( \nabla \underline{v} + (\nabla \underline{v})^T \right) \\ &= \mu \begin{pmatrix} 2\frac{\partial v_1}{\partial x_1} & \frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2} & \frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2} & 2\frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} \\ \frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3} & \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} & 2\frac{\partial v_3}{\partial x_3} \end{pmatrix}_{123} \end{aligned}$$

Total molecular fluid force on a finite surface $\mathcal{S}$

$$\underline{\mathcal{F}} = \iint_{\mathcal{S}} \left[ \hat{n} \cdot \underline{\underline{\tilde{\Pi}}} \right]_{\text{at surface}} dS$$

Stationary fluid $\left[ \hat{n} \cdot \underline{\underline{\tilde{\Pi}}} \right] = -p\hat{n}$

Moving fluid $\left[ \hat{n} \cdot \underline{\underline{\tilde{\Pi}}} \right] = -p\hat{n} + \hat{n} \cdot \underline{\underline{\tilde{\tau}}}$

Total fluid torque on a finite surface $\mathcal{S}$

$$\underline{\mathcal{T}} = \iint_{\mathcal{S}} \left[ \underline{R} \times \left( \hat{n} \cdot \underline{\underline{\tilde{\Pi}}} \right) \right]_{\text{at surface}} dS$$

Total flow rate out through a finite surface $\mathcal{S}$

$$Q = \dot{V} = \iint_{\mathcal{S}} \left[ \hat{n} \cdot \underline{v} \right]_{\text{at surface}} dS$$

Average velocity across a finite surface $\mathcal{S}$

$$\langle v \rangle = \frac{Q}{\mathcal{S}}$$

| Coordinate system | surface differential $dS$ |
|---|---|
| Cartesian (top, $\hat{n} = \hat{e}_z$) | $dS = dxdy$ |
| Cartesian (side a, $\hat{n} = \hat{e}_y$) | $dS = dxdz$ |
| Cartesian (side b, $\hat{n} = \hat{e}_x$) | $dS = dydz$ |
| cylindrical (top, $\hat{n} = \hat{e}_z$) | $dS = rdrd\theta$ |
| cylindrical (side, $\hat{n} = \hat{e}_r$) | $dS = Rd\theta dz$ |
| spherical, ($\hat{n} = \hat{e}_r$) | $dS = R^2 \sin\theta d\theta d\phi$ |

| Coordinate system | volume differential $dV$ |
|---|---|
| Cartesian | $dV = dxdydz$ |
| cylindrical | $dV = rdrd\theta dz$ |
| spherical | $dV = r^2 \sin\theta drd\theta d\phi$ |

| Coordinate system | coordinates | basis vectors |
|---|---|---|
| spherical | $x = r\sin\theta\cos\phi$ | $\hat{e}_r = (\sin\theta\cos\phi\hat{e}_x) + (\sin\theta\sin\phi\hat{e}_y) + \cos\theta\hat{e}_z$ |
| | $y = r\sin\theta\sin\phi$ | $\hat{e}_\theta = (\cos\theta\cos\phi)\hat{e}_x + (\cos\theta\sin\phi)\hat{e}_y + (-\sin\theta)\hat{e}_z$ |
| | $z = r\cos\theta$ | $\hat{e}_\phi = (-\sin\phi)\hat{e}_x + \cos\phi\hat{e}_y$ |
| cylindrical | $x = r\cos\theta$ | $\hat{e}_r = \cos\theta\hat{e}_x + \sin\theta\hat{e}_y$ |
| | $y = r\sin\theta$ | $\hat{e}_\theta = (-\sin\theta)\hat{e}_x + \cos\theta\hat{e}_y$ |
| | $z = z$ | $\hat{e}_z = \hat{e}_z$ |

$$\text{Divergence Theorem} \qquad \iint_{\mathcal{S}} \hat{n} \cdot \underline{F}\, dS = \iiint_{\mathcal{V}} \nabla \cdot \underline{F}\, dV$$

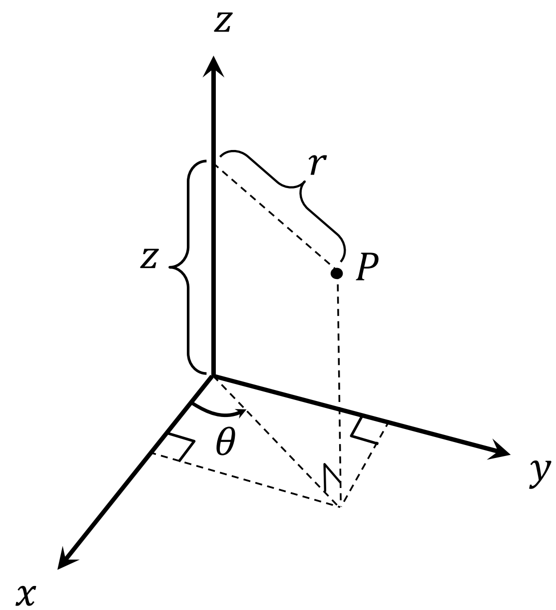
$$\text{Stokes Theorem} \qquad \oint_{\mathcal{C}} \hat{t} \cdot \underline{F}\, dl = \iint_{\mathcal{S}} \hat{n} \cdot (\nabla \times \underline{F})\, dS$$

Vector identities:

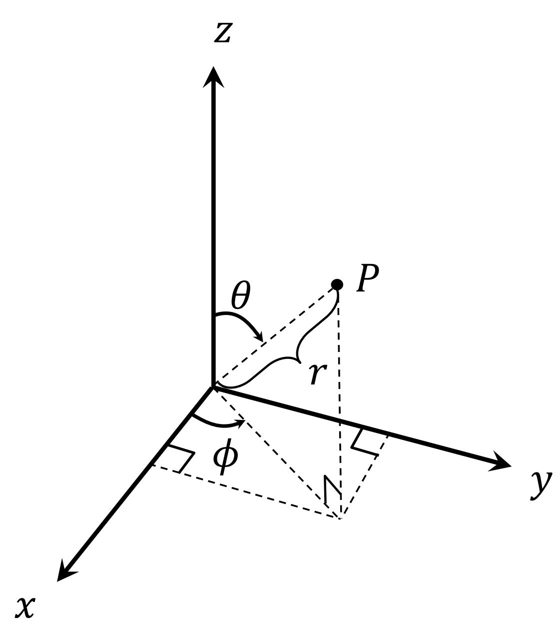
$$\begin{aligned}
\nabla \cdot \nabla \times \underline{F} &= 0 \quad \text{(Divergence of curl} = 0) \\
\nabla \times \nabla f &= 0 \quad \text{(Curl of gradient} = 0) \\
\nabla (fg) &= f\nabla g + g\nabla f \\
\underline{F} \cdot \nabla \underline{F} &= \frac{1}{2}\nabla \left(\underline{F}^2\right) - \underline{F} \times (\nabla \times \underline{F}) \\
\nabla \cdot (f\underline{F}) &= f\nabla \cdot \underline{F} + \underline{F} \cdot \nabla f \\
\nabla \times \nabla \times \underline{F} &= \nabla (\nabla \cdot \underline{F}) - \nabla^2 \underline{F} \\
\nabla \cdot (\underline{F} \times \underline{G}) &= \underline{G} \cdot (\nabla \times \underline{F}) - \underline{F} \cdot (\nabla \times \underline{G})
\end{aligned}$$

The equations in F. A. Morrison, ***An Introduction to Fluid Mechanics*** (Cambridge, 2013) assume the following definitions of the cylindrical and spherical coordinate systems.

**Cylindrical Coordinate System:** Note that the $\theta$-coordinate swings around the $z$-axis




**Spherical Coordinate System:** Note that the $\theta$-coordinate swings down from the $z$-axis; this is different from its definition in the cylindrical system above.



# The Equation of Continuity and the Equation of Motion in Cartesian, cylindrical, and spherical coordinates

**CM3110 Fall 2011 Faith A. Morrison**

**Continuity Equation**, Cartesian coordinates

$$\frac{\partial \rho}{\partial t} + \left( v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} \right) + \rho \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) = 0$$

**Continuity Equation**, cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial (\rho r v_r)}{\partial r} + \frac{1}{r}\frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{\partial (\rho v_z)}{\partial z} = 0$$

**Continuity Equation**, spherical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial (\rho r^2 v_r)}{\partial r} + \frac{1}{r \sin\theta}\frac{\partial (\rho v_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial (\rho v_\phi)}{\partial \phi} = 0$$

**Equation of Motion** for an incompressible fluid, 3 components in Cartesian coordinates

$$\begin{aligned}
\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) &= -\frac{\partial P}{\partial x} + \left(\frac{\partial \tilde{\tau}_{xx}}{\partial x} + \frac{\partial \tilde{\tau}_{yx}}{\partial y} + \frac{\partial \tilde{\tau}_{zx}}{\partial z}\right) + \rho g_x \\
\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) &= -\frac{\partial P}{\partial y} + \left(\frac{\partial \tilde{\tau}_{xy}}{\partial x} + \frac{\partial \tilde{\tau}_{yy}}{\partial y} + \frac{\partial \tilde{\tau}_{zy}}{\partial z}\right) + \rho g_y \\
\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) &= -\frac{\partial P}{\partial z} + \left(\frac{\partial \tilde{\tau}_{xz}}{\partial x} + \frac{\partial \tilde{\tau}_{yz}}{\partial y} + \frac{\partial \tilde{\tau}_{zz}}{\partial z}\right) + \rho g_z
\end{aligned}$$

**Equation of Motion** for an incompressible fluid, 3 components in cylindrical coordinates

$$\begin{aligned}
\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) &= -\frac{\partial P}{\partial r} + \left(\frac{1}{r}\frac{\partial (r\tilde{\tau}_{rr})}{\partial r} + \frac{1}{r}\frac{\partial \tilde{\tau}_{\theta r}}{\partial \theta} - \frac{\tilde{\tau}_{\theta\theta}}{r} + \frac{\partial \tilde{\tau}_{zr}}{\partial z}\right) + \rho g_r \\
\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta v_r}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) &= -\frac{1}{r}\frac{\partial P}{\partial \theta} + \left(\frac{1}{r^2}\frac{\partial (r^2\tilde{\tau}_{r\theta})}{\partial r} + \frac{1}{r}\frac{\partial \tilde{\tau}_{\theta\theta}}{\partial \theta} + \frac{\partial \tilde{\tau}_{z\theta}}{\partial z} + \frac{\tilde{\tau}_{\theta r} - \tilde{\tau}_{r\theta}}{r}\right) + \rho g_\theta \\
\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) &= -\frac{\partial P}{\partial z} + \left(\frac{1}{r}\frac{\partial (r\tilde{\tau}_{rz})}{\partial r} + \frac{1}{r}\frac{\partial \tilde{\tau}_{\theta z}}{\partial \theta} + \frac{\partial \tilde{\tau}_{zz}}{\partial z}\right) + \rho g_z
\end{aligned}$$

**Equation of Motion** for an incompressible fluid, 3 components in spherical coordinates

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right)$$

$$= -\frac{\partial P}{\partial r} + \left(\frac{1}{r^2}\frac{\partial (r^2\tilde{\tau}_{rr})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (\tilde{\tau}_{\theta r}\sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tilde{\tau}_{\phi r}}{\partial \phi} - \frac{\tilde{\tau}_{\theta\theta} + \tilde{\tau}_{\phi\phi}}{r}\right) + \rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2\cot\theta}{r}\right)$$

$$= -\frac{1}{r}\frac{\partial P}{\partial \theta} + \left(\frac{1}{r^3}\frac{\partial (r^3\tilde{\tau}_{r\theta})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (\tilde{\tau}_{\theta\theta}\sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tilde{\tau}_{\phi\theta}}{\partial \phi} + \frac{\tilde{\tau}_{\theta r} - \tilde{\tau}_{r\theta}}{r} - \frac{\tilde{\tau}_{\phi\phi}\cot\theta}{r}\right) + \rho g_\theta$$

$$\rho\left(\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{v_\phi v_\theta\cot\theta}{r}\right)$$

$$= -\frac{1}{r\sin\theta}\frac{\partial P}{\partial \phi} + \left(\frac{1}{r^3}\frac{\partial (r^3\tilde{\tau}_{r\phi})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial \left(\tilde{\tau}_{\theta\phi}\sin\theta\right)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tilde{\tau}_{\phi\phi}}{\partial \phi} + \frac{\tilde{\tau}_{\phi r} - \tilde{\tau}_{r\phi}}{r} + \frac{\tilde{\tau}_{\phi\theta}\cot\theta}{r}\right) + \rho g_\phi$$

**Equation of Motion** for incompressible, Newtonian fluid (Navier-Stokes equation) 3 components in Cartesian coordinates

$$\begin{aligned}
\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right) &= -\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)+\rho g_x \\
\rho\left(\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right) &= -\frac{\partial P}{\partial y}+\mu\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_y}{\partial z^2}\right)+\rho g_y \\
\rho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right) &= -\frac{\partial P}{\partial z}+\mu\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right)+\rho g_z
\end{aligned}$$

**Equation of Motion** for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in cylindrical coordinates

$$\begin{aligned}
\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right) &= -\frac{\partial P}{\partial r}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (rv_r)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}\right)+\rho g_r \\
\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right) &= -\frac{1}{r}\frac{\partial P}{\partial \theta}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (rv_\theta)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right)+\rho g_\theta \\
\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right) &= -\frac{\partial P}{\partial z}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right)+\rho g_z
\end{aligned}$$

**Equation of Motion** for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in spherical coordinates

$$\begin{aligned}
&\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi}-\frac{v_\theta^2+v_\phi^2}{r}\right) \\
&\quad= -\frac{\partial P}{\partial r}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right)\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v_r}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial \phi^2}\right. \\
&\qquad\left.-\frac{2}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(v_\theta\sin\theta\right)-\frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right)+\rho g_r \\
&\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}+\frac{v_r v_\theta}{r}-\frac{v_\phi^2\cot\theta}{r}\right) \\
&\quad= -\frac{1}{r}\frac{\partial P}{\partial \theta}+\mu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(v_\theta\sin\theta\right)\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial \phi^2}\right. \\
&\qquad\left.+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}-\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right)+\rho g_\theta \\
&\rho\left(\frac{\partial v_\phi}{\partial t}+v_r\frac{\partial v_\phi}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}+\frac{v_r v_\phi}{r}+\frac{v_\phi v_\theta\cot\theta}{r}\right) \\
&\quad= -\frac{1}{r\sin\theta}\frac{\partial P}{\partial \phi}+\mu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(v_\phi\sin\theta\right)\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial \phi^2}\right. \\
&\qquad\left.+\frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial \phi}+\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right)+\rho g_\phi
\end{aligned}$$

Note: the $r$-component of the Navier-Stokes equation in spherical coordinates may be simplified by adding $0 = \frac{2}{r}\nabla\cdot\underline{v}$ to the component shown above. This term is zero due to the continuity equation (mass conservation). See Bird et. al.

References:


1. R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena*, $2^{nd}$ edition, Wiley: NY, 2002.
2. R. B. Bird, R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric Fluids: Volume 1 Fluid Mechanics*, Wiley: NY, 1987.

Prof. Faith A. Morrison
Department of Chemical Engineering
MichiganTech

# FACTORS FOR UNIT CONVERSIONS

| Quantity | Equivalent Values |
|---|---|
| Mass | 1 kg = 1000 g = 0.001 metric ton = 2.20462 $lb_m$ = 35.27392 oz |
| | 1 $lb_m$ = 16 oz = 5 x $10^{-4}$ ton = 453.593 g = 0.453593 kg |
| Length | 1 m = 100 cm = 1000 mm = $10^6$ microns ($\mu$m) = $10^{10}$ angstroms (Å) |
| | = 39.37 in = 3.2808 ft = 1.0936 yd = 0.0006214 mile |
| | 1 ft = 12 in. = 1/3 yd = 0.3048 m = 30.48 cm |
| Volume | 1 $m^3$ = 1000 liters = $10^6$ $cm^3$ = $10^6$ ml |
| | = 35.3145 $ft^3$ = 220.83 imperial gallons = 264.17 gal |
| | = 1056.68 qt |
| | 1 $ft^3$ = 1728 $in^3$ = 7.4805 gal = 0.028317 $m^3$ = 28.317 liters |
| | = 28 317 $cm^3$ |
| Force | 1 N = 1 $kg \cdot m/s^2$ = $10^5$ dynes = $10^5$ $g \cdot cm/s^2$ = 0.22481 $lb_f$ |
| | 1 $lb_f$ = 32.174 $lb_m \cdot ft/s^2$ = 4.4482 N = 4.4482 x $10^5$ dynes |
| Pressure | 1 atm = 1.01325 x $10^5$ $N/m^2$ (Pa) = 101.325 kPa = 1.01325 bars |
| | = 1.01325 x $10^6$ $dynes/cm^2$ |
| | = 760 mm Hg at 0° C (torr) = 10.333 m $H_2O$ at 4° C |
| | = 14.696 $lb_f/in^2$ (psi) = 33.9 ft $H_2O$ at 4°C |
| | 100 kPa = 1 bar |
| Energy | 1 J = 1 $N \cdot m$ = $10^7$ ergs = $10^7$ $dyne \cdot cm$ |
| | = 2.778 x $10^{-7}$ $kW \cdot h$ = 0.23901 cal |
| | = 0.7376 $ft \cdot lb_f$ = 9.47817 x $10^{-4}$ Btu |
| Power | 1 W = 1 J/s = 0.23885 cal/s = 0.7376 $ft \cdot lb_f/s$ = 9.47817 x $10^{-4}$ Btu/s = 3.4121 Btu/h |
| | = 1.341 x $10^{-3}$ hp (horsepower) |
| Viscosity | 1 $Pa \cdot s$ = 1 $N \cdot s/m^2$ = 1 $kg/m \cdot s$ |
| | = 10 poise = 10 $dynes \cdot s/cm^2$ = 10 $g/cm \cdot s$ |
| | = $10^3$ cp (centipoise) |
| | = 0.67197 $lb_m/ft \cdot s$ = 2419.088 $lb_m/ft \cdot h$ |
| Density | 1 $kg/m^3$ = $10^{-3}$ $g/cm^3$ |
| | = 0.06243 $lb_m/ft^3$ |
| | $10^3$ $kg/m^3$ = 1 $g/cm^3$ = 62.428 $lb_m/ft^3$ |
| Volumetric Flow | 1 $m^3$/s= 35.3145 $ft^3$/s=15,850.2 gal/min (gpm) |
| | 1 gpm = 6.30907 x $10^{-5}$ $m^3$/s=2.22802 x $10^{-3}$ $ft^3$/s=3.7854 liter/min |
| | 1 liter/min=0.26417 gpm |

| | | |
|---|---|---|
| Temperature | $T(^oC) = \frac{5}{9}\left[T(^oF) - 32\right]$ | |
| | $T(^oF) = \frac{9}{5}T(^oC) + 32 = 1.8T(^oC) + 32$ | |
| Absolute Temperature | T(K) = T(˚C) + 273.15 | |
| | T(˚R) = T(˚F) + 459.67 | |
| Temperature Interval ($\Delta$T) | 1 C˚ = 1 K = 1.8 F˚ = 1.8 R˚ | |
| | 1F˚ = 1 R˚ = (5/9) C˚ = (5/9) K | |

## USEFUL QUANTITIES

SG = $\rho$(20˚C)/$\rho_{water}$ (4˚C)

$\rho_{water}$(4˚C) = 1000 kg/m$^3$ = 62.43 lb$_m$/ft$^3$ = 1.000 g/cm$^3$

$\rho_{water}$(25˚C) = 997.08 kg/m$^3$ = 62.25 lb$_m$/ft$^3$ = 0.99709 g/cm$^3$

g = 9.8066 m/s$^2$ = 980.66 cm/s$^2$ = 32.174 ft/s$^2$

$\mu_{water}$ (25˚C) = 8.937 x 10$^{-4}$ Pa·s = 8.937 x 10$^{-4}$ kg/m·s
= 0.8937 cp =0.8937 x 10$^{-2}$ g/cm·s = 6.005 x 10$^{-4}$ lb$_m$/ft·s

Composition of air:

| | |
|---|---|
| $N_2$ | 78.03% |
| $O_2$ | 20.99% |
| Ar | 0.94% |
| $CO_2$ | 0.03% |
| $H_2$, He, Ne, Kr, Xe | 0.01% |
| | 100.00% |

$M_{air}$ = 29 g/mol = 29 kg/kmol = 29 lb$_m$/lbmole

$\hat{C}_{p,water}$ (25˚C) = 4.182 kJ/kg K = 0.9989 cal/g˚C = 0.9997 Btu/lb$_m$˚F

R = 8.314 m$^3$·Pa/mol·K = 0.08314 liter·bar/mol·K = 0.08206 liter·atm/mol·K
= 62.36 liter·mm Hg/mol·K = 0.7302 ft$^3$·atm/lbmole·˚R
= 10.73 ft$^3$·psia/lbmole·˚R
= 8.314 J/mol·K
= 1.987 cal/mol·K = 1.987 Btu/lbmole·˚R

# Data Correlations for Examinations

CM3110 Transport Phenomena I
Michigan Technological University
Professor Faith A. Morrison

## I. Flow through Smooth Pipes

### A. All Reynolds numbers: Morrison

The correlation from Morrison (2013) fits the smooth pipe data for all Reynolds numbers; beyond $Re = 4000$ this correlation follows the Prandtl equation (see Figure 1; Morrison, equation 7.158). This correlation is explicit in $f$; when flow rate is known, $\Delta p$ may be found directly; when $\Delta p$ is known, $Q$ or $\langle v \rangle$ must be solved for iteratively.

Morrison (2013)

$$f = \left( \frac{0.0076 \left(\frac{3170}{Re}\right)^{0.165}}{1 + \left(\frac{3170}{Re}\right)^{7.0}} \right) + \frac{16}{Re} \qquad (1)$$

### B. $4,000 \leq Re \leq 1 \times 10^6$: Prandtl

The Prandtl correlation for $f(Re)$ in turbulent flow is not explicit in friction factor and must be solved iteratively except when $f$ is known (Morrison, equation 7.156). This is good only for $Re > 4,000$/

Prandtl
or VonKarman-Nikuradse
(Denn, 1980)

$$\frac{1}{\sqrt{f}} = 4.0 \log\left(Re\sqrt{f}\right) - 0.40 \qquad (2)$$

### C. $4,000 \leq Re \leq 1 \times 10^6$: A simplified Correlation

For the turbulent regime, an approximate correlation that is much simpler to work with (with a calculator on an exam, for example) is given here and shown in Figure 2 (Morrison, equation 7.157). This is good only for $Re > 4,000$.

Simplified Turbulent
(White, 1974)

$$f = \frac{1.02}{4} (\log Re)^{-2.5} \qquad (3)$$



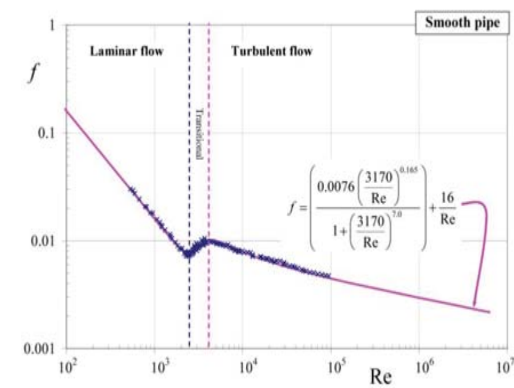



Figure 1: Equation 3 captures smooth pipe friction factor as a function of Reynolds number over the entire Reynolds-number range (Morrison, 2013) and is recommended for spreadsheet use. Also shown are Nikuradse's experimental data for flow in smooth pipes (Nikuradse, 1933). Use beyond $Re = 10^6$ is not recommended; for Re>4000 equation 3 follows the Prandtl equation. (Morrison, 2013, p532)

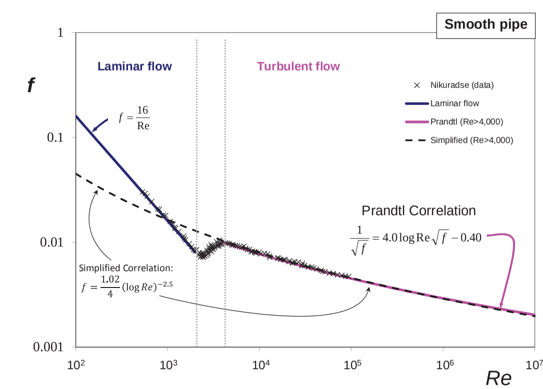



Figure 2: For turbulent flow, the simplified (equation 3) or Prandtl (equation 2) correlations may be used. For work with a calculator, the simplified correlation is perhaps the easiest to work with. (Morrison, 2013, p531)

## II. Flow Around a Sphere

### A. All Reynolds Numbers: Morrison

The correlation from Morrison (2013) fits the flow around a sphere for all Reynolds numbers (Figure 3; Morrison equation 8.83); beyond $Re = 10^6$ this correlation follows the curve shown in Figure 3.

Morrison (2013)

$$C_D = \frac{24}{Re} + \frac{2.6\left(\frac{Re}{5.0}\right)}{1+\left(\frac{Re}{5.0}\right)^{1.52}} + \frac{0.411\left(\frac{Re}{263{,}000}\right)^{-7.94}}{1+\left(\frac{Re}{263{,}000}\right)^{-8.00}} + \frac{0.25\left(\frac{Re}{10^6}\right)}{1+\left(\frac{Re}{10^6}\right)} \quad (4)$$

### Simplified Correlations

The correlations below (Morrison, 2013; equation 8.82) are simpler relationships more suitable to calculator/exam work.

$Re < 2$

$$C_D = \frac{24}{Re} \quad (5)$$

$0.1 \le Re \le 1{,}000$

$$C_D = \frac{24}{Re}(1 + 0.14 Re^{0.7}) \quad (6)$$

$1{,}000 \le Re \le 2.6 \times 10^5$

$$C_D = 0.445 \quad (7)$$

$2.8 \times 10^5 \le Re \le 10^6$

$$\frac{\log C_D}{\left(\frac{Re}{10^6}\right)} = 4.43 \log Re - 27.3 \quad (8)$$



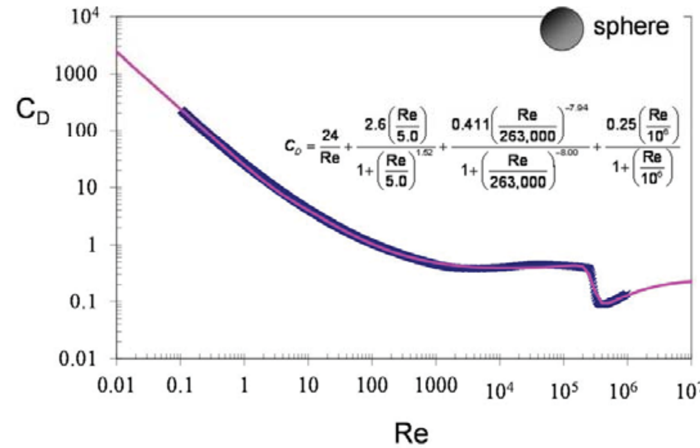



Figure 3: Equation 4 captures flow around a sphere as a function of Reynolds number over the entire Reynolds-number range (Morrison, 2013) and is recommended for spreadsheet use. Also shown are experimental data from White (1974). Use beyond $Re = 10^6$ is not recommended. (Morrison, 2013, p625)

**References**
M. Denn, Process Fluid Mechanics (Prentice Hall, Englewood Cliffs, NJ, 1980)
F. A. Morrison, *An Introduction to Fluid Mechanics* (Cambridge University Press, New York, 2013).
F. M. White, *Viscous Fluid Flow* (McGraw-Hill, Inc.: New York, 1974).

# The Newtonian Constitutive Equation in Cartesian, Cylindrical, and Spherical coordinates

Prof. Faith A. Morrison, Michigan Technological University

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## *Cartesian Coordinates*

$$\begin{pmatrix} \tilde{\tau}_{xx} & \tilde{\tau}_{xy} & \tilde{\tau}_{xz} \\ \tilde{\tau}_{yx} & \tilde{\tau}_{yy} & \tilde{\tau}_{yz} \\ \tilde{\tau}_{zx} & \tau_{zy} & \tilde{\tau}_{zz} \end{pmatrix}_{xyz} = \mu \begin{pmatrix} 2\frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} & \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \\ \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} & 2\frac{\partial v_y}{\partial y} & \frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \\ \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} & \frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} & 2\frac{\partial v_z}{\partial z} \end{pmatrix}_{xyz}$$

## *Cylindrical Coordinates*

$$\begin{pmatrix} \tilde{\tau}_{rr} & \tilde{\tau}_{r\theta} & \tilde{\tau}_{rz} \\ \tilde{\tau}_{\theta r} & \tilde{\tau}_{\theta\theta} & \tilde{\tau}_{\theta z} \\ \tilde{\tau}_{zr} & \tau_{z\theta} & \tilde{\tau}_{zz} \end{pmatrix}_{r\theta z} = \mu \begin{pmatrix} 2\frac{\partial v_r}{\partial r} & r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta} & \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \\ r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta} & 2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) & \frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} \\ \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} & \frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} & 2\frac{\partial v_z}{\partial z} \end{pmatrix}_{r\theta z}$$

## *Spherical Coordinates*

$$\begin{pmatrix} \tilde{\tau}_{rr} & \tilde{\tau}_{r\theta} & \tilde{\tau}_{r\phi} \\ \tilde{\tau}_{\theta r} & \tilde{\tau}_{\theta\theta} & \tilde{\tau}_{\theta\phi} \\ \tilde{\tau}_{\phi r} & \tau_{\phi\theta} & \tilde{\tau}_{\phi\phi} \end{pmatrix}_{r\theta\phi}$$

$$= \mu \begin{pmatrix} 2\frac{\partial v_r}{\partial r} & r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta} & \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right) \\ r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta} & 2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) & \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} \\ \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right) & \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} & 2\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta\cot\theta}{r}\right) \end{pmatrix}_{r\theta\phi}$$

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These expressions are general and are applicable to three-dimensional flows. For unidirectional flows they reduce to Newton's law of Viscosity. Reference: Faith A. Morrison, *An Introduction to Fluid Mechanics* (Cambridge University Press: New York, 2013)

16 April 2014

$$\text{hydraulic diameter (general)}, D_H \equiv \frac{4A_{xs}}{p}$$

$$\text{Po} = \text{Re}_{D_H} f_{D_H}$$

$$\frac{1}{\sqrt{f_{D_H}}} = 4.0 \log\left(\frac{\text{Re}_{D_H}\sqrt{f_{D_H}}}{\frac{\text{Po}_{\text{duct}}}{16}}\right) - 0.40$$

| Geometry | Po |
|---|---|
| Circle | 16 |
| Equilateral triangle | 13.33 |
| Slit | 24 |
| Ellipse (a,b) | $\frac{32\pi^2}{p}(a^2+b^2)$ |

---

$$\text{void fraction}, \varepsilon = \frac{\text{empty bed volume}}{\text{total bed volume}}$$

$$a_v = \frac{\text{total partical surface area}}{\text{particle volume}}$$

$$\text{superficial velocity}, v_0 = \frac{Q}{\mathcal{V}/L}$$

$$\text{hydraulic diameter for packed bed}, D_H = \frac{4\varepsilon}{(1-\varepsilon)a_v}$$

$$\text{Reynolds number for packed bed} = \frac{\rho(v_0/\varepsilon)D_H}{\mu}$$

$$\text{friction factor for packed bed} = \left(\frac{\Delta p}{L}\right)\left(\frac{D_H\varepsilon^2}{2\rho v_0^2}\right)$$

---

| Mechanism | $h, \frac{BTU}{hr\ ft^{2\,o}F}$ | $h, \frac{W}{m^2K}$ |
|---|---|---|
| Condensing steam | 1000-5000 | 5700-28,000 |
| Condensing organics | 200-500 | 1100-2800 |
| Boiling liquids | 300-5000 | 1700-28,000 |
| Moving water | 50-3000 | 280-17,000 |
| Moving hydrocarbons | 10-300 | 55-1700 |
| Still air | 0.5-4 | 2.8-23 |
| Moving air | 2-10 | 11.3-55 |

*Reference: C. J. Geankoplis, Magnitude of Some Heat-Transfer Coefficients, page 241*

**A.3-16 Thermal Conductivities, Densities, and Heat Capacities of Metals**

| *Material* | $t$ (°C) | $\rho$ $\left(\frac{kg}{m^3}\right)$ | $c_p$ $\left(\frac{kJ}{kg \cdot K}\right)$ | $k (W/m \cdot K)$ | | |
|---|---|---|---|---|---|---|
| Aluminum | 20 | 2707 | 0.896 | 202 (0°C)<br>230 (300°C) | 206 (100°C) | 215 (200°C) |
| Brass (70–30) | 20 | 8522 | 0.385 | 97 (0°C) | 104 (100°C) | 109 (200°C) |
| Cast iron | 20 | 7593 | 0.465 | 55 (0°C) | 52 (100°C) | 48 (200°C) |
| Copper | 20 | 8954 | 0.383 | 388 (0°C) | 377 (100°C) | 372 (200°C) |
| Lead | 20 | 11 370 | 0.130 | 35 (0°C) | 33 (100°C) | 31 (200°C) |
| Steel 1%C | 20 | 7801 | 0.473 | 45.3 (18°C)<br>43 (300°C) | 45 (100°C) | 45 (200°C) |
| 308 stainless | 20 | 7849 | 0.461 | 15.2 (100°C) | 21.6 (500°C) | |
| 304 stainless | 0 | 7817 | 0.461 | 13.8 (0°C) | 16.3 (100°C) | 18.9 (300°C) |
| Tin | 20 | 7304 | 0.227 | 62 (0°C) | 59 (100°C) | 57 (200°C) |

*Source:* L. S. Marks, *Mechanical Engineers' Handbook*, 5th ed. New York: McGraw-Hill Book Company, 1951; E. R. G. Eckert and R. M. Drake, *Heat and Mass Transfer*, 2nd ed. New York: McGraw-Hill Book Company, 1959; R. H. Perry and C. H. Chilton, *Chemical Engineers' Handbook*, 5th ed. New York: McGraw-Hill Book Company, 1973; National Research Council, *International Critical Tables*. New York: McGraw-Hill Book Company, 1929.

The **Equation of Energy** in Cartesian, cylindrical, and spherical coordinates for Newtonian fluids of constant density, with source term $S_e$. Source could be electrical energy due to current flow, chemical energy, etc. Two cases are presented: the general case where thermal conductivity may be a function of temperature (vector flux $\underline{\tilde{q}} = \underline{q}/area$ appears in the equations); and the more usual case, where thermal conductivity is constant.

Fall 2019 Faith A. Morrison, Michigan Technological University

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**Microscopic energy balance**, in terms of flux; Gibbs notation

$$\rho \hat{C}_p \left( \frac{\partial T}{\partial t} + \underline{v} \cdot \nabla T \right) = -\nabla \cdot \underline{\tilde{q}} + S_e$$

**Microscopic energy balance**, in terms of flux; Cartesian coordinates

$$\rho \hat{C}_p \left( \frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} \right) = -\left( \frac{\partial \tilde{q}_x}{\partial x} + \frac{\partial \tilde{q}_y}{\partial y} + \frac{\partial \tilde{q}_z}{\partial z} \right) + S_e$$

**Microscopic energy balance**, in terms of flux; cylindrical coordinates

$$\rho \hat{C}_p \left( \frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} \right) = -\left( \frac{1}{r} \frac{\partial (r \tilde{q}_r)}{\partial r} + \frac{1}{r} \frac{\partial \tilde{q}_\theta}{\partial \theta} + \frac{\partial \tilde{q}_z}{\partial z} \right) + S_e$$

**Microscopic energy balance**, in terms of flux; spherical coordinates

$$\rho \hat{C}_p \left( \frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + \frac{v_\phi}{r \sin\theta} \frac{\partial T}{\partial \phi} \right) = -\left( \frac{1}{r^2} \frac{\partial (r^2 \tilde{q}_r)}{\partial r} + \frac{1}{r \sin\theta} \frac{\partial (\tilde{q}_\theta \sin\theta)}{\partial \theta} + \frac{1}{r \sin\theta} \frac{\partial \tilde{q}_\phi}{\partial \phi} \right) + S_e$$

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**Fourier's law of heat conduction**, Gibbs notation: $\underline{\tilde{q}} = -k \nabla T$

**Fourier's law of heat conduction**, Cartesian coordinates: $\begin{pmatrix} \tilde{q}_x \\ \tilde{q}_y \\ \tilde{q}_z \end{pmatrix}_{xyz} = \begin{pmatrix} -k \frac{\partial T}{\partial x} \\ -k \frac{\partial T}{\partial y} \\ -k \frac{\partial T}{\partial z} \end{pmatrix}_{xyz}$

**Fourier's law of heat conduction**, cylindrical coordinates: $\begin{pmatrix} \tilde{q}_r \\ \tilde{q}_\theta \\ \tilde{q}_z \end{pmatrix}_{r\theta z} = \begin{pmatrix} -k \frac{\partial T}{\partial r} \\ -\frac{k}{r} \frac{\partial T}{\partial \theta} \\ -k \frac{\partial T}{\partial z} \end{pmatrix}_{r\theta z}$

**Fourier's law of heat conduction**, spherical coordinates: $\begin{pmatrix} \tilde{q}_r \\ \tilde{q}_\theta \\ \tilde{q}_\phi \end{pmatrix}_{r\theta\phi} = \begin{pmatrix} -k \frac{\partial T}{\partial r} \\ -\frac{k}{r} \frac{\partial T}{\partial \theta} \\ -\frac{k}{r \sin\theta} \frac{\partial T}{\partial \phi} \end{pmatrix}_{r\theta\phi}$

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The **Equation of Energy** for systems with **constant $k$**

**Microscopic energy balance**, constant thermal conductivity; Gibbs notation

$$\rho \hat{C}_p \left( \frac{\partial T}{\partial t} + \underline{v} \cdot \nabla T \right) = k \nabla^2 T + S_e$$

**Microscopic energy balance**, constant thermal conductivity; Cartesian coordinates

$$\rho \hat{C}_p \left( \frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} \right) = k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + S_e$$

**Microscopic energy balance**, constant thermal conductivity; cylindrical coordinates

$$\rho \hat{C}_p \left( \frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} \right) = k \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right) + S_e$$

**Microscopic energy balance**, constant thermal conductivity; spherical coordinates

$$\rho \hat{C}_p \left( \frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + \frac{v_\phi}{r \sin\theta} \frac{\partial T}{\partial \phi} \right)$$
$$= k \left( \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left( \sin\theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 T}{\partial \phi^2} \right) + S_e$$

Reference: F. A. Morrison, "Web Appendix to *An Introduction to Fluid Mechanics*," Cambridge University Press, New York, 2013 (*pages.mtu.edu/~fmorriso/cm310/IFMWebAppendixDMicroEBalanceMorrison.pdf*). This worksheet is on the web at *pages.mtu.edu/~fmorriso/cm310/energy.pdf*.

## A.2-11 Heat-Transfer Properties of Liquid Water, SI Units

| $T$ (°C) | $T$ (K) | $\rho$ (kg/m³) | $c_p$ (kJ/kg · K) | $\mu \times 10^3$ (Pa · s, or kg/m · s) | $k$ (W/m · K) | $N_{Pr}$ | $\beta \times 10^4$ (1/K) | $(g\beta\rho^2/\mu^2) \times 10^{-8}$ (1/K · m³) |
|---|---|---|---|---|---|---|---|---|
| 0 | 273.2 | 999.6 | 4.229 | 1.786 | 0.5694 | 13.3 | −0.630 | |
| 15.6 | 288.8 | 998.0 | 4.187 | 1.131 | 0.5884 | 8.07 | 1.44 | 10.93 |
| 26.7 | 299.9 | 996.4 | 4.183 | 0.860 | 0.6109 | 5.89 | 2.34 | 30.70 |
| 37.8 | 311.0 | 994.7 | 4.183 | 0.682 | 0.6283 | 4.51 | 3.24 | 68.0 |
| 65.6 | 338.8 | 981.9 | 4.187 | 0.432 | 0.6629 | 2.72 | 5.04 | 256.2 |
| 93.3 | 366.5 | 962.7 | 4.229 | 0.3066 | 0.6802 | 1.91 | 6.66 | 642 |
| 121.1 | 394.3 | 943.5 | 4.271 | 0.2381 | 0.6836 | 1.49 | 8.46 | 1300 |
| 148.9 | 422.1 | 917.9 | 4.312 | 0.1935 | 0.6836 | 1.22 | 10.08 | 2231 |
| 204.4 | 477.6 | 858.6 | 4.522 | 0.1384 | 0.6611 | 0.950 | 14.04 | 5308 |
| 260.0 | 533.2 | 784.9 | 4.982 | 0.1042 | 0.6040 | 0.859 | 19.8 | 11 030 |
| 315.6 | 588.8 | 679.2 | 6.322 | 0.0862 | 0.5071 | 1.07 | 31.5 | 19 260 |

## A.2-11 Heat-Transfer Properties of Liquid Water, English Units

| $T$ (°F) | $\rho$ $\left(\frac{lb_m}{ft^3}\right)$ | $c_p$ $\left(\frac{btu}{lb_m \cdot °F}\right)$ | $\mu \times 10^3$ $\left(\frac{lb_m}{ft \cdot s}\right)$ | $k$ $\left(\frac{btu}{h \cdot ft \cdot °F}\right)$ | $N_{Pr}$ | $\beta \times 10^4$ (1/°R) | $(g\beta\rho^2/\mu^2) \times 10^{-6}$ (1/°R · ft³) |
|---|---|---|---|---|---|---|---|
| 32 | 62.4 | 1.01 | 1.20 | 0.329 | 13.3 | −0.350 | |
| 60 | 62.3 | 1.00 | 0.760 | 0.340 | 8.07 | 0.800 | 17.2 |
| 80 | 62.2 | 0.999 | 0.578 | 0.353 | 5.89 | 1.30 | 48.3 |
| 100 | 62.1 | 0.999 | 0.458 | 0.363 | 4.51 | 1.80 | 107 |
| 150 | 61.3 | 1.00 | 0.290 | 0.383 | 2.72 | 2.80 | 403 |
| 200 | 60.1 | 1.01 | 0.206 | 0.393 | 1.91 | 3.70 | 1010 |
| 250 | 58.9 | 1.02 | 0.160 | 0.395 | 1.49 | 4.70 | 2045 |
| 300 | 57.3 | 1.03 | 0.130 | 0.395 | 1.22 | 5.60 | 3510 |
| 400 | 53.6 | 1.08 | 0.0930 | 0.382 | 0.950 | 7.80 | 8350 |
| 500 | 49.0 | 1.19 | 0.0700 | 0.349 | 0.859 | 11.0 | 17 350 |
| 600 | 42.4 | 1.51 | 0.0579 | 0.293 | 1.07 | 17.5 | 30 300 |

**Geankoplis, 4th edition**

**NOTE: Equate the label to the provided quantity in the supplied units. For example, for water at $0^oC$:**

$$\mu \times 10^3 = 1.786\ Pa\ s$$

$$\mu = 1.786 \times 10^{-3} Pa\ s$$

## A.3-3 Physical Properties of Air at 101.325 kPa (1 Atm Abs), SI Units

| $T$ (°C) | $T$ (K) | $\rho$ (kg/m³) | $c_p$ (kJ/kg·K) | $\mu \times 10^5$ (Pa·s, or kg/m·s) | $k$ (W/m·K) | $N_{Pr}$ | $\beta \times 10^3$ (1/K) | $g\beta\rho^2/\mu^2$ (1/K·m³) |
|---|---|---|---|---|---|---|---|---|
| −17.8 | 255.4 | 1.379 | 1.0048 | 1.62 | 0.02250 | 0.720 | 3.92 | $2.79 \times 10^8$ |
| 0 | 273.2 | 1.293 | 1.0048 | 1.72 | 0.02423 | 0.715 | 3.65 | $2.04 \times 10^8$ |
| 10.0 | 283.2 | 1.246 | 1.0048 | 1.78 | 0.02492 | 0.713 | 3.53 | $1.72 \times 10^8$ |
| 37.8 | 311.0 | 1.137 | 1.0048 | 1.90 | 0.02700 | 0.705 | 3.22 | $1.12 \times 10^8$ |
| 65.6 | 338.8 | 1.043 | 1.0090 | 2.03 | 0.02925 | 0.702 | 2.95 | $0.775 \times 10^8$ |
| 93.3 | 366.5 | 0.964 | 1.0090 | 2.15 | 0.03115 | 0.694 | 2.74 | $0.534 \times 10^8$ |
| 121.1 | 394.3 | 0.895 | 1.0132 | 2.27 | 0.03323 | 0.692 | 2.54 | $0.386 \times 10^8$ |
| 148.9 | 422.1 | 0.838 | 1.0174 | 2.37 | 0.03531 | 0.689 | 2.38 | $0.289 \times 10^8$ |
| 176.7 | 449.9 | 0.785 | 1.0216 | 2.50 | 0.03721 | 0.687 | 2.21 | $0.214 \times 10^8$ |
| 204.4 | 477.6 | 0.740 | 1.0258 | 2.60 | 0.03894 | 0.686 | 2.09 | $0.168 \times 10^8$ |
| 232.2 | 505.4 | 0.700 | 1.0300 | 2.71 | 0.04084 | 0.684 | 1.98 | $0.130 \times 10^8$ |
| 260.0 | 533.2 | 0.662 | 1.0341 | 2.80 | 0.04258 | 0.680 | 1.87 | $0.104 \times 10^8$ |

## A.3-3 Physical Properties of Air at 101.325 kPa (1 Atm Abs), English Units

| $T$ (°F) | $\rho$ ($lb_m/ft^3$) | $c_p$ ($btu/lb_m \cdot °F$) | $\mu$ (centipoise) | $k$ ($btu/h \cdot ft \cdot °F$) | $N_{Pr}$ | $\beta \times 10^3$ (1/°R) | $g\beta\rho^2/\mu^2$ (1/°R·ft³) |
|---|---|---|---|---|---|---|---|
| 0 | 0.0861 | 0.240 | 0.0162 | 0.0130 | 0.720 | 2.18 | $4.39 \times 10^6$ |
| 32 | 0.0807 | 0.240 | 0.0172 | 0.0140 | 0.715 | 2.03 | $3.21 \times 10^6$ |
| 50 | 0.0778 | 0.240 | 0.0178 | 0.0144 | 0.713 | 1.96 | $2.70 \times 10^6$ |
| 100 | 0.0710 | 0.240 | 0.0190 | 0.0156 | 0.705 | 1.79 | $1.76 \times 10^6$ |
| 150 | 0.0651 | 0.241 | 0.0203 | 0.0169 | 0.702 | 1.64 | $1.22 \times 10^6$ |
| 200 | 0.0602 | 0.241 | 0.0215 | 0.0180 | 0.694 | 1.52 | $0.840 \times 10^6$ |
| 250 | 0.0559 | 0.242 | 0.0227 | 0.0192 | 0.692 | 1.41 | $0.607 \times 10^6$ |
| 300 | 0.0523 | 0.243 | 0.0237 | 0.0204 | 0.689 | 1.32 | $0.454 \times 10^6$ |
| 350 | 0.0490 | 0.244 | 0.0250 | 0.0215 | 0.687 | 1.23 | $0.336 \times 10^6$ |
| 400 | 0.0462 | 0.245 | 0.0260 | 0.0225 | 0.686 | 1.16 | $0.264 \times 10^6$ |
| 450 | 0.0437 | 0.246 | 0.0271 | 0.0236 | 0.674 | 1.10 | $0.204 \times 10^6$ |
| 500 | 0.0413 | 0.247 | 0.0280 | 0.0246 | 0.680 | 1.04 | $0.163 \times 10^6$ |

*Source:* National Bureau of Standards, *Circular* 461C, 1947; 564, 1955; NBS–NACA, *Tables of Thermal Properties of Gases*, 1949; F. G. Keyes, *Trans. A.S.M.E.*, 73, 590, 597 (1951); 74, 1303 (1952); D. D. Wagman, *Selected Values of Chemical Thermodynamic Properties*. Washington, D.C.: National Bureau of Standards, 1953.

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**NOTE: Equate the label to the provided quantity in the supplied units. For example, for air at $0^oC$:**

$$\mu \times 10^5 = 1.72\ Pa\ s$$

$$\mu = 1.72 \times 10^{-5} Pa\ s$$