Exam 2
CM3110
Morrison
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① Total molecular fluid force

$$\underline{F} = \iint_{S'} \left(\hat{n} \cdot \underset{=}{\tilde{\Pi}}\right)_{\substack{\text{at} \\ \text{surface}}} dS$$

Total fluid torque

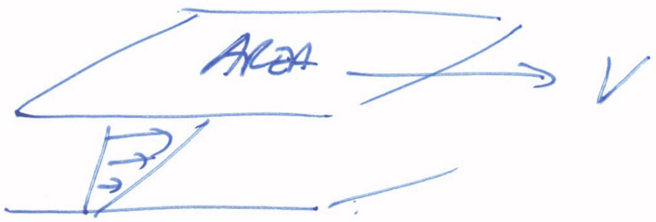
$$\underline{\mathcal{T}} = \iint_{S'} \underline{R} \times \left(n \cdot \underset{=}{\tilde{\Pi}}\right)_{\substack{\text{at} \\ \text{surface}}} dS$$

Flow rate

$$Q = \iint_{S'} \left(n \cdot \underline{v}\right)_{\substack{\text{at} \\ \text{surface}}} dS$$

2) a) Shear stress τ_{yx}

force on a y surface in the x-direction per unit area



$$\text{Shear stress} = \frac{\text{Force to pull plate at } V}{\text{AREA}}$$

produces the shear flow that allows us to measure viscosity

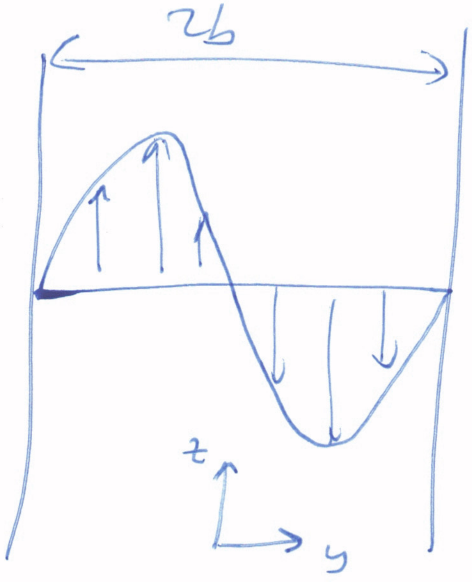
ⓑ Newton's law of viscosity

$$\tau_{21} = \mu \frac{dv_1}{dx_2} = \mu \frac{V}{H}$$

$$= \left(8.937 \times 10^{-4}\ \mathrm{Pa\,s}\right)\left(\frac{10^{-2}\ \mathrm{m}}{\mathrm{s}}\right)\left(\frac{1}{10^{-3}\ \mathrm{m}}\right)$$

$$= \boxed{8.9 \times 10^{-3}\ \mathrm{Pa}}$$

3)

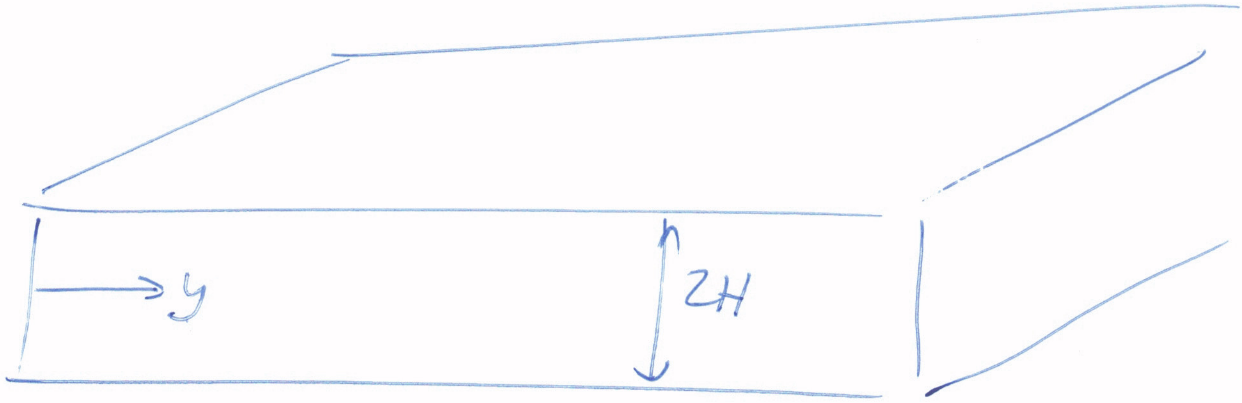


$y = b \quad v_z = 0$

$y = -b \quad v_z = 0$

no slip BC

4)



$X=0 \quad P=P_0$

$X=L \quad P=P_L$

What is $\tilde{\underline{\underline{\Pi}}}$

$$\underline{v} = \begin{pmatrix} 0 \\ v_y \\ 0 \end{pmatrix}_{xyz}$$

$$\tilde{\underline{\underline{\Pi}}} = -P\,\underline{\underline{I}} + \underline{\underline{\tau}}$$

$$v_y = v_y(x)$$

$$-P\,\underline{\underline{I}} = \begin{pmatrix} -P(y) & 0 & 0 \\ 0 & -P(y) & 0 \\ 0 & 0 & -P(y) \end{pmatrix}_{xyz}$$

$\tilde{\underline{\underline{\tau}}}$ (see sheet)

The Newtonian Constitutive Equation in Cartesian, Cylindrical, and Spherical coordinates

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Cartesian Coordinates

$$\begin{pmatrix} \tilde{\tau}_{xx} & \tilde{\tau}_{xy} & \tilde{\tau}_{xz} \\ \tilde{\tau}_{yx} & \tilde{\tau}_{yy} & \tilde{\tau}_{yz} \\ \tilde{\tau}_{zx} & \tau_{zy} & \tilde{\tau}_{zz} \end{pmatrix}_{xyz} = \mu \begin{pmatrix} \cancel{2\frac{\partial v_x}{\partial x}} & \cancel{\frac{\partial v_x}{\partial y}} + \frac{\partial v_y}{\partial x} & \cancel{\frac{\partial v_x}{\partial z}} + \cancel{\frac{\partial v_z}{\partial x}} \\ \cancel{\frac{\partial v_x}{\partial y}} + \frac{\partial v_y}{\partial x} & \cancel{2\frac{\partial v_y}{\partial y}} & \cancel{\frac{\partial v_y}{\partial z}} + \cancel{\frac{\partial v_z}{\partial y}} \\ \cancel{\frac{\partial v_x}{\partial z}} + \cancel{\frac{\partial v_z}{\partial x}} & \cancel{\frac{\partial v_y}{\partial z}} + \cancel{\frac{\partial v_z}{\partial y}} & \cancel{2\frac{\partial v_z}{\partial z}} \end{pmatrix}_{xyz}$$

Cylindrical Coordinates

$$\begin{pmatrix} \tilde{\tau}_{rr} & \tilde{\tau}_{r\theta} & \tilde{\tau}_{rz} \\ \tilde{\tau}_{\theta r} & \tilde{\tau}_{\theta\theta} & \tilde{\tau}_{\theta z} \\ \tilde{\tau}_{zr} & \tau_{z\theta} & \tilde{\tau}_{zz} \end{pmatrix}_{r\theta z} = \mu \begin{pmatrix} 2\frac{\partial v_r}{\partial r} & r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta} & \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \\ r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta} & 2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) & \frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} \\ \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} & \frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} & 2\frac{\partial v_z}{\partial z} \end{pmatrix}_{r\theta z}$$

Spherical Coordinates

$V_x = 0$

$V_z = 0$

$\frac{\partial V_y}{\partial y} = 0$

$\frac{\partial V_y}{\partial z} = 0$

$$\begin{pmatrix} \tilde{\tau}_{rr} & \tilde{\tau}_{r\theta} & \tilde{\tau}_{r\phi} \\ \tilde{\tau}_{\theta r} & \tilde{\tau}_{\theta\theta} & \tilde{\tau}_{\theta\phi} \\ \tilde{\tau}_{\phi r} & \tau_{\phi\theta} & \tilde{\tau}_{\phi\phi} \end{pmatrix}_{r\theta\phi}$$

$$= \mu \begin{pmatrix} 2\frac{\partial v_r}{\partial r} & r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta} & \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right) \\ r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta} & 2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) & \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} \\ \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right) & \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} & 2\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta\cot\theta}{r}\right) \end{pmatrix}_{r\theta\phi}$$

These expressions are general and are applicable to three-dimensional flows. For unidirectional flows they reduce to Newton's law of Viscosity. Reference: Faith A. Morrison, *An Introduction to Fluid Mechanics* (Cambridge University Press: New York, 2013)

16 April 2014

$$\underset{\approx}{\tau} = \begin{pmatrix} 0 & \mu \frac{\partial v_y}{\partial x} & 0 \\ \mu \frac{\partial v_y}{\partial x} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{xyz}$$

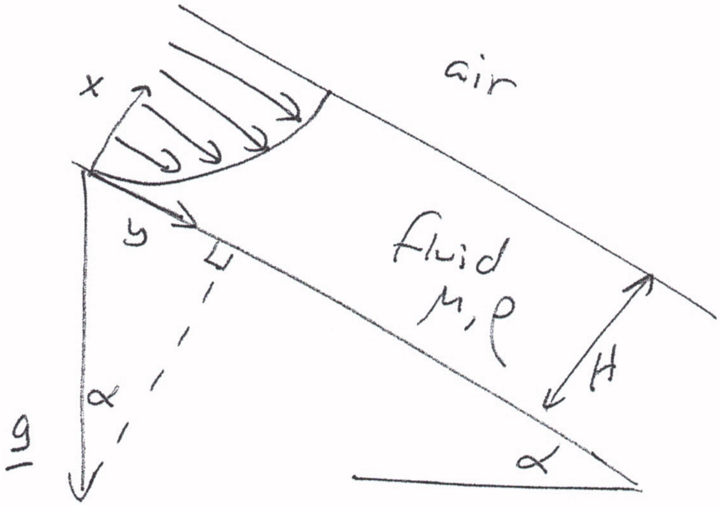
$$\mu \frac{\partial v_y}{\partial x} = \frac{\cancel{H^2}(P_L - P_0)}{\cancel{2}\cancel{\mu}L}\left(\frac{1}{\cancel{H^2}}\right)(\cancel{2}x)\cancel{\mu}$$

$$\mu \frac{\partial v_y}{\partial x} = \left(\frac{P_L - P_0}{L}\right) x$$

$$\underset{\approx}{\Pi} = \begin{pmatrix} \frac{(P_0 - P_L)y}{L} - P_0 & \frac{P_L - P_0}{L} x & 0 \\ \frac{P_L - P_0}{L} x & \frac{(P_0 - P_L)y}{L} - P_0 & 0 \\ 0 & 0 & \frac{(P_0 - P_L)y}{L} - P_0 \end{pmatrix}_{xyz}$$

check units? ✓

5) calculate $\underline{v}$



$$\underline{g} = \begin{pmatrix} -g\cos\alpha \\ g\sin\alpha \\ 0 \end{pmatrix}_{xyz}$$

$$\underline{v} = \begin{pmatrix} 0 \\ v_y \\ 0 \end{pmatrix}_{xyz}$$

1. continuity eqn
2. Navier Stokes
3. Solve
4. Boundary Conditions

- steady
- ρ = constant
- Newtonian
- wide flow $\frac{\partial v_y}{\partial z} = 0$

The Equation of Continuity and the Equation of Motion in Cartesian, cylindrical, and spherical coordinates

CM3110 Fall 2011 Faith A. Morrison

Continuity Equation, Cartesian coordinates

$$\underbrace{\cancel{\frac{\partial \rho}{\partial t}}}_{\text{Steady}} + \underbrace{\left(\cancel{v_x \frac{\partial \rho}{\partial x}} + \cancel{v_y \frac{\partial \rho}{\partial y}} + \cancel{v_z \frac{\partial \rho}{\partial z}} \right)}_{\text{constant density}} + \rho \left(\overset{v_x = 0}{\cancel{\frac{\partial v_x}{\partial x}}} + \frac{\partial v_y}{\partial y} + \underset{v_z = 0}{\cancel{\frac{\partial v_z}{\partial z}}} \right) = 0$$

Continuity Equation, cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial (\rho r v_r)}{\partial r} + \frac{1}{r}\frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{\partial (\rho v_z)}{\partial z} = 0$$

Continuity Equation, spherical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial (\rho r^2 v_r)}{\partial r} + \frac{1}{r \sin\theta}\frac{\partial (\rho v_\theta \sin\theta)}{\partial \theta} + \frac{1}{r \sin\theta}\frac{\partial (\rho v_\phi)}{\partial \phi} = 0$$

Equation of Motion for an incompressible fluid, 3 components in Cartesian coordinates

$$\Rightarrow \boxed{\frac{\partial v_y}{\partial y} = 0}$$

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial P}{\partial x} + \left(\frac{\partial \tilde{\tau}_{xx}}{\partial x} + \frac{\partial \tilde{\tau}_{yx}}{\partial y} + \frac{\partial \tilde{\tau}_{zx}}{\partial z} \right) + \rho g_x$$

$$\rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = -\frac{\partial P}{\partial y} + \left(\frac{\partial \tilde{\tau}_{xy}}{\partial x} + \frac{\partial \tilde{\tau}_{yy}}{\partial y} + \frac{\partial \tilde{\tau}_{zy}}{\partial z} \right) + \rho g_y$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial P}{\partial z} + \left(\frac{\partial \tilde{\tau}_{xz}}{\partial x} + \frac{\partial \tilde{\tau}_{yz}}{\partial y} + \frac{\partial \tilde{\tau}_{zz}}{\partial z} \right) + \rho g_z$$

Equation of Motion for an incompressible fluid, 3 components in cylindrical coordinates

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) = -\frac{\partial P}{\partial r} + \left(\frac{1}{r}\frac{\partial (r \tilde{\tau}_{rr})}{\partial r} + \frac{1}{r}\frac{\partial \tilde{\tau}_{\theta r}}{\partial \theta} - \frac{\tilde{\tau}_{\theta\theta}}{r} + \frac{\partial \tilde{\tau}_{zr}}{\partial z} \right) + \rho g_r$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta v_r}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \left(\frac{1}{r^2}\frac{\partial (r^2 \tilde{\tau}_{r\theta})}{\partial r} + \frac{1}{r}\frac{\partial \tilde{\tau}_{\theta\theta}}{\partial \theta} + \frac{\partial \tilde{\tau}_{z\theta}}{\partial z} + \frac{\tilde{\tau}_{\theta r} - \tilde{\tau}_{r\theta}}{r} \right) + \rho g_\theta$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial P}{\partial z} + \left(\frac{1}{r}\frac{\partial (r \tilde{\tau}_{rz})}{\partial r} + \frac{1}{r}\frac{\partial \tilde{\tau}_{\theta z}}{\partial \theta} + \frac{\partial \tilde{\tau}_{zz}}{\partial z} \right) + \rho g_z$$

Equation of Motion for an incompressible fluid, 3 components in spherical coordinates

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r} \right)$$

$$= -\frac{\partial P}{\partial r} + \left(\frac{1}{r^2}\frac{\partial (r^2 \tilde{\tau}_{rr})}{\partial r} + \frac{1}{r \sin\theta}\frac{\partial (\tilde{\tau}_{\theta r} \sin\theta)}{\partial \theta} + \frac{1}{r \sin\theta}\frac{\partial \tilde{\tau}_{\phi r}}{\partial \phi} - \frac{\tilde{\tau}_{\theta\theta} + \tilde{\tau}_{\phi\phi}}{r} \right) + \rho g_r$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2 \cot\theta}{r} \right)$$

$$= -\frac{1}{r}\frac{\partial P}{\partial \theta} + \left(\frac{1}{r^3}\frac{\partial (r^3 \tilde{\tau}_{r\theta})}{\partial r} + \frac{1}{r \sin\theta}\frac{\partial (\tilde{\tau}_{\theta\theta} \sin\theta)}{\partial \theta} + \frac{1}{r \sin\theta}\frac{\partial \tilde{\tau}_{\phi\theta}}{\partial \phi} + \frac{\tilde{\tau}_{\theta r} - \tilde{\tau}_{r\theta}}{r} - \frac{\tilde{\tau}_{\phi\phi} \cot\theta}{r} \right) + \rho g_\theta$$

$$\rho \left(\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{v_\phi v_\theta \cot\theta}{r} \right)$$

$$= -\frac{1}{r \sin\theta}\frac{\partial P}{\partial \phi} + \left(\frac{1}{r^3}\frac{\partial (r^3 \tilde{\tau}_{r\phi})}{\partial r} + \frac{1}{r \sin\theta}\frac{\partial (\tilde{\tau}_{\theta\phi} \sin\theta)}{\partial \theta} + \frac{1}{r \sin\theta}\frac{\partial \tilde{\tau}_{\phi\phi}}{\partial \phi} + \frac{\tilde{\tau}_{\phi r} - \tilde{\tau}_{r\phi}}{r} + \frac{\tilde{\tau}_{\phi\theta} \cot\theta}{r} \right) + \rho g_\phi$$

$\frac{\partial V_y}{\partial 5} = 0$ $V_x = 0$ $\frac{\partial V_y}{\partial y} = 0$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation) 3 components in Cartesian coordinates

$$\rho\left(\cancel{\frac{\partial v_x}{\partial t}} + \cancel{v_x\frac{\partial v_x}{\partial x}} + \cancel{v_y\frac{\partial v_x}{\partial y}} + \cancel{v_z\frac{\partial v_x}{\partial z}}\right) = -\frac{\partial P}{\partial x} + \mu\left(\cancel{\frac{\partial^2 v_x}{\partial x^2}} + \cancel{\frac{\partial^2 v_x}{\partial y^2}} + \cancel{\frac{\partial^2 v_x}{\partial z^2}}\right) + \rho g_x$$

$$\rho\left(\cancel{\frac{\partial v_y}{\partial t}} + \cancel{v_x\frac{\partial v_y}{\partial x}} + \boxed{v_y\frac{\partial v_y}{\partial y}} + \cancel{v_z\frac{\partial v_y}{\partial z}}\right) = -\frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \boxed{\frac{\partial^2 v_y}{\partial y^2}} + \cancel{\frac{\partial^2 v_y}{\partial z^2}}\right) + \rho g_y$$

Wide no variation

$$\rho\left(\cancel{\frac{\partial v_z}{\partial t}} + \cancel{v_x\frac{\partial v_z}{\partial x}} + \cancel{v_y\frac{\partial v_z}{\partial y}} + \cancel{v_z\frac{\partial v_z}{\partial z}}\right) = -\frac{\partial P}{\partial z} + \mu\left(\cancel{\frac{\partial^2 v_z}{\partial x^2}} + \cancel{\frac{\partial^2 v_z}{\partial y^2}} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}\right) + \cancel{\rho g_z}$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in cylindrical coordinates

Steady $V_z = 0$ $g_z = 0$

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial P}{\partial r} + \mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rv_r)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right) + \rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rv_\theta)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right) + \rho g_\theta$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right) + \rho g_z$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in spherical coordinates

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right)$$
$$= -\frac{\partial P}{\partial r} + \mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right)\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2}\right.$$
$$\left. - \frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(v_\theta\sin\theta\right) - \frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right) + \rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2\cot\theta}{r}\right)$$
$$= -\frac{1}{r}\frac{\partial P}{\partial \theta} + \mu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(v_\theta\sin\theta\right)\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial\phi^2}\right.$$
$$\left. + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right) + \rho g_\theta$$

$$\rho\left(\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{v_\phi v_\theta\cot\theta}{r}\right)$$
$$= -\frac{1}{r\sin\theta}\frac{\partial P}{\partial \phi} + \mu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(v_\phi\sin\theta\right)\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial\phi^2}\right.$$
$$\left. + \frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial\phi} + \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\theta}{\partial\phi}\right) + \rho g_\phi$$

Note: the r-component of the Navier-Stokes equation in spherical coordinates may be simplified by adding $0 = \frac{2}{r}\nabla\cdot\underline{v}$ to the component shown above. This term is zero due to the continuity equation (mass conservation). See Bird et. al.

References:

1. R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena*, 2nd edition, Wiley: NY, 2002.
2. R. B. Bird, R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric Fluids: Volume 1 Fluid Mechanics*, Wiley: NY, 1987.

x-component of Navier-Stokes:

$$\boxed{0 = -\frac{\partial P}{\partial x} + \rho(-g\cos\alpha)}$$ (hydro-static)

y-component of Navier Stokes

$$\boxed{0 = -\frac{\partial P}{\partial y} + \mu\frac{\partial^2 v_y}{\partial x^2} + \rho(g\sin\alpha)}$$

z-component of Navier Stokes

$$\boxed{0 = -\frac{\partial P}{\partial z}}$$ ⇒ $P \neq$ function of z (width direction)

$\frac{\partial P}{\partial y} = 0$ because at $x = H$ $P = P_{atm}$ and although $P = P(x)$ due to x-momentum (see above) this affects all positions y the same.

Solve

$$-\rho g \sin\alpha = \mu \frac{d^2 v_y}{dx^2}$$

$$\frac{d}{dx}\underbrace{\left(\frac{dv_y}{dx}\right)}_{\equiv \Phi} = \underbrace{\left(\frac{-\rho g \sin\alpha}{\mu}\right)}_{\equiv K}$$

$$\frac{d\Phi}{dx} = K$$

$$\frac{dv_y}{dx} = \Phi = Kx + C_1$$

$$\boxed{v_y = \frac{K}{2}x^2 + C_1 x + C_2}$$

Boundary Conditions:

$x = 0 \quad v_y = 0$ (no slip)

$x = H \quad \frac{dv_y}{dx} = 0$ (stress match at free surface)

BC1: $\boxed{0 = C_2}$

BC 2: $0 = \frac{dv_y}{dx} = KH + C_1$

$$\boxed{C_1 = -KH}$$

$$v_y = \frac{K}{2}X^2 - KHX$$

$$\boxed{v_y = \frac{\rho g \sin\alpha}{\mu}\left(HX - \frac{X^2}{2}\right)}$$

check BC: $X = 0 \quad v_y = 0$ ✓

$X = H \quad \frac{dv_y}{dx} = \left(\frac{\rho g \sin\alpha}{\mu}\right)\left(H - \frac{\cancel{2}\overset{H}{X}}{\cancel{2}}\right) = 0$ ✓

check units: $\frac{\cancel{kg}}{\cancel{m^3}}\,\frac{m}{s^{\cancel{2}}}\,\frac{m\,s\,\cancel{m^2}}{\cancel{kg}} = \frac{m}{s}$ ✓

//