Appendix D

Special Topics

D.1 Microscopic Energy Balance

A fundamental law of nature is that energy is conserved. The first law of thermodynamics relates the time rate-of-change of energy of a body to the heat flow into the system and the work done by the system[176].

$$\text{First law of thermodynamics:} \quad \frac{dE_B}{dt} = Q_{in,B} - W_{by,B} \qquad \text{(D.1)}$$
Energy of a body is conserved

The term $Q_{in,B}$ is the total rate of heat flow into the body, $W_{by,B}$ is the total rate of work done by the body, and E_B is the total energy of the body, comprised of internal, kinetic, and potential energy[186, 176] (Figure 6.3):

$$E_B = U + E_k + E_p \qquad \text{(D.2)}$$

We discuss the meaning of these terms and how to quantify them in Chapter 6.

To write equation D.1 on fluid systems, we need to adapt this expression from being the energy balance on a body to being an energy balance on a control volume. We carry out this derivation in this section of the web appendix. At the end of the discussion below, we arrive at equation 6.78 in the text, which is equation D.31 below.

We seek to write the energy balance on a control volume rather than on a body. As with the mass and momentum balances on a control volume (equations 6.28 and 6.29), changing from the balance on a body to the balance on a control volume results in the addition of a convective term. The analysis of Chapter 3 (Figure 3.22) that resulted in the Reynolds transport theorem can be carried out for energy balances too. The correct convective energy term that emerges is analogous to the convective term for mass, with energy per unit volume

$\rho\hat{E}$ replacing density ρ (mass per unit volume).

Energy balance on a CV
(First law of thermodynamics;
Reynolds transport theorem
for energy)

$$\frac{dE_{CV}}{dt} = Q_{in,CV} - W_{by,CV} + \iint_{CS} -(\hat{n} \cdot \underline{v})\rho\hat{E} \; dS \qquad \text{(D.3)}$$

In this equation E_{CV} is the total energy of the control volume, $\hat{E}$ is the energy per unit mass of the fluid, $\hat{E} = \hat{U} + \hat{E}_k + \hat{E}_p$, and the terms $Q_{in,CV}$ and $W_{by,CV}$ are the rate of heat addition to the control volume and the rate of work done by the control volume, respectively. The convective term (the integral) represents the net flow of energy into the control volume per unit time. Equation D.3 has the same form as the mass balance on a control volume (equation 6.28) and the momentum balance on a control volume (equation 6.29). The three balance equations written for a body and for a control volume are compared in Figure 6.4.

In fluid mechanics, the energy balance in equation D.3 applied to macroscopic control volumes leads to important relationships between pressure, fluid velocity, and work by engineering devices such as pumps and turbines (see discussions of the mechanical energy balance and the Bernoulli equation in Chapter 9). Equation D.3 applied to a microscopic control volume gives the microscopic energy balance, which is a fundamental relationship used to calculate properties in non-isothermal flows. In this text we concentrate on isothermal flows, and thus the microscopic energy balance is not of central importance to us. We derive the microscopic energy balance here for completeness and as background to the macroscopic energy balance discussion in Chapter 9. For more on non-isothermal flows see the literature[18].

To apply the energy balance given in equation D.3 to the arbitrary, microscopic control volume in Figure 6.1, we work term by term to write the indicated quantities as volume and surface integrals, and we subsequently perform mathematical manipulations similar to those used in deriving the microscopic mass and momentum equations. Beginning at the left of equation D.3, the rate-of-change-of-energy term written for the arbitrary control volume becomes an integral over the arbitrary control volume CV.

$$E_{CV} = \iiint_{CV} \rho\hat{E} \; dV \qquad \text{(D.4)}$$

$$\frac{dE_{CV}}{dt} = \frac{d}{dt} \iiint_{CV} \rho\hat{E} \; dV \qquad \text{(D.5)}$$

$$= \iiint_{CV} \frac{\partial}{\partial t}\left(\rho\hat{E}\right) dV \qquad \text{(D.6)}$$

We have used Leibniz rule for a constant volume to bring the time derivative inside the integral.

The heat flow into a control volume $Q_{in,CV}$ is due to conduction and source terms. Source terms are, for example, the heat generated by reaction or the heat generated by electrical dissipation in a wire.

$$Q_{in,CV} = Q_{in,conduction} + Q_{in,sources} \qquad \text{(D.7)}$$

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Conduction appears as a flux of energy through a surface. First we define the vector energy flux as $\underline{\tilde{q}}$.

$$
\begin{array}{c}
\text{Energy flux} \\
\text{due to conduction} \\
\text{in } \hat{q} \text{ direction}
\end{array}
\qquad
\underline{\tilde{q}} \;=\; \left(\frac{\text{energy}}{\text{area} \cdot \text{time}} \right) \hat{q}
\qquad (D.8)
$$

where $\hat{q}$ is the direction of the energy flux. To write net conduction into the arbitrary control volume of Figure 6.1, we consider small pieces of the tangent plane ΔS_i with outwardly pointing unit normal $\hat{n}$. Only the portion of $\underline{\tilde{q}}$ that is normal to ΔS_i brings conducted energy across the surface. Thus,

$$
\begin{array}{c}
\text{Energy flux through } \Delta S_i \\
\text{due to conduction} \\
(\hat{n} \text{ normal to } \Delta S_i)
\end{array}
\qquad = \quad [\hat{n} \cdot \hat{q}]\, \Delta S_i
\qquad (D.9)
$$

We can write the total conductive energy into CV as a sum over the fluxes through these small pieces of the tangent plane ΔS_i.

$$
\begin{array}{c}
\text{Heat flow into } CV \\
\text{due to conduction}
\end{array}
\qquad
Q_{in,conduction} \;\approx\; - \sum_{\substack{\text{control} \\ \text{surfaces} \\ \Delta S_i}} \Delta S_i \; \left[\hat{n} \cdot \underline{\tilde{q}} \right]_i
\qquad (D.10)
$$

The negative sign changes the outflow (due to the sign convention on $\hat{n}$) to inflow. Following the derivations early in Chapter 6 on similar quantities, we observe that equation D.10 is in the form of equation 6.8; thus

$$
Q_{in,conduction} \;=\; - \lim_{\Delta A \longrightarrow 0} \left[\sum_{i=1}^{N} \Delta S_i \; \left[\hat{n} \cdot \underline{\tilde{q}} \right]_i \right]
\qquad (D.11)
$$

$$
=\; - \iint_{CS} (\hat{n} \cdot \underline{\tilde{q}})\, dS
\qquad (D.12)
$$

$$
=\; - \iiint_{CV} \nabla \cdot \underline{\tilde{q}}\, dV
\qquad (D.13)
$$

where we have used the divergence theorem to arrive at the volume integral from the surface integral.

The heat flow into a control volume due electrical sources is a straightforward term that may be written as an integral over the control volume.

$$
\begin{array}{c}
\text{Heat flow in} \\
\text{due to chemical} \\
\text{or electrical sources}
\end{array}
\qquad
Q_{in,sources} \;=\; \iiint_{CV} \left(\frac{\text{source energy}}{\text{time} \cdot \text{volume}} \right) dV
\qquad (D.14)
$$

$$
Q_{in,sources} \;=\; \iiint_{CV} \mathcal{S}_e\, dV
\qquad (D.15)
$$

where $\mathcal{S}_e$ is the energy produced per time per volume by the sources.

The rate-of-work done by the fluid in the control volume $W_{by,CV}$ may be calculated from the definition of work, the inner (dot) product of force and displacement; rate-of-work is force dotted with velocity[186].

$$\text{Work defined} \qquad (\text{force}) \cdot (\text{displacement}) \;=\; \underline{f} \cdot \underline{x} \qquad \text{(D.16)}$$
$$\text{Rate-of-work defined} \qquad\qquad W \;\equiv\; \underline{f} \cdot \underline{v} \qquad \text{(D.17)}$$

The forces on the control volume were identified during the momentum-balance discussion: the surface contact forces due to molecular interactions and the body forces. Body force (gravity) has already been accounted for in equation D.3 in the $d(m\hat{E})_{CV}/dt$ term by virtue of including the potential energy there. This leaves just the work done the molecular forces to be included in $W_{by,CV}$. We showed in Chapter 4 that the molecular force $\underline{\mathcal{F}}$ on a surface $\mathcal{S}$ may be written as

$$\underline{\mathcal{F}} = \iint_{\mathcal{S}} \left[\hat{n} \cdot \underline{\tilde{\underline{\Pi}}} \right]_{\text{at surface}} dS \qquad \text{(D.18)}$$

The rate-of-work done by these forces is therefore

$$\begin{pmatrix} \text{rate of work} \\ \text{on CV due to} \\ \text{molecular forces} \end{pmatrix} \quad W_{by,molecular\,forces} \;=\; -\iint_{CS} (\hat{n} \cdot \underline{\tilde{\underline{\Pi}}}) \cdot \underline{v}\, dS \qquad \text{(D.19)}$$

Note that since $\underline{\mathcal{F}}$ is the molecular force *on* the control volume, we must introduce a negative sign to indicate work *by* the fluid in the control volume. As usual, we can break the total molecular stress $\underline{\tilde{\underline{\Pi}}}$ up into two parts, $-p\underline{\underline{I}}$ and $\underline{\tilde{\tau}}$. With the assistance of Table B.1 we obtain

$$W_{by,molecular\,forces} \;=\; -\iint_{CS} \left(\hat{n} \cdot (-p\underline{\underline{I}} + \underline{\tilde{\tau}}) \right) \cdot \underline{v}\, dS \qquad \text{(D.20)}$$

$$=\; \iint_{CS} p\hat{n} \cdot \underline{v}\, dS - \iint_{CS} \hat{n} \cdot \left(\underline{\tilde{\tau}} \cdot \underline{v} \right)\, dS \qquad \text{(D.21)}$$

$$=\; \iiint_{CV} \nabla \cdot (p\underline{v})\, dV - \iiint_{CV} \nabla \cdot \left(\underline{\tilde{\tau}} \cdot \underline{v} \right)\, dV \qquad \text{(D.22)}$$

Again we have used the divergence theorem to arrive at the final expression above.

We now assemble the individual terms we have derived into the energy balance on a microscopic control volume, equation D.3; our development parallels the derivation of the

microscopic mass and momentum balances.

Energy balance on CV

$$\frac{dE_{CV}}{dt} = Q_{in,CV} - W_{by,CV} + \iint_{CS} -(\hat{n} \cdot \underline{v})\rho\hat{E} \, dS \quad \text{(D.23)}$$

$$\frac{dE_{CV}}{dt} = Q_{in,conduction} + Q_{in,sources}$$

$$-W_{by,molecular\,forces}$$

$$+ \iint_{CS} -(\hat{n} \cdot \underline{v})\rho\hat{E} \, dS \quad \text{(D.24)}$$

$$\iiint_{CV} \frac{\partial\rho\hat{E}}{\partial t} \, dV = -\iiint_{CV} \nabla \cdot \tilde{\underline{q}} \, dV + \iiint_{CV} \mathcal{S}_e \, dV$$

$$-\iiint_{CV} \nabla \cdot (p\underline{v}) \, dV + \iiint_{CV} \nabla \cdot \left(\tilde{\underline{\underline{\tau}}} \cdot \underline{v}\right) \, dV$$

$$-\iiint_{CV} \nabla \cdot \left(\underline{v}\rho\hat{E}\right) \, dV \quad \text{(D.25)}$$

We have applied the divergence theorem to the convective term. Combining the terms of equation D.25 under one integral we obtain,

$$\iiint_{CV} \left[\frac{\partial\left(\rho\hat{E}\right)}{\partial t} + \nabla \cdot \tilde{\underline{q}} - \mathcal{S}_e + \nabla \cdot (p\underline{v}) - \nabla \cdot \left(\tilde{\underline{\underline{\tau}}} \cdot \underline{v}\right) + \nabla \cdot \left(\underline{v}\rho\hat{E}\right)\right] \, dV = 0 \quad \text{(D.26)}$$

Because the control volume, which serves as the limits of the integration, is arbitrary, the integrand must be everywhere zero, resulting in a microscopic energy balance.

$$\frac{\partial\left(\rho\hat{E}\right)}{\partial t} + \nabla \cdot \tilde{\underline{q}} - \mathcal{S}_e + \nabla \cdot (p\underline{v}) - \nabla \cdot \left(\tilde{\underline{\underline{\tau}}} \cdot \underline{v}\right) + \nabla \cdot \left(\underline{v}\rho\hat{E}\right) = 0 \quad \text{(D.27)}$$

Microscopic energy balance

$$\frac{\partial\left(\rho\hat{E}\right)}{\partial t} + \nabla \cdot \left(\underline{v}\rho\hat{E}\right) = -\nabla \cdot \tilde{\underline{q}} - \nabla \cdot (p\underline{v}) + \nabla \cdot \left(\tilde{\underline{\underline{\tau}}} \cdot \underline{v}\right) + \mathcal{S}_e \quad \text{(D.28)}$$

Equation D.28 is the microscopic energy balance, but a more useful form maybe derived by working a bit on several terms. The left-hand side of equation D.28 may be simplified if both the time and spatial derivative expressions are expanded using the product rule of differentiation. Using Einstein notation we can show that the left-hand side of equation D.28 is equivalent to (see Table B.1 and problem 6.14)

$$\frac{\partial\left(\rho\hat{E}\right)}{\partial t} + \nabla \cdot \left(\underline{v}\rho\hat{E}\right) = \rho\frac{\partial\hat{E}}{\partial t} + \hat{E}\left[\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\underline{v})\right] + \rho\left(\underline{v} \cdot \nabla\hat{E}\right) \quad \text{(D.29)}$$

$$= \rho\left(\frac{\partial\hat{E}}{\partial t} + \underline{v} \cdot \nabla\hat{E}\right) \quad \text{(D.30)}$$

The terms in square brackets in equation D.29 are all the terms of the continuity equation (equation 6.41), and thus this combination sums to zero. The energy equation is thus

Microscopic
energy
balance

$$\boxed{\rho\left(\frac{\partial \hat{E}}{\partial t} + \underline{v} \cdot \nabla \hat{E}\right) = -\nabla \cdot \underline{\tilde{q}} - \nabla \cdot (p\underline{v}) + \nabla \cdot (\underline{\underline{\tilde{\tau}}} \cdot \underline{v}) + \mathcal{S}_e} \qquad \text{(D.31)}$$

$$\begin{pmatrix} \text{rate of} \\ \text{change of} \\ \text{energy} \end{pmatrix} + \begin{pmatrix} \text{net} \\ \text{convection} \\ \text{out} \end{pmatrix} = \begin{pmatrix} \text{net} \\ \text{conduction} \\ \text{in} \end{pmatrix} - \begin{pmatrix} \text{pressure/} \\ \text{volume} \\ \text{work by} \end{pmatrix}$$

$$- \begin{pmatrix} \text{viscous} \\ \text{work} \\ \text{by} \end{pmatrix} + \begin{pmatrix} \text{current/} \\ \text{reaction} \\ \text{heat in} \end{pmatrix}$$

Equation D.31 expresses the balance of total energy at a point in a flow. The time rate-of-change of the energy plus the net convection in is balanced by heat flow in (due to conduction and sources) and work done by the fluid (through pressure/volume changes or through viscous action). This is the first law of thermodynamics applied at a point.

The left-hand side of equation D.31 is the substantial derivative of specific energy $\hat{E} = \hat{U} + \hat{E}_k + \hat{E}_p$. If we compare this equation to the microscopic mass balance (continuity equation) and the microscopic momentum balance (Cauchy momentum equation), we see that these too have substantial derivatives on the left-hand side (see Figure 6.5). The substantial derivative is composed of the time-rate-of-change of the function and the convective term. As discussed in section 1.3, the substantial derivative represents the changes taking place along a very particular path, the path of a particle of fluid traveling along a streamline in the flow. From our derivations we know that the convective term arises in all three microscopic balances from converting the balance equations from balances on a body to balances on a control volume (Figure 6.4). Further development of the microscopic energy balance is given in section D.2.

The behavior of isothermal flows is mostly determined by the mass and momentum balances, and the energy balance is of limited use. One important use is that the energy balance on an arbitrary control volume (equation D.3) is the starting point for the derivation of the macroscopic energy balances of Chapters 1 and 9, including the mechanical energy balance and the macroscopic Bernoulli equation. The main value of the microscopic energy balance in equation D.31 is in non-isothermal flow problems to account for the interaction of the flow field with the temperature field. An in-depth discussion of non-isothermal flow problems is beyond the scope of this text; the reader is encouraged to consult the literature for more on this topic[18, 23].

D.2 Further Development of the Microscopic Energy Equation

In section D.1 we derived the microscopic energy balance (equation D.31):

Microscopic energy balance
$$\rho\left(\frac{\partial \hat{E}}{\partial t} + \underline{v}\cdot\nabla\hat{E}\right) = -\nabla\cdot\underline{\tilde{q}} - \nabla\cdot(p\underline{v}) + \nabla\cdot\underline{\underline{\tilde{\tau}}}\cdot\underline{v} + \mathcal{S}_e \qquad (D.32)$$

$$\begin{pmatrix} \text{rate of} \\ \text{change of} \\ \text{energy} \end{pmatrix} + \begin{pmatrix} \text{net} \\ \text{convection} \\ \text{in} \end{pmatrix} = \begin{pmatrix} \text{net} \\ \text{conduction} \\ \text{in} \end{pmatrix} + \begin{pmatrix} \text{pressure/} \\ \text{volume} \\ \text{work by} \end{pmatrix}$$

$$+ \begin{pmatrix} \text{viscous} \\ \text{work} \\ \text{by} \end{pmatrix} + \begin{pmatrix} \text{current/} \\ \text{reaction} \\ \text{heat in} \end{pmatrix}$$

The left-hand side gives the time rate-of-change and the convective rate-of-change of the specific energy $\hat{E} = \hat{E}_k + \hat{E}_p + \hat{U}$. These terms together are the substantial derivative of $\hat{E}$ (section 1.3.3). On the right-hand side of equation D.32 there are terms to account for heat flow in due to conduction, work done by the fluid due to pressure and viscous forces, and heat flow in due to sources such as reaction or electrical current.

We can also write individual expressions for the rate of change of internal, kinetic, and potential energy, and we do this in this section. As we see below, writing the expressions for the individual types of energy shows us that there are some extra terms that appear in these individual expressions. When the different energy terms are summed together, the total energy balance equation D.32 is recovered. Writing the individual rate of change of energy expressions helps us to understand the energy exchanges that occur in a fluid among the three types of energy.

D.2.1 Kinetic Energy

Kinetic energy is the energy associated with the fluid's velocity. To find the kinetic energy, we begin then with the momentum balance; we choose the form that includes the stress tensor (the Cauchy momentum equation).

Cauchy momentum equation (microscopic momentum balance)
$$\rho\frac{\partial \underline{v}}{\partial t} + \rho\underline{v}\cdot\nabla\underline{v} = -\nabla p + \nabla\cdot\underline{\underline{\tilde{\tau}}} + \rho\underline{g} \qquad (D.33)$$

Energy is the capacity to do work, and work is force times displacement[186]. To convert the force terms of the microscopic momentum balance into work terms, we need to dot the Cauchy momentum equation with the velocity $\underline{v}$[18].

$$\text{Rate of work defined:} \qquad W \;=\; \underline{f}\cdot\underline{v} \qquad\qquad (D.34)$$

$$\underline{v} \cdot \left(\rho \frac{\partial \underline{v}}{\partial t} + \rho \underline{v} \cdot \nabla \underline{v} = -\nabla p + \nabla \cdot \underline{\tilde{\tau}} + \rho \underline{g} \right) \tag{D.35}$$

$$\rho \underline{v} \cdot \frac{\partial \underline{v}}{\partial t} + \rho \underline{v} \cdot (\underline{v} \cdot \nabla \underline{v}) = -\underline{v} \cdot \nabla p + \underline{v} \cdot \left(\nabla \cdot \underline{\tilde{\tau}} \right) + \rho \underline{v} \cdot \underline{g} \tag{D.36}$$

We can identify the left-hand side of the result in equation D.36 as the total rate of change of the kinetic energy, both the time derivative and the convective term; the derivation follows. The quantity $\frac{1}{2} m_B v^2$ is the kinetic energy associated with a body of mass m_B moving with a speed v. For fluid of density ρ moving with velocity $\underline{v}$ the kinetic energy per unit mass is given by

$$\begin{array}{c} \text{Kinetic energy} \\ \text{per unit mass} \end{array} \qquad \hat{E}_k = \frac{1}{2} v^2 = \frac{1}{2} \underline{v} \cdot \underline{v} = \frac{1}{2} \underline{v}^2 \tag{D.37}$$

where $\underline{v}^2 = \underline{v} \cdot \underline{v}$. Through vector manipulations, the left-hand side of equation D.36 becomes

$$\rho \frac{\partial \left(\frac{1}{2} \underline{v}^2 \right)}{\partial t} + \rho \underline{v} \cdot \left(\frac{1}{2} \underline{v}^2 \right) = -\underline{v} \cdot \nabla p + \underline{v} \cdot \left(\nabla \cdot \underline{\tilde{\tau}} \right) + \rho \underline{v} \cdot \underline{g} \tag{D.38}$$

$$\rho \left(\frac{\partial \hat{E}_k}{\partial t} + \underline{v} \cdot \nabla \hat{E}_k \right) = -\underline{v} \cdot \nabla p + \underline{v} \cdot \left(\nabla \cdot \underline{\tilde{\tau}} \right) + \rho \underline{v} \cdot \underline{g} \tag{D.39}$$

where $\hat{E}_k = \underline{v}^2/2$. The right-hand side of equation D.39 may be expanded by using the following two identities from Table B.1, which result from applying the product rule of differentiation to the appropriate quantities.

$$\nabla \cdot (p\underline{v}) = p(\nabla \cdot \underline{v}) + \underline{v} \cdot \nabla p \tag{D.40}$$

$$\nabla \cdot (\underline{\tilde{\tau}} \cdot \underline{v}) = \underline{\tilde{\tau}}^T : \nabla \underline{v} + \underline{v} \cdot \left(\nabla \cdot \underline{\tilde{\tau}} \right) \tag{D.41}$$

Solving for the terms that appear in equation D.39 and substituting these into the that equation we obtain

$$\begin{array}{c} \text{Kinetic} \\ \text{energy} \\ \text{equation} \end{array} \quad \rho \left(\frac{\partial \hat{E}_k}{\partial t} + \underline{v} \cdot \nabla \hat{E}_k \right) = -\nabla \cdot (p\underline{v}) + p(\nabla \cdot \underline{v})$$

$$+ \nabla \cdot \left(\underline{\tilde{\tau}} \cdot \underline{v} \right) - \underline{\tilde{\tau}}^T : \nabla \underline{v}$$

$$+ \rho \underline{v} \cdot \underline{g} \tag{D.42}$$

Equation D.42 is an equation for the kinetic energy changes in the fluid. The left-hand side is the substantial derivative of $\hat{E}_k$. The right-hand side terms describe changes in kinetic energy due to two types of pressure effects, two types of viscous effects, and due to kinetic energy storage into gravitational potential energy. We can distinguish between the two types of pressure and viscous effects later in our discussion, once we have arrived at the equation for internal energy.

D.2.2   Potential Energy

We can isolate the potential energy effects by considering the term $\rho \underline{v} \cdot \underline{g}$, which appears in the kinetic energy equation (equation D.42). This term represents the kinetic energy to be gained or lost as fluid moves in a gravity potential field; in this context the acceleration due to gravity is a force per unit mass.

$$\underline{f}_{gravity} = m_B \underline{g} \tag{D.43}$$

$$\frac{\text{force}}{\text{mass}} = \underline{g} \tag{D.44}$$

and thus rate-of-work $\underline{f} \cdot \underline{v}$ of gravity per unit volume is $\rho(\underline{g} \cdot \underline{v})$. For a conservative force such as gravity[186], we can relate the force per unit mass $\underline{g}$ to the gradient of an associated potential energy $\hat{E}_p$:

$$\begin{array}{cc}\text{Potential energy} \\ \text{due to gravity}\end{array} \qquad \underline{g} = -\nabla \hat{E}_p \tag{D.45}$$

If we now dot multiply $\underline{v}$ on both sides of equation D.45 and multiply by ρ we obtain

$$\rho \underline{v} \cdot \nabla \hat{E}_p = -\rho \underline{v} \cdot \underline{g} \tag{D.46}$$

Since the potential energy field (the gravity field) does not change with time, $\partial \hat{E}_p / \partial t = 0$. It does no harm, therefore, to incorporate the time-derivative of potential energy into our equations. Multiplying $\partial \hat{E}_p / \partial t = 0$ by ρ and adding it to both sides of equation D.46 we obtain

$$\text{Potential energy equation} \quad \rho \left(\frac{\partial \hat{E}_p}{\partial t} + \underline{v} \cdot \nabla \hat{E}_p \right) = -\rho \underline{v} \cdot \underline{g} \tag{D.47}$$

which is an equation for the substantial derivative of potential energy. Equation D.47 is analogous to the kinetic energy equation, equation D.42.

D.2.3   Internal Energy

Internal energy is the energy that is stored in the temperature of the material or in the material's chemical arrangement, such as whether the material is a solid, a liquid, or a gas or what chemical structure the material has. We can obtain the expression for the rate-of-change of the internal energy of a fluid by subtracting the previously determined expressions for kinetic and potential energy.

Before proceeding to this subtraction, we first work a bit on the conductive-flux term of the total energy balance, equation D.32. The first term on the right-hand side of equation D.32 is the heat flow in due to conduction. The conductive flux term may be written

in terms of temperature by using Fourier's law of heat conduction[17],

$$\text{Fourier's law of heat conduction} \qquad \boxed{\underline{\tilde{q}} = -k\nabla T} \qquad \text{(D.48)}$$

where k is the thermal conductivity. Fourier's law is one of the fundamental transport laws of nature[18] and tells us the direction of the heat flux – heat moves down a temperature gradient. Fourier's law for one-dimensional heat conduction is analogous to Newton's law of viscosity for unidirectional flow.

$$\begin{array}{cc} \text{Newton's law of viscosity}^1 \\ \text{(unidirectional flow in 3-direction)} \end{array} \qquad -\tilde{\tau}_{13} = \tau_{13} \;=\; -\mu\frac{\partial v_3}{\partial x_1} \qquad \text{(D.49)}$$

$$\text{Fourier's law of conduction} \qquad \tilde{q}_1 \;=\; -k\frac{\partial T}{\partial x_1} \qquad \text{(D.50)}$$

The analogy between Newton's and Fourier's laws results from a shared physics: heat conduction and Newtonian momentum flux are byproducts of Brownian motion. Brownian motion is the microscopic thermal motion of molecules. This motion, when combined with a gradient – of temperature for energy conduction, or of velocity for momentum transfer – causes flux of energy or momentum. Diffusion of a chemical species down a concentration gradient is also caused by Brownian motion, and mass transport is the third of the transport processes. For more on transport phenomena, see the literature[18].

Returning to the conductive term of the energy balance, we can incorporate Fourier's law to write $\nabla\cdot\underline{\tilde{q}}$ in terms of temperature. With the definition $\nabla\cdot\nabla T = \nabla^2 T$, the conduction term of the microscopic energy balance becomes

$$\begin{aligned} -\nabla\cdot\underline{\tilde{q}} \;&=\; -\nabla\cdot(-k\nabla T) \\ &=\; k\nabla^2 T \end{aligned} \qquad \text{(D.51)}$$

The thermal conductivity k is constant.

The final rate-of-change of energy expression we seek is for the internal energy. With equations D.42 and D.47 we have expressions for the substantial derivative of kinetic and potential energies; we can subtract these equations from the overall energy equation (equation D.32) to isolate an equation for the internal energy $\hat{U}$.

$$\begin{array}{c} \text{Thermal} \\ \text{energy} \\ \text{equation} \end{array} \qquad \rho\left(\frac{\partial \hat{U}}{\partial t} + \underline{v}\cdot\nabla\hat{U}\right) = k\nabla^2 T - p(\nabla\cdot\underline{v}) + \underline{\underline{\tilde{\tau}}}^T : \nabla\underline{v} + \mathcal{S}_e \qquad \text{(D.52)}$$

The thermal energy equation indicates that the changes in internal energy are due to: conduction; one type of pressure effect; one type of viscous effect; and heat flow into a point due to sources (current or chemical reaction).

The appearance of $p(\nabla \cdot \underline{v})$ and $\tilde{\underline{\underline{\tau}}}^{T} : \nabla\underline{v}$ in both the kinetic and internal energy equations but with opposite signs helps us to identify the meaning of these terms. These two terms represent pathways by which kinetic energy is transformed into internal energy. The term $p(\nabla\cdot\underline{v})$, which may be positive or negative, represents energy exchange between kinetic and internal energy by virtue of volume change. The term $\tilde{\underline{\underline{\tau}}}^{T} : \nabla\underline{v}$, which is always positive (this is not shown here, but may be easily shown using Einstein notation, problem 6.15) and is therefore irreversible, represents kinetic energy conversion to internal energy by viscous dissipation.

The other instance of one type of energy converting to another is the conversion of kinetic energy into potential energy: the term $\rho\underline{v}\cdot\underline{g}$ appears in the kinetic energy equation and with the opposite sign in the potential energy equation. When a portion of fluid moves upward against gravity, its energy converts from kinetic to potential; if the fluid moves downward pulled by gravity, its energy reverses and goes from potential to kinetic. This term does not appear in the total energy equation because no overall change of energy occurs; rather, energy switches between two types of energy through this contribution.

The microscopic energy balance and the equations for the three contributing energies, internal, kinetic, and potential, are summarized in Figure D.1. In the next section we develop

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Energy Equations

kinetic $\qquad \rho\left(\dfrac{\partial \hat{E}_k}{\partial t} + \underline{v}\cdot\nabla\hat{E}_k\right) = -\nabla\cdot(p\underline{v}) + p(\nabla\cdot\underline{v}) + \nabla\cdot\left(\tilde{\underline{\underline{\tau}}}\cdot\underline{v}\right) - \tilde{\underline{\underline{\tau}}}^{T} : \nabla\underline{v} + \rho\underline{v}\cdot\underline{g}$

potential $\qquad \rho\left(\dfrac{\partial \hat{E}_p}{\partial t} + \underline{v}\cdot\nabla\hat{E}_p\right) = -\rho\underline{v}\cdot\underline{g}$

internal $\qquad \rho\left(\dfrac{\partial \hat{U}}{\partial t} + \underline{v}\cdot\nabla\hat{U}\right) = k\nabla^2 T - p(\nabla\cdot\underline{v}) + \tilde{\underline{\underline{\tau}}}^{T} : \nabla\underline{v} + S_e$

___

total $\qquad \rho\left(\dfrac{\partial \hat{E}}{\partial t} + \underline{v}\cdot\nabla\hat{E}\right) = k\nabla^2 T - \nabla\cdot(p\underline{v}) + \nabla\cdot\left(\tilde{\underline{\underline{\tau}}}\cdot\underline{v}\right) + S_e$

</div>

Figure D.1: The balance of energy on a control volume is governed by the first law of thermodynamics. Individual equations for internal, kinetic, and potential energy are derived in this section.

one additional energy equation, that for internal energy under some conditions that are quite

common for incompressible fluids.

D.2.4 Thermal Energy Equation

One final version of the microscopic energy balance is worth mentioning. For certain circumstances, we can arrive at a version of the microscopic energy balance that is explicit in temperature. A common circumstance is to consider an incompressible fluid ($\nabla \cdot \underline{v} = 0$) under constant pressure. For this circumstance we can write the internal energy in terms of the temperature and the heat capacity at constant pressure, $\hat{U} = \hat{C}_p T$[176]. With these assumptions, the left-hand-side of equation D.52 becomes

$$\rho \left(\frac{\partial \hat{U}}{\partial t} + \underline{v} \cdot \nabla \hat{U} \right) = \rho \hat{C}_p \left(\frac{\partial T}{\partial t} + \nabla \cdot T \right) \tag{D.53}$$

If we further neglect viscous dissipation ($\underline{\underline{\tilde{\tau}}}^T : \nabla v \approx 0$), equation D.52 becomes

Thermal energy equation
(no viscous dissipation,
fluid at constant p or $\rho \neq \rho(T)$)
$$\boxed{\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + \underline{v} \cdot \nabla T \right) = k \nabla^2 T + \mathcal{S}_e} \tag{D.54}$$

The thermal energy equation is a single equation, which may be written in any coordinate system, as shown below for Cartesian coordinates for the version in equation D.54. The thermal energy equation written in cylindrical and spherical coordinates is listed in Table B.12.

$$\text{Cartesian} \quad \frac{\partial T}{\partial t} + \left(v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} \right) = \frac{k}{\rho \hat{C}_p} \left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right] + \frac{S}{\rho \hat{C}_p} \tag{D.55}$$

The meaning of each term in the energy equations are summarized in Table D.1.

D.3 Momentum Transport in Moving Control Volumes

In Chapter 3 we were able to derive a version of Newton's second law that applied to fixed, stationary control volumes (Figure D.2). The result was the Reynolds transport theorem as applied to momentum transport.

Reynolds
Transport
Theorem
$$\boxed{\sum_{\substack{\text{on} \\ \text{CV}}} \underline{f} = \frac{d\mathcal{P}}{dt} + \iint_S (\hat{n} \cdot \underline{v}) \, \rho \underline{v} \, dS} \tag{D.56}$$