An Improvement in the Calculation of Turbulent Friction in Rectangular Ducts

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Frictional pressure drop in rectangular ducts is examined. Using correspondence between theory and experiment in laminar flow as a means for acceptance of published data, turbulent flow data for smooth rectangular ducts were compared with smooth circular tube data. Data for ducts having aspect ratios between unity and 39:1 were obtained in the literature and, in conjunction with new experimental data, were examined for deviations from the smooth circular tube line (smooth Moody). It was found that at constant Reynolds number based on hydraulic diameter the friction factor increases monotonically with increasing aspect ratio. It was thus concluded that the hydraulic diameter is not the proper length dimension to use in the Reynolds number to insure similarity between the circular and rectangular ducts. Instead, it was determined that if a modified Reynolds number Re was obtained so that geometric similarity was provided in laminar flow by the relation $f = 64/Re^*$ for all geometries, that this Reynolds number also provided good similarity in fully developed turbulent flow within a ~ 5 percent scatter band about the smooth tube line. By using this "laminar equivalent" Reynolds number, Re*, it is demonstrated that circular tube methods may be readily applied to rectangular ducts eliminating large errors in estimation of friction factor.*

Introduction

It has been common practice in the fields of fluid mechanics to utilize the hydraulic or equivalent diameter in predicting turbulent pressure drop along duct lengths having noncircular cross sections. This wide-spread practice is due mainly to the general lack of an exact analytical description of the losses in turbulent flow although the laminar case has been well known for some years. During the initial preparation for the work of reference [1][1] single-phase frictional pressure loss data were obtained for rectangular channels of aspect ratios[2] between 12.8:1 and 31:1 for the Reynolds number range between 10 and 10^5 utilizing both air and water as the working fluids. Significant discrepancies were found in turbulent flow between the data and the theory of Prandtl and von Karman as embodied in the Colebrook equation [2] even though quite excellent agreement with the well known laminar flow theory [3] was obtained using consistent measurement techniques. These discrepancies were not able to be explained in terms of roughness nor in terms of incomplete development of the flow field. Examination of verifiable turbulent data in the literature showed an unmistakable aspect ratio effect not recognized even as late as 1962 [4, 12]. Indeed it was found that a monotonic increase in friction factor with increasing aspect ratio occurred and that this increase could be easily determined on the basis of laminar flow theory. It is the purpose of this paper to review the work to date, present new data at high aspect ratios, demonstrate the monotonic effect of aspect ratio on the turbulent friction factor in rectangular channels, and to describe a simple method of calculating turbulent frictional pressure drop in rectangular ducts. This new method is found to correlate the fully developed turbulent data to a mean deviation of -0.23 percent with an rms deviation of 3.5 percent for all culled data and 0.38 percent independent rms deviation for the new data presented herein. Finally, concerns regarding calculation of roughness effects are expressed, some tentative conclusions drawn, and suggestions for further work are made.

Historical Review

Ever since the milestone works of Stanton and Pannell [5] and of Nikuradse [6], the two parameters of relative roughness, ϵ/d, and Reynolds number, Re, have been considered sufficient to correlate friction factors in circular ducts and, with suitable empirical modification, in noncircular tubes also. This in spite of the evidence of Nunner [7], Deissler [8], Eckert and Irvine [9], Colebrook [2] (when compared with Nikuradse's data), Schlichting [10] and others have demonstrated that other factors are important. The standard procedure has been to calculate the "hydraulic" or "equivalent" diameter and then to use this as the "correct" length dimension in the Reynolds number to obtain

[1]Numbers in brackets deisgnate References at end of paper.

[2]Ratios of width-to-spacing dimensions.

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circular tube equivalency [11]. Extensive studies described in 1952 [12], and 1964 [4], advocated this procedure even though the latter reference included a modification of the Deissler method [8], adapated for rectangular ducts, which predicted significant monotonic effects of aspect ratio on the turbulent friction factor. The major effect leading to this conclusion has been the large scatter of the data when compared with the smooth tube line predicted by Prandtl and von Karman [14], given by

$$\frac{1}{\sqrt{f}} = 2.0 \log_{10} \mathrm{Re} \sqrt{f} - 0.8, \tag{1}$$

this scatter spanning the approximate range −23 percent to +37 percent. Herein, the D'Arcy friction factor [11] is used, defined as

$$f = \frac{2\, D_e(-dp/dz)}{\rho v^2} \tag{2}$$

Clearly some method of objectively separating good from questionable data had to be found if these or similar conditions were not to be reproduced. The data found in the open literature for adiabatic friction factors in rectangular ducts are shown in Fig. 1. These data were reviewed by Hartnett, Koh, and McComas [4] and in more detail in reference [1]. Only a brief summary shall be given herein.

The data of Schiller [13] for square and rectangular (3.5:1) ducts agree reasonably well with equation (1), whereas those data of Davies and White [15] for aspect ratios to 160:1 cross the "smooth tube" line at a Reynolds number of ~4300 being approximately 8 percent higher at a Reynolds number of 9000. Their laminar flow data were, however, inconsistent. For an aspect ratio of 2.92:1, Cornish [3] obtained data which also agreed well with equation (1) while both the laminar and turbulent flow data of Allen and Grunberg [16] are below the accepted values.

The data of Washington and Marks [17] exhibited considerable scatter and inconsistent trends for both laminar and turbulent regimes. Since raw data were included in this paper, their channel spacings were altered to give agreement with theory in laminar flow: 0.129 versus 0.125 given, and 0.234 versus 0.250 given for respective aspect ratios of 21.4:1 and 38.9:1. The trends become quite similar to those of Davis and White [15], as seen in Fig. 2.

The small aspect ratio data of Lea and Tadros [18], (2.3:1 and 2:1), as well as their square duct data are slightly below the data of Cornish [3] and nominally 10 percent below equation (1). The data of Nikuradse [6] for aspect ratios of 3.5:1 as well as those of Heubescher (8:1 from reference [13]), agree well with (1). The latter data are not shown as there are no laminar or near laminar flow data included. Lowdermilk, Wieland, and Livingood [19] obtained data on both square and rectangular (5:1), ducts, the latter agreeing with equation (1) while the former lay below the circular tube line by 10 to 15 percent. This agrees with the results of Lea and Tadros [18] but not with Shiller [13] or Nikuradse [6]. Eckert and Irvine [9] obtained reasonably good agreement with equation (1) for a duct having an aspect ratio of 3:1. On the other hand, Wilkie, et al. [20] obtained data for a 12:1 duct (not shown), consistently above the "smooth-tube" line by 10 percent for Reynolds numbers up to 160,000. Likewise,

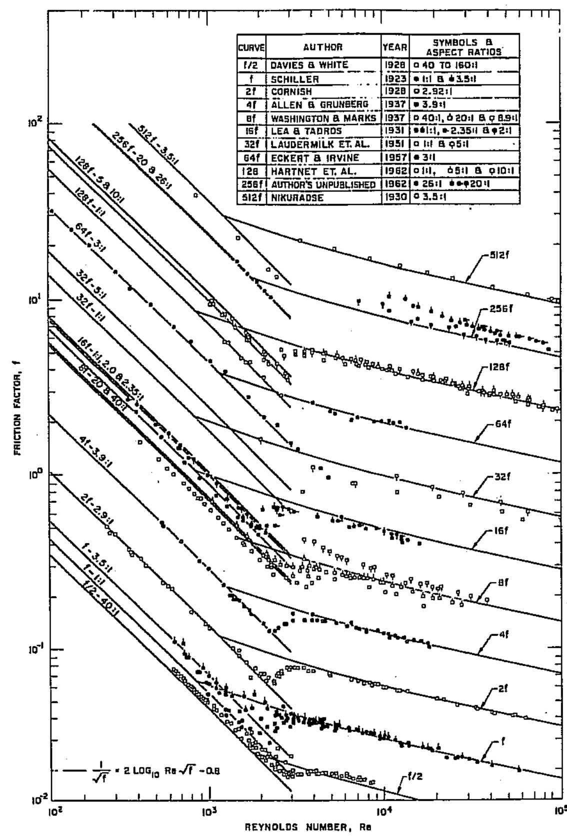

Fig. 1 Composite of frictional pressure drop data for rectangular ducts

Hartnett, Koh, and McComas [4] obtained extensive data on square and rectangular ducts (5:1 and 10:1) tending to show increasing friction factor with increasing aspect ratio at constant Reynolds number.

In addition, some early data were obtained by the author on thin ducts having aspect ratios of 20:1 and 26:1. Those data taken in 1962 (Fig. 3) were not as carefully controlled as were the later data but all showed friction factors consistently above that predicted by equation (1).

In summary, while it appears that the small aspect ratio data in general tend to either agree with the predictions or fall slightly low, the high aspect ratio data tend to agree or lie above the estimates of equation (1). These trends appear to agree with the rather sparse amount of theoretical work available. Most notable is the method of Deissler and Taylor as applied by Hartnett, Koh, and McComas [4] resulting in predictions of monotonically increasing friction factor with increasing aspect ratio, in approximate agreement with the trends exhibited by the data. Lohrenz and Kurata [30] devised an empirical method, similar to the one discussed herein, but involving redefinition of the hydraulic diameter used in both Reynolds number and friction factor. This method, however, alters the definition of the friction factor from the dimensionless wall shear stress currently employed

Nomenclature

A = cross-sectional flow area
C = coefficient in laminar friction equation
D_e = hydraulic diameter $4A/P_w$
f = D'Arcy friction factor (equation (2))
L = length between pressure sensing points
p = pressure
Δp = differential pressure
P_w = wetted perimeter
Re = Reynolds number—$\rho v D_e/\mu$
Re* = modified Reynolds number $= \dfrac{\phi^* \rho v D_e}{\mu}$
s = channel spacing
v = velocity
w = channel width
z = axial coordinate
μ = viscosity
ρ = density
ϕ^* = shape function (equation (6))
τ_w = wall shear stress

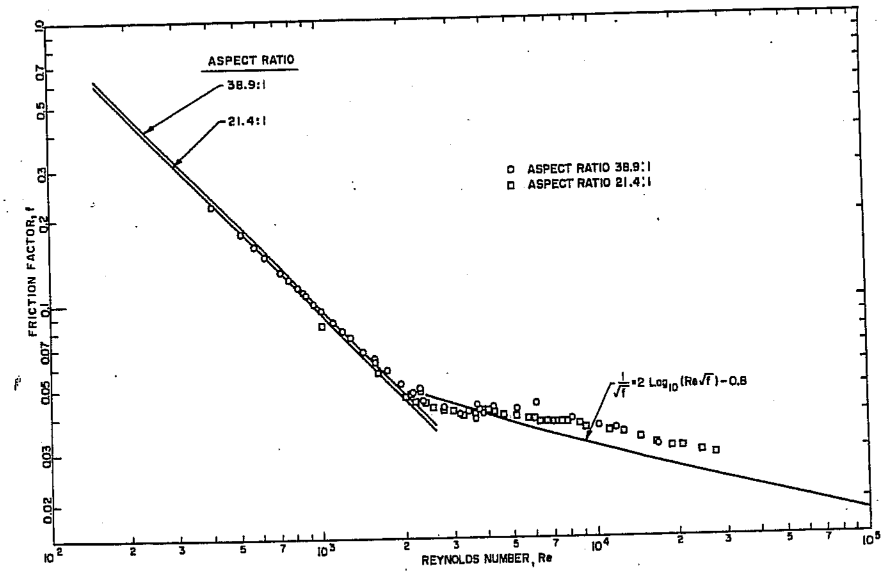

Fig. 2 Corrected frictional pressure drop data of Washington and Marks [17]

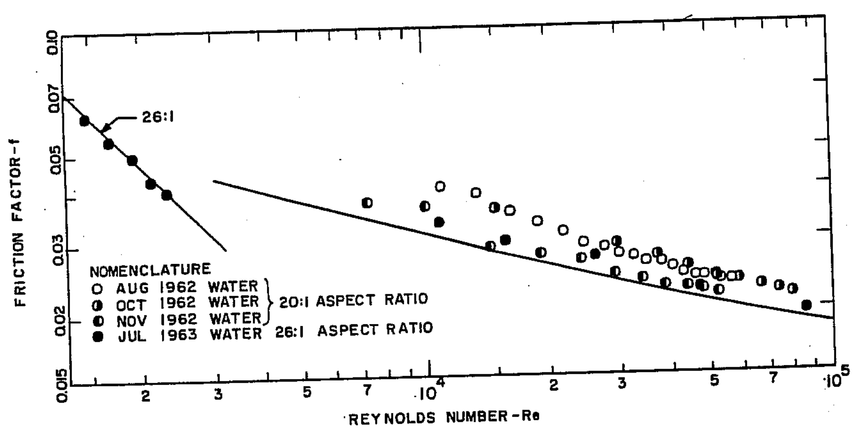

Fig. 3 Unpublished 1962 and 1963 data obtained by the author

wherein the hydraulic diameter naturally appears due to averaging over the wetted perimeter of the conduit, and makes this definition different for each different geometry. As a result, similarity was not obtained in the turbulent regime. Wilkins [23] and Ahmed and Brundrett [24] both suggested semiempirical corrections to the hydraulic diameter method but these do not, however, adequately explain the variations noted. Hanks [25] presented a theoretical approach toward the calculation of turbulent friction factors for infinite parallel plates but did not extend the method to rectangular ducts. Aranovich [26] derived a method for generalized geometries and applied it to equilateral triangles, annuli, and rod cluster geometries. For thin annuli, corresponding to infinite parallel plates, the turbulent friction factor was predicted to be about 13 percent higher than for the equivalent circular tube. Rehme [27] developed a method for calculating turbulent friction factors in rod bundles, triangular ducts, and eccentric annuli based on the work of Maubach [28]. This method was not applied to rectangular geometries, nor did Rehme determine a method for providing geometric equivalency with circular ducts as shall be developed in he next section.

Data Selection

The previous section presents a considerable quantity of data obtained for frictional flow of an isothermal fluid in rectangular geometries. Reference [4] presents a good tabular summary of these experiments.

In general, the picture appears quite chaotic with no immediately apparent trend of the data other than general scatter about the smooth tube line which vary from 23 percent below to 37 percent above the smooth tube line, a spread of 60 percent. Such a situation is clearly unacceptable if the data are to be taken at face value. Likewise, unless the magnitude of these large variations is acceptable from a design standpoint, these data cannot be accepted as evidence that pressure drop in rectangular geometries is adequately described by the use of circular geometry methods through the use of the hydraulic diameter concept. Various possible reasons for the discrepancies with pertinent remarks are included in the following:

(*a*) Poor control of geometry, especially the small dimension upon which the friction factor is especially dependent. For the experiments previously referenced the spacings varied from 0.686 mm to 200 mm. The high aspect ratio channels required small dimensions for expedience sake and hence lack of tolerance was a definite possibility.

(*b*) The experiment was in error. Discussion regarding this possibility shall be reserved for another section.

(*c*) The flow fields were not fully developed. For laminar flow this would cause measured friction factors to be larger than the theoretical values while in turbulent flow the reverse would be generally true. Hartnett, Koh, and McComas [4] showed experimentally that the critical length-to-hydraulic diameter ratio varied linearly with Reynolds number and decreased with increasing aspect ratio. Their results showed that for high aspect ratio channels greater than 10:1, the flow was fully developed at about 33 diameters at a Reynolds number of 1000. This is essentially confirmed mathematically by Han [21].

(*d*) The hydraulic diameter is not the correct length dimension to obtain geometric similarity between round and rectangular ducts.

Before the latter possibility could be explored, the first three had to be eliminated as a possible source of problems. Two alternatives presented themselves. First, all data could be discarded and new experiments performed. This was not felt to be a viable choice since to cover the range of geometries would obviously be a considerable undertaking. The second alternative was to attempt to devise a test for acceptability of data for comparison purposes. To be a valid indicator of the worth of a set of data, the indicator had to be obvious, be hopefully independent of any assumptions of a questionable nature, and, of course, not be so restrictive as to eliminate all or most of the data. The indicator chosen for this purpose was that the experiment included sufficient laminar flow data to demonstrate independently the adequacy of the experimental method and that the geometry was probably as reported. This test would automatically eliminate problems (*a*) and (*b*) above while involving only the assumption that the reasonably well established laminar flow theory was accurate. While this test was somewhat restrictive and may have eliminated some otherwise adequate data from consideration, it did not leave much room for erroneous data to be included to confuse the picture. The data thus culled included that of references [2, 4, 9, and 13], and the corrected data of [17]. Also some additional data were obtained for high aspect ratios and are presented in the next section. A total of 263 data points were obtained in the Reynolds number range of $7000 \leq \text{Re} \leq 103{,}000$, 152 of which were reported in the open literature. These data included aspect ratios between 1:1 and 38.9:1.

Description of Supplementary Data

Two additional sets of data were obtained, one to add to the high aspect ratio data where only the corrected data of Washington and Marks, and the unpublished 1963 data existed, and the other to check to see if any long-length developing flow effects were noticed. Both sets of data were obtained using the experimental techniques described in reference [1]. Briefly, tests were with either air or water, the air vented directly to atmosphere from the duct exit, and the low pressure, ambient water recirculated. Differential pressures were taken on a combined micro-standard manometry system having minimum indications of 250 μm of fluid, and a maximum range of 5 m of mercury. Manometry fluids included 0.785 sg fluid, water, 1.75 sg fluid, and mercury. Air and water flow rates were metered through 0.9 μm and 30 μm filters respectively by Cox logarithmic rotameters with 1 percent and 1/2 percent local accuracy, respectively. The water was demineralized and maintained at pH-7. Temperatures were measured with sheathed, iron-constantan thermocouples to a 0°C standard with a Rubicon type 2745 potentiometer while pressures were measured by a 0–10 bar Heise gauge having calibrated accuracy of 0.15 percent of scale. Calculation procedures for the air-flow data used an iterative Fanno flow theory thoroughly described in reference [1]. The two sets of data were obtained on separate, carefully controlled geometries fabricated from type 316 stainless steel having aspect ratios of 26:1 and 31:1. The former had an overall length of 260 diameters and an inlet length of 25 diameters while the latter had a length of 500 diameters with an inlet length of 15 diameters. Seven incremental pressure differences were measured on the latter test section. These data shall be discussed in the succeeding sections.

A third set of data, not specifically obtained for the purposes of this test but also having an adequate laminar flow range, will be included to provide data at an intermediate aspect ratio of 12.8:1. This test section was designed for the visual studies of reference [1], so was made of Plexiglas. Thus, the geometry was not as well controlled as for the other two sets of data due to the hygroscopic nature of this plastic. While the average channel spacing was mechanically measured to be 5.029 mm (3 tracks of measurement taken each 2.5 cm over a 3.05-m length), the hydraulic determination using the laminar flow data was 4.978 mm, a difference of 51 μm. This discrepancy was an order of magnitude worse than for that determined on the other two channels. The latter dimension, 4.978 mm, was used to reduce the data. It should also be noted that four sets of fluid injection ports on this plastic channel were made of sintered stainless steel plate. The resulting roughness ratio (ϵ/D) was determined to be 0.00005.

All the supplementary data for aspect ratios of 12.8:1, 26:1, and 31:1 are shown in Fig. 4. The middle four curves in this figure are for different increments on the 31:1 duct. For the 26:1 aspect ratio data (f versus Re), agreement between the average channel spacing calculated from the laminar flow measurements and the average of 150 separate mechanical measurements was within 5 μm out of a total of 2.438 mm, indicating good knowledge of the geometry, and adequate experimental procedures. Agreement between the compressible air and incompressible water data is also good providing a check on the compressible flow calculations for local Mach numbers up to 0.6, beyond which exit choking occurred. It is seen that these data for fully established turbulent flow are significantly above ($\sim$12 percent) and parallel to the "smooth-tube" line. The measured roughness (profilometer) in this case was less by more than an order of magnitude than that required to produce the measured friction factors in the vicinity of 100,000 Reynolds number.

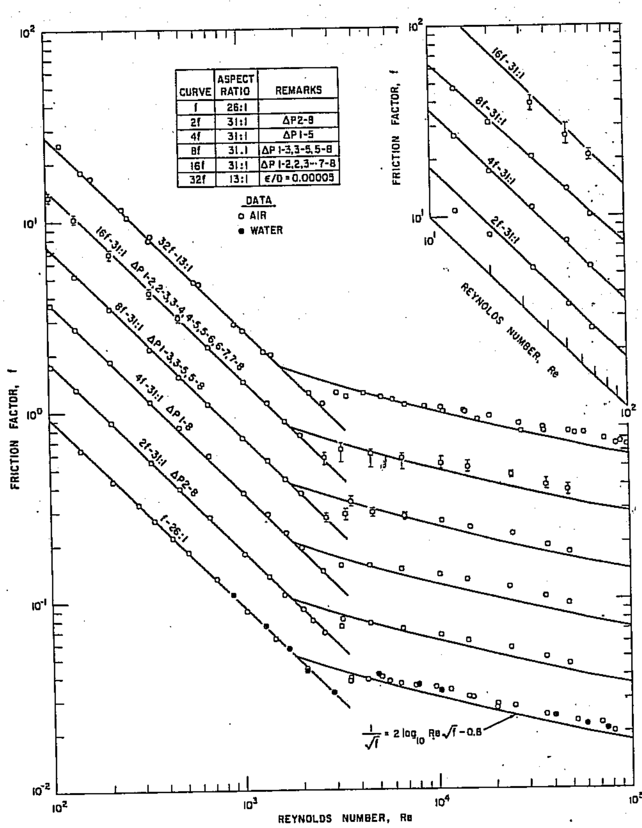

Fig. 4 Supplementary frictional pressure drop data obtained by the author

For the higher, 31:1 aspect ratio data ($2f$, $4f$, $8f$, and $16f$ versus Re), the numbers next to the lines indicate the pressure tap increments, 1–8 being between Taps 1 and 8. Tap 1 was located 15 diameters from the inlet and Tap 2 was 100 diameters from the inlet. The smoothness of the channel was practically identical to that of the 26:1 channel. Except where Tap 1 was used and where entrance effects could exist, the hydraulic and mechanical spacing measurements agreed everywhere within 25 μm. For Taps 2–8, the most accurate differential pressure increment, the agreement was 2.5 μm, comparable to that obtained with the other supplementary test. Again it is seen that the turbulent flow data are approximately 12 percent above the "smooth-tube" line. There is no evidence of any developing length effect in any of the increments except, perhaps, in the 1–8 increment which included data within 15 diameters of the inlet and which appear to be biased slightly higher than the rest of the data. These observations are in agreement with those of Hartnett, Koh, and McComas [4]. It is interesting to note that the scatter observed in the incremental measurements as indicated by the vertical bars on the data points is considerably greater in turbulent flow than in laminar flow where the differential pressures were much smaller.

Comparison of Culled Data

The data sorted as a result of the culling process are shown plotted in Fig. 5. These data have been segregated into similar aspect ratio groups near 1:1, 3:1, 10:1, and 26:1, for the four lower curves. Examination shows that the data of Hartnett, et al. [4] at 5:1 aspect ratio have definitely higher friction factors than the data of Schiller [13] at 3.5:1 and of Cornish at 2.9:1. Likewise, the author's 12.8:1 data are clearly greater than Hartnett's, et al., 10:1 data, although this could be accounted for by roughness. Both these latter sets have friction factors greater than the lower aspect ratio data. Also, the 26:1 and 31:1 data are all obviously in a higher friction range than the other data with the exception of the author's 12.8:1 data.

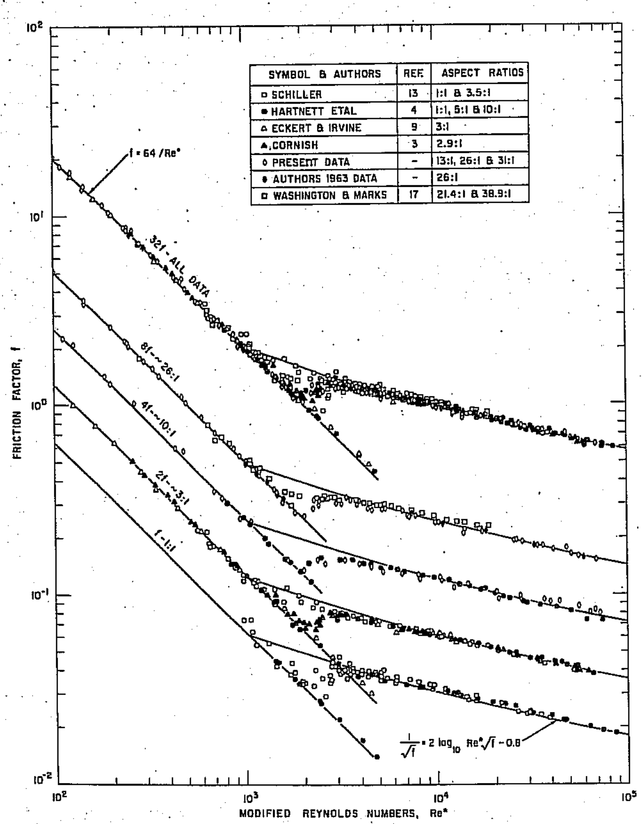

Fig. 5 Culled frictional pressure drop data for aspect ratios near 1:1, 3:1, 10:1, and 26:1, and a complete curve showing all data together

It is noted that the author's smooth-channel data obtained on three separate test sections, having two different aspect ratios (26:1 and 31:1), with two different fluids (air and water), taken over a span of six years, all agree in their behavior. They all have accurate laminar flow data showing agreement with the theory of Cornish [3]. They all have transition to turbulent flow at similar Reynolds numbers, and they all fall approximately the same amount above the "smooth-tube" correlation.

Finally, the corrected 21.4:1 and 38.9:1 aspect ratio data of Washington and Marks [17] seem to fall slightly above the present 26:1 and 31:1 data even though their laminar flow data agree reasonably well with theory. Thus, there still may be a problem with these data. Unfortunately, these data were the only high aspect ratio data, except those presented herein, which survived the culling procedure and so were maintained for comparison purposes.

All the culled data are gathered together in one curve in Fig. 5: the top, $32f$ versus Re, line. It is seen that, as expected, the laminar flow data generally lie between the predictions of Cornish [3] for square ducts and infinite parallel plate. In turbulent flow, the data scatter about the smooth-tube line by approximately -5 to $+20$ percent, a band 25 percent wide. Even neglecting the Washington and Marks [17] data, the scatter is -5 to $+15$ percent, a 20 percent band. It appears, neglecting these latter data, that there may well be a monotonic effect of aspect ratio on friction factor at constant Reynolds number.

A monotonic effect of aspect ratio is universally accepted for laminar flow due to the work of Cornish [3]. Similar trends have been predicted for rectangular geometries by Hartnett, Koh, and McComas [4] based on the method of Deissler and Taylor [8]. This latter method makes use of the assumption that the universal velocity profile for turbulent flow exists everywhere perpendicular to a solid boundary. The derived method involves a graphical iterative scheme which results in cross-sectional isokinetic contours from which the friction factor and Reynolds number are obtained. The method is quite tedious and time-consuming, and the results not in good enough agreement with data to make the effort worthwhile. Fig. 6, taken from reference 4, shows that the predictions for the square duct, using the Deissler-Taylor analyses, fall about 12 percent low whereas for the 10:1 duct the predictions are as much as 10 percent high. In addition, where the slope of the reference [4] data closely follows that of the smooth tube line, the extended Deissler-Taylor analysis crosses the line. The trends of increasing friction factor with aspect ratio appear to be the same. The analysis is noteworthy in that it indicates the hydraulic diameter does not necessarily yield equivalent round tube friction factors.

Modified Reynolds Number and Laminar Equivalent Diameter

The previous section showed, by data comparisons, that the Reynolds number based on hydraulic diameter is insufficient to fully correlate the turbulent flow friction factor and that a monotonic increase in friction factor with aspect ratio is quite likely to exist similar to the trends predicted by the Deissler-Taylor method. These variations are qualitatively similar to those seen in laminar flow. It will be shown empirically that the same geometry factor which produces equivalency between circular and rectangular geometries in laminar flow, works for turbulent flow.

It is quite a simple matter to provide similarity between round and rectangular ducts in laminar flow. The appropriate dimensional factor is taken to provide a modified Reynolds number, Re*, which yields the identical friction factor relation as would be obtained for a round tube. That is,

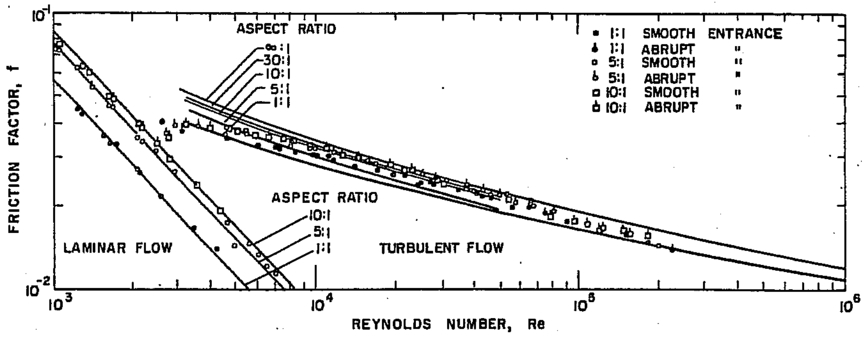

Fig. 6 Modification of the method of Deissler and Taylor by Hartnett, Koh, and McComas [4] for rectangular channels

$$f = \frac{64}{\mathrm{Re}^*} \tag{3}$$

By comparison with the standard form for rectangular ducts [14] it is easily seen that

$$\mathrm{Re}^* = \frac{64}{C\left(\frac{w}{s}\right)}\mathrm{Re} = \phi^*\left(\frac{w}{s}\right)\mathrm{Re} \tag{4}$$

where the standard Reynolds number is based on the hydraulic diameter. It is thus seen that a simple function of aspect ratio is all that is necessary to find the proper similarity between circular and rectangular geometries in laminar flow. A "laminar-equivalent" diameter may be defined based on the work of Cornish [3] as

$$D_L = \phi^*\left(\frac{w}{s}\right)D_e \tag{5}$$

where the geometry function ϕ^* is given by[3]

$$\phi^*\left(\frac{w}{s}\right) = \frac{2}{3}\left\{1 + \frac{s}{w}\right\}^2\left\{1 - \frac{192s}{\pi^5 w}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^5}\tanh\frac{(2n+1)\pi w}{2s}\right\} \tag{6}$$

Thus, if laminar flow data are correlated as a function of $\mathrm{Re}^* = \phi^* vD_e\rho/\mu$ on an f versus Re^* plane, it is insured through equation (2) that both round tube data and rectangular channel data will be correlated by the same line, $f = 64/\mathrm{Re}^*$, where $\phi^* = 1$ for round tubes.

The function $\phi^*(w/s)$ is shown graphically in Fig. 7. An approximate relationship which will give ϕ^* within about 2 percent is

$$\phi^*(w/s) \sim \frac{2}{3} + \frac{11}{24}\frac{s}{w}\left(2 - \frac{s}{w}\right) \tag{7}$$

which has the correct limits and slope at $s/w \to 1.0$. The largest deviation is near an aspect ratio of 2:1 and will produce an error in estimation of friction factor which is negligible compared with the expected scatter of the data.

At this point it is noted that nothing has to be done to the friction factors defined by equation (2) in terms of the measured pressure drop. This is reasonable since the pressure gradient is directly calculated as a function of the perimeter-averaged wall shear stress $\bar{\tau}_w$ as

$$-\frac{dp}{dz} = \bar{\tau}_w\frac{P_w}{A} = \frac{4\bar{\tau}_w}{D_e} \tag{8}$$

where the hydraulic diameter enters in a straightforward manner. The friction factor becomes the dimensionless averaged wall shear stress in the normal fashion

$$f = \frac{4\bar{\tau}_w}{\frac{1}{2}\rho v^2} \tag{9}$$

normalized to the kinetic head.

Since the single parameter, Re, is known to correlate all smooth circular tube friction data for both laminar and turbulent flow, it may be expected that the single parameter Re^* will do the same for both smooth round tubes and smooth rectangular ducts. On this basis, a simple replacement of Re^* for Re in the Colebrook equation would give

$$\frac{1}{\sqrt{f}} = 2.0\log_{10}\mathrm{Re}^*\sqrt{f} - 0.8 \tag{10}$$

so that $f = f(\mathrm{Re}^*)$ for smooth ducts. The effect of this would

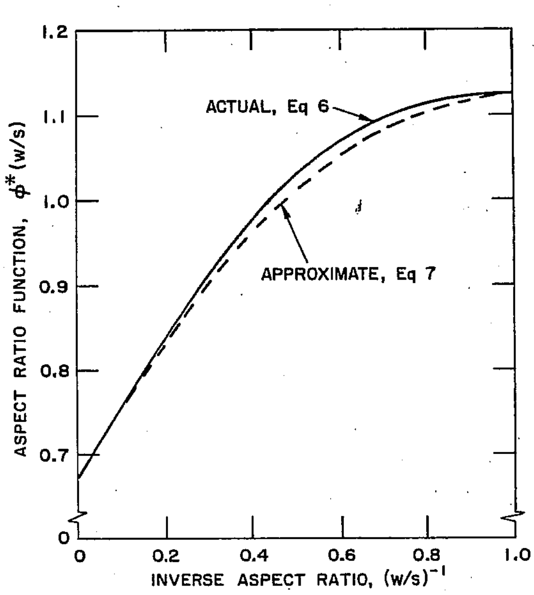

Fig. 7 Geometry function for calculation of the laminar equivalent diameter

[3]Note: The work of reference [1] mistakenly used a series of alternating signs which, because of the rapid convergence of equation 6 was not detected prior to its release. This error, less than 1 percent decreasing with increasing aspect ratios to under 0.1 percent, had no effect on the results of conclusions drawn therefrom. Thanks are due to S. W. Barnes and C. L. Gregory of the General Electric Co. for bringing this error to my attention.

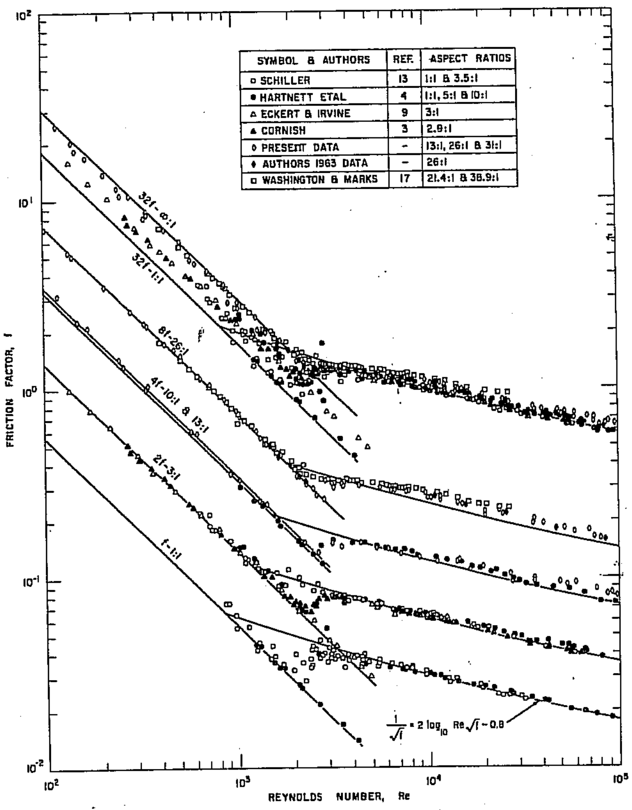

Fig. 8 Comparison of the modified Reynolds number concept with all culled data for frictional pressure drop in rectangular channels

be to reduce the effective Reynolds number of high aspect ratio rectangular channels by an amount approaching 2/3, sliding these data to the left on a plane such as in Fig. 4. It is seen that this is in the correct direction to improve the agreement between the data and the round tube theory.

The results of this concept may be seen in Fig. 8 where all culled data shown originally in Fig. 5 are replotted on the basis of f versus Re*. The improvement is immediately obvious by comparison. With the exception of the Washington and Marks data, nearly all fall within ± 5 percent of the line given by equation (10). For the supplementary 26:1 aspect ratio data and the 31:1 aspect ratio overall measurements not including the first developing length increment, the agreement was within 0.4 percent rms for all Re* greater than 5000. For all the culled data above a modified Reynolds number of 7000, the mean deviation is -0.23 percent while the rms deviation is 3.5 percent. It is felt that this agreement validates the culling procedures and the concepts of the modified Reynolds number and laminar equivalent diameter for correlation of rectangular channel data.

Discussion and Conclusions

It has been demonstrated, in agreement with the work of other investigators, that the hydraulic diameter is inappropriate for accurate estimation of turbulent friction factor in rectangular ducts. The confirmable data in the literature coupled with the new data presented herein, clearly show a monotonic increase in friction factor with increasing aspect ratio. It has been shown that the empirical use of the "laminar equivalent" diameter to calculate a modified Reynolds number yields excellent similarity between round tubes and rectangular ducts in laminar flow, and results in excellent similarity in turbulent flow. The original data shown in Fig. 1 had a scatter band between -23 and $+37$ percent around the Colebrook equation (1). Application of the culling procedure which eliminated all data having no laminar flow agreement with the well established theory of Cornish [3], reduced this scatter band to the approximate range of -5 to $+20$ percent for Reynolds numbers greater than 7000. Use of the proposed correlation parameter Re* further reduced this scatter to a 3.5 percent rms deviation with a mean deviation of -0.23 percent for modified Reynolds numbers over 7000. In fact, if the still questionable "corrected" data of Washington and Marks are neglected, virtually all data fall within ± 5 percent of the predictions based on equation (10).

It is interesting that this method appears to work in view of the known differences between round tube and rectangular duct flow structure. For instance it is well known [10] that the Reynolds stresses set up secondary flow patterns in the corners which do not exist in laminar flow. This is expected to yield an additional loss component beyond what occurs in round tubes at similar conditions. Perhaps this effect is counteracted by another, compensating effect which the writer has not considered. On the other hand, this effect may be of negligible importance.

New data for aspect ratios of 26:1 and 31:1 are presented for the range of Reynolds numbers between 10 and 100,000 in Fig. 4. Of these data, those obtained on different increments for the 31:1 duct are all plotted on the same two curves, $8f$ and $16f$ versus Re. It is seen that the scatter in the measured friction factors is significantly greater in turbulent flow than in laminar flow, especially in the transition range. Since measurement of higher differential pressures is generally more accurate than for the lower magnitudes, it is suggested that this scatter may be an inherent unsteadiness in the pressure gradient. In fact, while the author was able to consistently obtain laminar measurements within the estimated accuracy limits of his instrumentation and methodology, this was not true of the turbulent flow data. Likewise, the correlated data shown in Fig. 8 show this trend. The scatter of the laminar flow data is noticeably less than that of the turbulent flow regime. For design purposes, then, it is recommended that a 5 percent uncertainty be placed on calculations obtained by equation (10).

In addition to the previous considerations, one should consider the implications regarding heat and mass transfer in turbulent flow. Since heat, mass, and momentum transfers are all analogous, one might expect to see aspect ratio dependencies in the transfer coefficients for the appropriate process. This possibility should be investigated.

Finally, and perhaps of more fundamental importance, is the idea of similarity as applied to frictional pressure drop in rough rectangular channels. It may easily be argued that if the laminar equivalent diameter, D_L, is the appropriate length dimension to use in correlating adiabatic, single-phase pressure losses in rectangular ducts, then for rough channels, $f = f(V, D_L, \rho, \mu, \epsilon)$. The Buckingham-pi theorem shows immediately that $f = f(\text{Re}^*, \epsilon/D_L)$ In other words, the roughness ratio for high aspect ratio rectangular ducts may be almost 50 percent larger than what it would be based on hydraulic diameter with a commensurate increase in friction factor predicted as a result. Certainly while this is more conjecture than anything at this point, this question could be answered through a relatively straightforward series of experiments. The possibility has important implications in design applications where commercial-grade finishes or surface degradation due to wear, corrosion, or deposition may occur.

Acknowledgments

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DISCUSSION

E. Brundrett[4]

The author is to be congratulated on his very thought-provoking re-evaluation of the concept of the equivalent hydraulic diameter. For this writer there are, as with all interesting research, certain questions arising from the analysis for the best characteristic diameter for rectangular ducts as presented by this paper.

This writer would feel more comfortable with the new concept if a greater experimental base could be used for the confirmation of the technique, particularly since a laminar flow analysis is being extended to the turbulent flow regime. Also this writer would feel more comfortable with the concept if some further discussions of the rejected data had occurred for those cases where obvious experimental errors could not be detected.

Also the utilization of the correction parameter ϕ^* on the classical hydraulic diameter, D_e, to yield D_L, does not appear to provide the classical asymptotic solution when the duct becomes quite large. For example with an aspect ratio of 10:1, D_L should be very nearly equal to $2S$ if classical arguments are at all valid. By computation for an aspect ratio of 10:1 $De = 4A/P = (4SW/2(W + S)) = 1.818S$ which of course gives rise to the dissatisfaction of the author, and of this writer with the hydraulic diameter. The author's proposed characteristic dimension at a 10:1 aspect ratio is by equations (5) and (7)

$$D_L = \phi^* \left(\frac{W}{S}\right) D_e = 0.7537\ D_e = 0.753 \times 1.818S = 1.302S$$

This value is still considerably less than $2S$, and shows no ability to converge to $2S$ as the aspect ratio increases. Perhaps as with all currently available characteristic diameters, there is a best range of application of the author's proposed effective diameter.

Finally this writer is inclined to treat with some caution the author's stated accuracy of prediction of ± 5 percent in view of the very extensive amount of data rejection that has occurred. The writer is however, impressed with the extent of the analysis that has been undertaken, and with the very thought-provoking comments, particularly those associated with the effect of roughness upon similarity in rectangular ducts at high aspect ratios.

P. S. Barna[5]

In recent years, the vexing question of friction in noncircular ducts has received renewed attention and it is gratifying that the author, Dr. Owen C. Jones, has so carefully reviewed the

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relevant literature from the early days of Blasius and Nikuradse up to date.

Friction is as complex as it is intricate and the writer had his share of hassle with it. He also had serious misgivings about using the "equivalent" or "hydraulic" diameter D_e in the past especially for turbulent flow between flat plates for which, for the case of infinite aspect ratio, $D_e = 2h$, where h is the distance between the plates.

For fully developed flow one may explain the discrepancy in friction with increasing aspect ratio, as discussed in the paper, by simply considering the contributions by the four walls on which the boundary layer is formed. For a square duct these contributions must be equal because of symmetry, but for other rectangular ducts they become unequal. In the case of moderate aspect ratio, say up to three, while the contribution to friction of the *near* walls increases, of the *far* walls it decreases and presumably these may compensate for each other. It may then be anticipated that the friction factor based on the equivalent hydraulic diameter may indeed agree with the results based on a circular pipe. With increasing aspect ratio, however, the contributions of the far walls to the overall friction may become negligible while the increasing influence of the near walls become predominant.

For the parallel plates with no far walls present one has had to contend with $D_e = 2h$ and with this value-upon substitution one calculates a lower friction and higher Reynolds number than the value $D_e = h$ would otherwise predict when using the relevant formulas. Therefore, it is not surprising to learn from the paper that high aspect ratio ducts yield higher friction, while so many earlier investigators found good agreement with circular pipe-friction for lower and even intermediate aspect ratios when using the hydraulic diameter. It is then refreshing to learn that the author has found through his studies an explanation for the discrepancies, a sort of reconciliation between the different views, and was able to establish a method which appears satisfactory. By replacing the hydraulic diameter D_e with the "laminar equivalent" diameter $D_e\phi^*$, the author concludes that by calculating a modified Reynolds number $R_e^* = vD_e\phi^*\rho/\mu$ an excellent similarity between round tubes and rectangular ducts in both laminar and turbulent flow may be obtained for all aspect ratios. A critical examination appears to prove his point.

The painstaking work is highly commendable and the paper is an excellent contribution to literature for which the author must be wholeheartedly congratulated.

N. Madsen[6]

The author of this paper has performed an extremely useful service for engineers interested in turbulent flow through smooth rectangular ducts. The collection of the extensive literature data, the contribution of original data, and the empirical correlation are all valuable additions to the arsenal of the designer. The ingenious use of an "equivalent laminar" diameter for calculation of a modified Reynolds number in turbulent flow seems to fit the chosen data very well indeed.

As with the introduction of any new concepts, new questions arise. The author has discussed some of these; however, a very basic question remains. According to the reasoning presented under the heading "Modified Reynolds Number and Laminar Equivalent Diameter" the concept of an "equivalent laminar" diameter should be applicable to all uniform, noncircular ducts for which an equation analogous to the Cornish equation exists.

An equation for pressure drop during laminar flow can always be derived for a given uniform noncircular cross section, either analytically or numerically, and it can be represented by an equation $f = 64/\text{Re}^*$. It is thus always possible to calculate a modified Reynolds number and an "equivalent laminar" diameter for any uniform cross section, whatever; and by force of the arguments presented in the paper, equation (10) should apply to turbulent flow through a smooth duct with the given cross section.

Of course, purely speculative ideas of this kind especially invented to match the requirements of a specific data set, can have no broad scientific value until they acquire support from theoretical evidence or at least are tested with other duct cross sections. Extension of the concept to other cross sections and the application to rough ducts, as suggested by the author, deserve further investigation. He correctly points out that the procedure does not allow for pressure drop caused by secondary flow.

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Author's Closure

The author wishes to thank Professors Brundrett, Madsen, and Barna for their kind remarks concerning this work, and agree completely with Professors Brundrett and Madsen in their desire for a larger data base and extension to other geometries. Other, independent data on rectangular ducts, especially at high aspect ratios is needed. Sufficient data in the laminar regime must be obtained to insure that adequate experimental procedures are followed. This can be especially troublesome for high aspect ratios in ducts of small gap, due to problems of dimensional control and stability. As for other geometries, this method has now been successfully extended to the case of smooth, concentric annuli and will be the subject of a forthcoming paper [31].

The experiments were reviewed in detail with the results documented in reference [1] of this paper. Any attempt to reject data for cause, except in a few cases, was a hopeless task. Having abandoned that attempt, the alternative was to accept data based on a culling procedure without prejudice to those data not accepted. This, of course, was done. It is certain that some perfectly valid data were not accepted. It is almost equally certain that no bad data were accepted, (with the exception of the questionable data of Washington and Marks [17] which deviated from the laminar theory above Re = 1000). It is the latter point which led directly to development of the method.

In answer to Professor Brundrett's remarks pertaining to the recommended 5 percent uncertainty, the author is convinced that a well-controlled experiment on an accurately known rectangular geometry will yield results within this uncertainty band. This is because uncertainties such as those associated with errors of flow, geometry, and thermodynamic state have hopefully been minimized in development of the correlation. Deviation of the friction factor outside of the recommended 5 percent range due to variations of similar factors within a given design tolerance must be considered by the designer in whatever context is required and over which the correlator has no control.

Finally, regarding the proper limiting value of the laminar equivalent diameter, the asymptotic value is seen to be correct when the variations of the laminar coefficient are considered. Since $f = 64/\text{Re}^*$, always, and since $f \to 96/\text{Re}$ as $w/s \to \infty$, it is easily seen that $D_L \to 2D_e/3$ as $w/s \to \infty$. The limiting value of D_e is twice the spacing, $2s$, so that $D_L \to 4s/3$ as $w/s \to \infty$ in agreement with (5), (6), and (7).

Additional Reference

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