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# XXII. *On the Theories of the Internal Friction of Fluids in Motion, and of the Equilibrium and Motion of Elastic Solids.* By G. G. STOKES, M.A., *Fellow of Pembroke College.*

[Read *April* 14, 1845.]

THE equations of Fluid Motion commonly employed depend upon the fundamental hypothesis that the mutual action of two adjacent elements of the fluid is normal to the surface which separates them. From this assumption the equality of pressure in all directions is easily deduced, and then the equations of motion are formed according to D'Alembert's principle. This appears to me the most natural light in which to view the subject; for the two principles of the absence of tangential action, and of the equality of pressure in all directions ought not to be assumed as independent hypotheses, as is sometimes done, inasmuch as the latter is a necessary consequence of the former*. The equations of motion so formed are very complicated, but yet they admit of solution in some instances, especially in the case of small oscillations. The results of the theory agree on the whole with observation, so far as the time of oscillation is concerned. But there is a whole class of motions of which the common theory takes no cognizance whatever, namely, those which depend on the tangential action called into play by the sliding of one portion of a fluid along another, or of a fluid along the surface of a solid, or of a different fluid, that action in fact which performs the same part with fluids that friction does with solids.

Thus, when a ball pendulum oscillates in an indefinitely extended fluid, the common theory gives the arc of oscillation constant. Observation however shows that it diminishes very rapidly in the case of a liquid, and diminishes, but less rapidly, in the case of an elastic fluid. It has indeed been attempted to explain this diminution by supposing a friction to act on the ball, and this hypothesis may be approximately true, but the imperfection of the theory is shown from the circumstance that no account is taken of the equal and opposite friction of the ball on the fluid.

Again, suppose that water is flowing down a straight aqueduct of uniform slope, what will be the discharge corresponding to a given slope, and a given form of the bed? Of what magnitude must an aqueduct be, in order to supply a given place with a given quantity of water? Of what form must it be, in order to ensure a given supply of water with the least expense of materials in the construction? These, and similar questions are wholly out of the reach of the common theory of Fluid Motion, since they entirely depend on the laws of the transmission of that tangential action which in it is wholly neglected. In fact, according to the common theory the water ought to flow on with uniformly accelerated velocity; for even the supposition of a certain friction against the bed would be of no avail, for such friction could not be transmitted through the mass. The practical importance of such questions as those above mentioned has made them the object of numerous experiments, from which empirical formulæ have been constructed. But such formulæ, although fulfilling well enough the purposes for which they were

* This may be easily shown by the consideration of a tetrahedron of the fluid, as in Art. 4.

constructed, can hardly be considered as affording us any material insight into the laws of nature; nor will they enable us to pass from the consideration of the phenomena from which they were derived to that of others of a different class, although depending on the same causes.

In reflecting on the principles according to which the motion of a fluid ought to be calculated when account is taken of the tangential force, and consequently the pressure not supposed the same in all directions, I was led to construct the theory explained in the first section of this paper, or at least the main part of it, which consists of equations (13), and of the principles on which they are formed. I afterwards found that Poisson had written a memoir on the same subject, and on referring to it I found that he had arrived at the same equations. The method which he employed was however so different from mine that I feel justified in laying the latter before this Society*. The leading principles of my theory will be found in the hypotheses of Art. 1, and in Art. 3.

The second section forms a digression from the main object of this paper, and at first sight may appear to have little connexion with it. In this section I have, I think, succeeded in shewing that Lagrange's proof of an important theorem in the ordinary theory of Hydrodynamics is untenable. The theorem to which I refer is the one of which the object is to show that $udx + vdy + wdz$, (using the common notation,) is always an exact differential when it is so at one instant. I have mentioned the principles of M. Cauchy's proof, a proof, I think, liable to no sort of objection. I have also given a new proof of the theorem, which would have served to establish it had M. Cauchy not been so fortunate as to obtain three first integrals of the general equations of motion. As it is, this proof may possibly be not altogether useless.

Poisson, in the memoir to which I have referred, begins with establishing, according to his theory, the equations of equilibrium and motion of elastic solids, and makes the equations of motion of fluids depend on this theory. On reading his memoir, I was led to apply to the theory of elastic solids principles precisely analogous to those which I have employed in the case of fluids. The formation of the equations, according to these principles, forms the subject of Sect. III.

The equations at which I have thus arrived contain two arbitrary constants, whereas Poisson's equations contain but one. In Sect. IV. I have explained the principles of Poisson's theories of elastic solids, and of the motion of fluids, and pointed out what appear to me serious objections against the truth of one of the hypotheses which he employs in the former. This theory seems to be very generally received, and in consequence it is usual to deduce the measure of the cubical compressibility of elastic solids from that of their extensibility, when formed into rods or wires, or from some quantity of the same nature. If the views which I have explained in this section be correct, the cubical compressibility deduced in this manner is too great, much too great in the case of the softer substances, and even the softer metals. The equations of Sect. III. have, I find, been already obtained by M. Cauchy in his *Exercises Mathématiques*, except that he has not considered the effect of the heat developed by sudden compression. The method which I have employed is different from his, although in some respects it much resembles it.

The equations of motion of elastic solids given in Sect. III. are the same as those to which different authors have been led, as being the equations of motion of the luminiferous ether in vacuum. It may seem strange that the same equations should have been arrived at for cases so different; and I believe this has appeared to some a serious objection to the employment of those equations in the case of light. I think the reflections which I have made at the end of Sect. IV., where I have examined the consequences of the law of continuity, a law which seems to pervade nature, may tend to remove the difficulty.

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* The same equations have also been obtained by Navier in the case of an incompressible fluid, (*Mém. de l'Institut*, T. VI.) but his principles differ from mine still more than do Poisson's.

## SECTION I.

### *Explanation of the Theory of Fluid Motion proposed. Formation of the Differential Equations. Application of these Equations to a few simple cases.*

1. Before entering on the explanation of this theory, it will be necessary to define, or fix the precise meaning of a few terms which I shall have occasion to employ.

In the first place, the expression "the velocity of a fluid at any particular point" will require some notice. If we suppose a fluid to be made up of ultimate molecules, it is easy to see that these molecules must, in general, move among one another in an irregular manner, through spaces comparable with the distances between them, when the fluid is in motion. But since there is no doubt that the distance between two adjacent molecules is quite insensible, we may neglect the irregular part of the velocity, compared with the common velocity with which all the molecules in the neighbourhood of the one considered are moving. Or, we may consider the mean velocity of the molecules in the neighbourhood of the one considered, apart from the velocity due to the irregular motion. It is this regular velocity which I shall understand by the *velocity of a fluid at any point*, and I shall accordingly regard it as varying continuously with the co-ordinates of the point.

Let $P$ be any material point in the fluid, and consider the instantaneous motion of a very small element $E$ of the fluid about $P$. This motion is compounded of a motion of translation, the same as that of $P$, and of the motion of the several points of $E$ relatively to $P$. If we conceive a velocity equal and opposite to that of $P$ impressed on the whole element, the remaining velocities form what I shall call the *relative velocities* of the points of the fluid about $P$; and the motion expressed by these velocities is what I shall call the *relative motion* in the neighbourhood of $P$.

It is an undoubted result of observation that the molecular forces, whether in solids, liquids, or gases, are forces of enormous intensity, but which are sensible at only insensible distances. Let $E'$ be a very small element of the fluid circumscribing $E$, and of a thickness greater than the distance to which the molecular forces are sensible. The forces acting on the element $E$ are the external forces, and the pressures arising from the molecular action of $E'$. If the molecules of $E$ were in positions in which they could remain at rest if $E$ were acted on by no external force and the molecules of $E'$ were held in their actual positions, they would be in what I shall call a state of *relative equilibrium*. Of course they may be far from being in a state of actual equilibrium. Thus, an element of fluid at the top of a wave may be sensibly in a state of relative equilibrium, although far removed from its position of equilibrium. Now, in consequence of the intensity of the molecular forces, the pressures arising from the molecular action on $E$ will be very great compared with the external moving forces acting on $E$. Consequently the state of relative equilibrium, or of relative motion, of the molecules of $E$ will not be sensibly affected by the external forces acting on $E$. But the pressures in different directions about the point $P$ depend on that state of relative equilibrium or motion, and consequently will not be sensibly affected by the external moving forces acting on $E$. For the same reason they will not be sensibly affected by any motion of rotation common to all the points of $E$; and it is a direct consequence of the second law of motion, that they will not be affected by any motion of translation common to the whole element. If the molecules of $E$ were in a state of relative equilibrium, the pressure would be equal in all directions about $P$, as in the case of fluids at rest. Hence I shall assume the following principle:—

*That the difference between the pressure on a plane in a given direction passing through any point* P *of a fluid in motion and the pressure which would exist in all directions about* P *if the fluid in its neighbourhood were in a state of relative equilibrium depends only on the relative motion of the fluid immediately about* P; *and that the relative motion* *due to any motion of rotation may be eliminated without affecting the differences of the pressures above mentioned.*

Let us see how far this principle will lead us when it is carried out.

2. It will be necessary now to examine the nature of the most general instantaneous motion of an element of a fluid. The proposition in this article is however purely geometrical, and may be thus enunciated:—"Supposing space, or any portion of space, to be filled with an infinite number of points which move in any continuous manner, retaining their identity, to examine the nature of the instantaneous motion of any elementary portion of these points."

Let $u$, $v$, $w$ be the resolved parts, parallel to the rectangular axes $Ox$, $Oy$, $Oz$, of the velocity of the point $P$, whose co-ordinates at the instant considered are $x$, $y$, $z$. Then the relative velocities at the point $P'$, whose co-ordinates are $x + x'$, $y + y'$, $z + z'$, will be

$$\frac{du}{dx}x' + \frac{du}{dy}y' + \frac{du}{dz}z' \text{ parallel to } x,$$

$$\frac{dv}{dx}x' + \frac{dv}{dy}y' + \frac{dv}{dz}z' \ldots\ldots\ldots\ldots y,$$

$$\frac{dw}{dx}x' + \frac{dw}{dy}y' + \frac{dw}{dz}z' \ldots\ldots\ldots\ldots z,$$

neglecting squares and products of $x'$, $y'$, $z'$. Let these velocities be compounded of those due to the angular velocities $\omega'$, $\omega''$, $\omega'''$ about the axes of $x$, $y$, $z$, and of the velocities $U$, $V$, $W$ parallel to $x$, $y$, $z$. The linear velocities due to the angular velocities being $\omega''z' - \omega'''y$, $\omega'''x' - \omega'z'$, $\omega'y' - \omega''x'$ parallel to the axes of $x$, $y$, $z$, we shall therefore have

$$U = \frac{du}{dx}x' + \left(\frac{du}{dy} + \omega'''\right)y' + \left(\frac{du}{dz} - \omega''\right)z',$$

$$V = \left(\frac{dv}{dx} - \omega'''\right)x' + \frac{dv}{dy}y' + \left(\frac{dv}{dz} + \omega'\right)z',$$

$$W = \left(\frac{dw}{dx} + \omega''\right)x' + \left(\frac{dw}{dy} - \omega'\right)y' + \frac{dw}{dz}z'.$$

Since $\omega'$, $\omega''$, $\omega'''$ are arbitrary, let them be so assumed that

$$\frac{dU}{dy'} = \frac{dV}{dx'},\quad \frac{dV}{dz'} = \frac{dW}{dy'},\quad \frac{dW}{dx'} = \frac{dU}{dz'},$$

which gives

$$\omega' = \tfrac{1}{2}\left(\frac{dw}{dy} - \frac{dv}{dz}\right),\quad \omega'' = \tfrac{1}{2}\left(\frac{du}{dz} - \frac{dw}{dx}\right),\quad \omega''' = \tfrac{1}{2}\left(\frac{dv}{dx} - \frac{du}{dy}\right), \ldots\ldots\ldots(1)$$

$$\left.\begin{aligned} U &= \frac{du}{dx}x' + \tfrac{1}{2}\left(\frac{du}{dy} + \frac{dv}{dx}\right)y' + \tfrac{1}{2}\left(\frac{du}{dz} + \frac{dw}{dx}\right)z', \\ V &= \tfrac{1}{2}\left(\frac{dv}{dx} + \frac{du}{dy}\right)x' + \frac{dv}{dy}y' + \tfrac{1}{2}\left(\frac{dv}{dz} + \frac{dw}{dy}\right)z', \\ W &= \tfrac{1}{2}\left(\frac{dw}{dx} + \frac{du}{dz}\right)x' + \tfrac{1}{2}\left(\frac{dw}{dy} + \frac{dv}{dz}\right)y' + \frac{dw}{dz}z'. \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

The quantities $\omega'$, $\omega''$, $\omega'''$ are what I shall call the *angular velocities of the fluid* at the point considered. This is evidently an allowable definition, since, in the particular case in which the element considered moves as a solid might do, these quantities coincide with the angular velocities considered in rigid dynamics. A further reason for this definition will appear in Sect. III.

Let us now investigate whether it is possible to determine $x'$, $y'$, $z'$ so that, considering only the relative velocities $U$, $V$, $W$, the line joining the points $P$, $P'$ shall have no angular motion. The conditions to be satisfied, in order that this may be the case, are evidently that the increments of the relative co-ordinates $x'$, $y'$, $z'$ of the second point shall be ultimately proportional to those co-ordinates. If $e$ be the rate of extension of the line joining the two points considered, we shall therefore have

$$\left.\begin{aligned} Fx' + hy' + gz' &= ex', \\ hx' + Gy' + fz' &= ey', \\ gx' + fy' + Hz' &= ez'; \end{aligned}\right\} \quad \text{.................................}(3)$$

where $F = \dfrac{du}{dx}$, $G = \dfrac{dv}{dy}$, $H = \dfrac{dw}{dz}$, $2f = \dfrac{dv}{dz} + \dfrac{dw}{dy}$, $2g = \dfrac{dw}{dx} + \dfrac{du}{dz}$, $2h = \dfrac{du}{dy} + \dfrac{dv}{dx}$.

If we eliminate from equations (3) the two ratios which exist between the three quantities $x'$, $y'$, $z'$, we get the well known cubic equation

$$(e - F)(e - G)(e - H) - f^2(e - F) - g^2(e - G) - h^2(e - H) - 2fgh = 0, \text{......}(4)$$

which occurs in the investigation of the principal axes of a rigid body, and in various others. As in these investigations, it may be shewn that there are in general three directions, at right angles to each other, in which the point $P'$ may be situated so as to satisfy the required conditions. If two of the roots of (4) are equal, there is one such direction corresponding to the third root, and an infinite number of others situated in a plane perpendicular to the former; and if the three roots of (4) are equal, a line drawn in any direction will satisfy the required conditions.

The three directions which have just been determined I shall call *axes of extension*. They will in general vary from one point to another, and from one instant of time to another. If we denote the three roots of (4) by $e'$, $e''$, $e'''$, and if we take new rectangular axes $Ox_,$, $Oy_,$, $Oz_,$, parallel to the axes of extension, and denote by $u_,$, $U_,$, &c. the quantities referred to these axes corresponding to $u$, $U$, &c., equations (3) must be satisfied by $y_,' = 0$, $z_,' = 0$, $e = e'$, by $x_,' = 0$, $z_,' = 0$, $e = e''$, and by $x_,' = 0$, $y_,' = 0$, $e = e'''$, which requires that $f_, = 0$, $g_, = 0$, $h_, = 0$, and we have

$$e' = F_, = \frac{du_,}{dx_,}, \quad e'' = G_, = \frac{dv_,}{dy_,}, \quad e''' = H_, = \frac{dw_,}{dz_,}.$$

The values of $U_,$, $V_,$, $W_,$, which correspond to the residual motion after the elimination of the motion of rotation corresponding to $\omega'$, $\omega''$ and $\omega'''$, are

$$U_, = e'x_,', \quad V_, = e''y_,', \quad W_, = e'''z_,'.$$

The angular velocity of which $\omega'$, $\omega''$, $\omega'''$ are the components is independent of the arbitrary directions of the co-ordinate axes: the same is true of the directions of the axes of extension, and of the values of the roots of equation (4). This might be proved in various ways; perhaps the following is the simplest. The conditions by which $\omega'$, $\omega''$, $\omega'''$ are determined are those which express that the relative velocities $U$, $V$, $W$, which remain after eliminating a certain angular velocity, are such that $Udx' + Vdy' + Wdz'$ is ultimately an exact differential, that is to say when squares and products of $x'$, $y'$ and $z'$ are neglected. It appears moreover from the solution that there is only one way in which these conditions can be satisfied for a given system of co-ordinate axes. Let us take new rectangular axes $O\mathrm{x}$, $O\mathrm{y}$, $O\mathrm{z}$, and let $\mathrm{U}$, $\mathrm{V}$, $\mathrm{W}$ be the resolved parts along these axes of the velocities $U$, $V$, $W$, and $\mathrm{x}'$, $\mathrm{y}'$, $\mathrm{z}'$, the relative co-ordinates of $P'$; then

$$U = \text{U} \cos x\text{x} + \text{V} \cos x\text{y} + \text{W} \cos x\text{z},$$

$$dx' = \cos x\text{x}\, d\text{x}' + \cos x\text{y}\, d\text{y}' + \cos x\text{z}\, d\text{z}', \text{ \&c.};$$

whence, taking account of the well known relations between the cosines involved in these equations, we easily find

$$U dx' + V dy' + W dz' = \text{U}\, d\text{x}' + \text{V}\, d\text{y}' + \text{W}\, d\text{z}'.$$

It appears therefore that the relative velocities U, V, W, which remain after eliminating a certain angular velocity, are such that $\text{U}\,d\text{x}' + \text{V}\,d\text{y}' + \text{W}\,d\text{z}'$ is ultimately an exact differential. Hence the values of U, V, W are the same as would have been obtained from equations (2) applied directly to the new axes, whence the truth of the proposition enunciated at the head of this paragraph is manifest.

The motion corresponding to the velocities $U_{,}$, $V_{,}$, $W_{,}$ may be further decomposed into a motion of dilatation, positive or negative, which is alike in all directions, and two motions which I shall call *motions of shifting*, each of the latter being in two dimensions, and not affecting the density. For let $\delta$ be the rate of linear extension corresponding to a uniform dilatation; let $\sigma x_{,}'$, $-\sigma y_{,}'$ be the velocities parallel to $x_{,}$, $y_{,}$, corresponding to a motion of shifting parallel to the the plane $x_{,}y_{,}$, and let $\sigma' x_{,}'$, $-\sigma' z_{,}'$ be the velocities parallel to $x_{,}$, $z_{,}$, corresponding to a similar motion of shifting parallel to the plane $x_{,}z_{,}$. The velocities parallel to $x_{,}$, $y_{,}$, $z_{,}$ respectively corresponding to the quantities $\delta$, $\sigma$ and $\sigma'$ will be $(\delta + \sigma + \sigma') x_{,}'$, $(\delta - \sigma) y_{,}'$, $(\delta - \sigma') z_{,}'$, and equating these to $U_{,}$, $V_{,}$, $W_{,}$ we shall get

$$\delta = \tfrac{1}{3}(e' + e'' + e'''), \quad \sigma = \tfrac{1}{3}(e' + e''' - 2e''), \quad \sigma' = \tfrac{1}{3}(e' + e'' - 2e''').$$

Hence the most general instantaneous motion of an elementary portion of a fluid is compounded of a motion of translation, a motion of rotation, a motion of uniform dilatation, and two motions of shifting of the kind just mentioned.

3. Having determined the nature of the most general instantaneous motion of an element of a fluid, we are now prepared to consider the normal pressures and tangential forces called into play by the relative displacements of the particles. Let $p$ be the pressure which would exist about the point $P$ if the neighbouring molecules were in a state of relative equilibrium: let $p + p_{,}$ be the normal pressure, and $t_{,}$ the tangential action, both referred to a unit of surface, on a plane passing through $P$ and having a given direction. By the hypotheses of Art. I. the quantities $p_{,}$, $t_{,}$ will be independent of the angular velocities $\omega'$, $\omega''$, $\omega'''$, depending only on the residual relative velocities $U, V, W$, or, which comes to the same, on $e'$, $e''$ and $e'''$, or on $\sigma, \sigma'$ and $\delta$. Since this residual motion is symmetrical with respect to the axes of extension, it follows that if the plane considered is perpendicular to any one of these axes the tangential action on it is zero, since there is no more reason why it should act in one direction rather than in the opposite; for by the hypotheses of Art. I. the change of density and temperature about the point $P$ is to be neglected, the constitution of the fluid being ultimately uniform about that point. Denoting then by $p + p'$, $p + p''$, $p + p'''$ the pressures on planes perpendicular to the axes of $x_{,}$, $y_{,}$, $z_{,}$, we must have

$$p' = \phi(e', e'', e'''), \quad p'' = \phi(e'', e''', e'), \quad p''' = \phi(e''', e', e''),$$

$\phi(e', e'', e''')$ denoting a function of $e'$, $e''$ and $e'''$ which is symmetrical with respect to the two latter quantities. The question is now to determine, on whatever may seem the most probable hypothesis, the form of the function $\phi$.

Let us first take the simpler case in which there is no dilatation, and only one motion of shifting, or in which $e'' = -e'$, $e''' = 0$, and let us consider what would take place if the fluid consisted of smooth molecules acting on each other by actual contact. On this supposition, it is clear, considering the magnitude of the pressures acting on the molecules compared with their masses, that they would be sensibly in a position of relative equilibrium, except when the equilibrium of any one of them became impossible from the displacement of the adjoining ones, in which case the molecule in question would start into a new position of equilibrium. This start would cause a corresponding displacement in the molecules immediately about the one which started, and this disturbance would be propagated immediately in all directions, the nature of the displacement however being different in different directions, and would soon become insensible. During the continuance of this disturbance, the pressure on a small plane drawn through the element considered would not be the same in all directions, nor normal to the plane: or, which comes to the same, we may suppose a uniform normal pressure $p$ to act, together with a normal pressure $p_{\prime\prime}$ and a tangential force $t_{\prime\prime}$, $p_{\prime\prime}$ and $t_{\prime\prime}$ being forces of great intensity and short duration, that is being of the nature of impulsive forces. As the number of molecules comprised in the element considered has been supposed extremely great, we may take a time $\tau$ so short that all summations with respect to such intervals of time may be replaced without sensible error by integrations, and yet so long that a very great number of starts shall take place in it. Consequently we have only to consider the average effect of such starts, and moreover we may without sensible error replace the impulsive forces such as $p_{\prime\prime}$ and $t_{\prime\prime}$, which succeed one another with great rapidity, by continuous forces. For planes perpendicular to the axes of extension these continuous forces will be the normal pressures $p'$, $p''$, $p'''$.

Let us now consider a motion of shifting differing from the former only in having $e'$ increased in the ratio of $m$ to 1. Then, if we suppose each start completed before the starts which would be sensibly affected by it are begun, it is clear that the same series of starts will take place in the second case as in the first, but at intervals of time which are less in the ratio of 1 to $m$. Consequently the continuous pressures by which the impulsive actions due to these starts may be replaced must be increased in the ratio of $m$ to 1. Hence the pressures $p'$, $p''$, $p'''$ must be proportional to $e'$, or we must have

$$p' = Ce', \quad p'' = C'e', \quad p''' = C''e'.$$

It is natural to suppose that these formulæ held good for negative as well as positive values of $e'$. Assuming this to be true, let the sign of $e'$ be changed. This comes to interchanging $x$ and $y$, and consequently $p'''$ must remain the same, and $p'$ and $p''$ must be interchanged. We must therefore have $C'' = 0$, $C' = -C$. Putting then $C = -2\mu$ we have

$$p' = -2\mu e', \quad p'' = 2\mu e', \quad p''' = 0.$$

It has hitherto been supposed that the molecules of a fluid are in actual contact. We have every reason to suppose that this is not the case. But precisely the same reasoning will apply if they are separated by intervals as great as we please compared with their magnitudes, provided only we suppose the force of restitution called into play by a small displacement of *any one* molecule to be very great.

Let us now take the case of two motions of shifting which coexist, and let us suppose $e' = \sigma + \sigma'$, $e'' = -\sigma$, $e''' = -\sigma'$. Let the small time $\tau$ be divided into $2n$ equal portions, and let us suppose that in the first interval a shifting motion corresponding to $e' = 2\sigma$, $e'' = -2\sigma$ takes place parallel to the plane $x_{\prime}\, y_{\prime}$, and that in the second interval a shifting motion corresponding to $e' = 2\sigma'$, $e''' = -2\sigma'$ takes place parallel to the plane $x_{\prime}\, z_{\prime}$, and so on alternately. On this supposition it is clear that if we suppose the time $\frac{\tau}{2n}$ to be extremely small, the continuous forces by which the effect of the starts may be replaced will be $p' = -2\mu(\sigma + \sigma')$, $p'' = 2\mu\sigma$, $p''' = 2\mu\sigma'$. By supposing $n$ indefinitely increased, we may make the motion considered approach as near as we please to that in which the two motions of shifting coexist; but we are not at liberty to do so, for in order to apply the above reasoning we must suppose the time $\frac{\tau}{2n}$ to be so large that the average effect of the starts which occur in it may be taken. Consequently it must be taken as an additional assumption, and not a matter of absolute demonstration, that the effects of the two motions of shifting are superimposed.

Hence if $\delta = 0$, *i.e.* if $e' + e'' + e''' = 0$, we shall have in general

$$p' = -2\mu e', \quad p'' = -2\mu e'', \quad p''' = -2\mu e''' \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

It was by this hypothesis of starts that I first arrived at these equations, and the differential equations of motion which result from them. On reading Poisson's memoir however, to which I shall have occasion to refer in Section IV., I was led to reflect that however intense we may suppose the molecular forces to be, and however near we may suppose the molecules to be to their positions of relative equilibrium, we are not therefore at liberty to suppose them *in* those positions, and consequently not at liberty to suppose the pressure equal in all directions in the intervals of time between the starts. In fact, by supposing the molecular forces indefinitely increased, retaining the same ratios to each other, we may suppose the displacements of the molecules from their positions of relative equilibrium indefinitely diminished, but on the other hand the force of restitution called into action by a given displacement is indefinitely increased in the same proportion. But be these displacements what they may, we know that the forces of restitution make equilibrium with forces equal and opposite to the effective forces; and in calculating the effective forces we may neglect the above displacements, or suppose the molecules to move in the paths in which they would move if the shifting motion took place with indefinite slowness. Let us first consider a single motion of shifting, or one for which $e'' = -e'$, $e''' = 0$, and let $p_{,}$ and $t_{,}$ denote the same quantities as before. If we now suppose $e'$ increased in the ratio of $m$ to 1, all the effective forces will be increased in that ratio, and consequently $p_{,}$ and $t_{,}$ will be increased in the same ratio. We may deduce the values of $p'$, $p''$ and $p'''$ just as before, and then pass by the same reasoning to the case of two motions of shifting which coexist, only that in this case the reasoning will be demonstrative, since we *may* suppose the time $\frac{\tau}{2n}$ indefinitely diminished. If we suppose the state of things considered in this paragraph to exist along with the motions of starting already considered, it is easy to see that the expressions for $p'$, $p''$ and $p'''$ will still retain the same form.

There remains yet to be considered the effect of the dilatation. Let us first suppose the dilatation to exist without any shifting: then it is easily seen that the relative motion of the fluid at the point considered is the same in all directions. Consequently the only effect which such a dilatation could have would be to introduce a normal pressure $p_{,}$, alike in all directions, in addition to that due to the action of the molecules supposed to be in a state of relative equilibrium. Now the pressure $p_{,}$ could only arise from the aggregate of the molecular actions called into play by the displacements of the molecules from their positions of relative equilibrium; but since these displacements take place, on an average, indifferently in all directions, it follows that the actions of which $p_{,}$ is composed neutralize each other, so that $p_{,} = 0$. The same conclusion might be drawn from the hypothesis of starts, supposing, as it is natural to do, that each start calls into action as much increase of pressure in some directions as diminution of pressure in others.

If the motion of uniform dilatation coexists with two motions of shifting, I shall suppose, for the same reason as before, that the effects of these different motions are superimposed. Hence subtracting $\delta$ from each of the three quantities $e'$, $e''$ and $e'''$, and putting the remainders in the place of $e'$, $e''$ and $e'''$ in equations (5), we have

$$p' = \tfrac{2}{3}\mu(e'' + e''' - 2e'), \quad p'' = \tfrac{2}{3}\mu(e''' + e' - 2e''), \quad p''' = \tfrac{2}{3}\mu(e' + e'' - 2e''') \ldots\ldots\ldots (6)$$

If we had started with assuming $\phi(e', e'', e''')$ to be a linear function of $e'$, $e''$ and $e'''$, avoiding all speculation as to the molecular constitution of a fluid, we should have had at once $p' = Ce' + C'(e'' + e''')$, since $p'$ is symmetrical with respect to $e''$ and $e'''$; or, changing the constants, $p' = \frac{2}{3}\mu(e'' + e''' - 2e') + \kappa(e' + e'' + e''')$. The expressions for $p''$ and $p'''$ would be obtained by interchanging the requisite quantities. Of course we may at once put $\kappa = 0$ if we assume that in the case of a uniform motion of dilatation the pressure at any instant depends only on the actual density and temperature at that instant, and not on the rate at which the former changes with the time. In most cases to which it would be interesting to apply the theory of the friction of fluids the density of the fluid is either constant, or may without sensible error be regarded as constant, or else changes slowly with the time. In the first two cases the results would be the same, and in the third case nearly the same, whether $\kappa$ were equal to zero or not. Consequently, if theory and experiment should in such cases agree, the experiments must not be regarded as confirming that part of the theory which relates to supposing $\kappa$ to be equal to zero.

4. It will be easy now to determine the oblique pressure, or resultant of the normal pressure and tangential action, on any plane. Let us first consider a plane drawn through the point $P$ parallel to the plane $yz$. Let $Ox_{,}$ make with the axes of $x$, $y$, $z$ angles whose cosines are $l'$, $m'$, $n'$; let $l''$, $m''$, $n''$ be the same for $Oy_{,}$, and $l'''$, $m'''$, $n'''$ the same for $Oz_{,}$. Let $P_1$ be the pressure, and $(xty)$, $(xtz)$ the resolved parts, parallel to $y$, $z$ respectively, of the tangential force on the plane considered, all referred to a unit of surface, $(xty)$ being reckoned positive when the part of the fluid towards $-x$ urges that towards $+x$ in the positive direction of $y$, and similarly for $(xtz)$. Consider the portion of the fluid comprised within a tetrahedron having its vertex in the point $P$, its base parallel to the plane $yz$, and its three sides parallel to the planes $x_{,}y_{,}$, $y_{,}z_{,}$, $z_{,}x_{,}$ respectively. Let $A$ be the area of the base, and therefore $l'A$, $l''A$, $l'''A$ the areas of the faces perpendicular to the axes of $x_{,}$, $y_{,}$, $z_{,}$. By D'Alembert's principle, the pressures and tangential actions on the faces of this tetrahedron, the moving forces arising from the external attractions, not including the molecular forces, and forces equal and opposite to the effective moving forces will be in equilibrium, and therefore the sums of the resolved parts of these forces in the directions of $x$, $y$ and $z$ will each be zero. Suppose now the dimensions of the tetrahedron indefinitely diminished, then the resolved parts of the external, and of the effective moving forces will vary ultimately as the cubes, and those of the pressures and tangential forces on the sides as the squares of homologous lines. Dividing therefore the three equations arising from equating to zero the resolved parts of the above forces by $A$, and taking the limit, we have

$$P_1 = \Sigma l'^2 (p + p'), \quad (xty) = \Sigma l'm' (p + p'), \quad (xtz) = \Sigma l'n' (p + p'),$$

the sign $\Sigma$ denoting the sum obtained by taking the quantities corresponding to the three axes of extension in succession. Putting for $p'$, $p''$ and $p'''$ their values given by (6), putting $e' + e'' + e''' = 3\delta$, and observing that $\Sigma l'^2 = 1$, $\Sigma l'm' = 0$, $\Sigma l'n' = 0$, the above equations become

$$P_1 = p - 2\mu \Sigma l'^2 e' + 2\mu\delta, \quad (xty) = -2\mu \Sigma l'm'e', \quad (xtz) = -2\mu \Sigma l'n'e'.$$

The method of determining the pressure on any plane from the pressures on three planes at right angles to each other, which has just been given, has already been employed by MM. Cauchy and Poisson.

The most direct way of obtaining the values of $\Sigma l'^2 e'$ &c. would be to express $l'$, $m'$ and $n'$ in terms of $e'$ by any two of equations (3), in which $x'$, $y'$, $z'$ are proportional to $l'$, $m'$, $n'$, together with the equation $l'^2 + m'^2 + n'^2 = 1$, and then to express the resulting symmetrical function of the roots of the cubic equation (4) in terms of the coefficients. But this method would be excessively laborious, and need not be resorted to. For after eliminating the angular motion of the element of fluid considered the remaining velocities are $e'x_{,}'$, $e''y_{,}'$, $e'''z_{,}'$, parallel to the axes of $x_{,}$, $y_{,}$, $z_{,}$. The sum of the resolved parts of these parallel to the axis of $x$ is $l'e'x_{,}' + l''e''y_{,}' + l'''e'''z_{,}'$. Putting for $x_{,}'$, $y_{,}'$, $z_{,}'$ their values $l'x' + m'y' + n'z'$ &c., the above sum becomes

$$x'\Sigma l'^2 e' + y'\Sigma l'm'e' + z'\Sigma l'n'e';$$

but this sum is the same thing as the velocity $U$ in equation (2), and therefore we have

$$\Sigma l'^2 e' = \frac{du}{dx}, \quad \Sigma l'm'e' = \frac{1}{2}\left(\frac{du}{dy} + \frac{dv}{dx}\right), \quad \Sigma l'n'e' = \frac{1}{2}\left(\frac{du}{dz} + \frac{dw}{px}\right).$$

It may also be very easily proved directly that the value of $3\delta$, the rate of cubical dilatation, satisfies the equation

$$3\delta = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \ldots\ldots\ldots\ldots (7)$$

Let $P_2$, $(ytz)$, $(ytx)$ be the quantities referring to the axis of $y$, and $P_3$, $(ztx)$, $(zty)$ those referring to the axis of $z$, which correspond to $P_1$ &c. referring to the axis of $x$. Then we see that $(ytz) = (zty)$, $(ztx) = (xtz)$, $(xty) = (ytx)$. Denoting these three quantities by $T_1$, $T_2$, $T_3$, and making the requisite substitutions and interchanges, we have

$$\left.\begin{aligned} P_1 &= p - 2\mu\left(\frac{du}{dx} - \delta\right), \\ P_2 &= p - 2\mu\left(\frac{dv}{dy} - \delta\right), \\ P_3 &= p - 2\mu\left(\frac{dw}{dz} - \delta\right), \\ T_1 = -\mu\left(\frac{dv}{dz} + \frac{dw}{dy}\right),\ T_2 &= -\mu\left(\frac{dw}{dx} + \frac{du}{dz}\right),\ T_3 = -\mu\left(\frac{du}{dy} + \frac{dv}{dx}\right). \end{aligned}\right\} \ldots\ldots\ldots\ldots (8)$$

It may also be useful to know the components, parallel to $x$, $y$, $z$, of the oblique pressure on a plane passing through the point $P$, and having a given direction. Let $l$, $m$, $n$ be the cosines of the angles which a normal to the given plane makes with the axes of $x$, $y$, $z$; let $P$, $Q$, $R$ be the components, referred to a unit of surface, of the oblique pressure on this plane, $P$, $Q$, $R$ being reckoned positive when the part of the fluid in which is situated the normal to which $l$, $m$ and $n$ refer is urged by the other part in the positive directions of $x$, $y$, $z$, when $l$, $m$ and $n$ are positive. Then considering as before a tetrahedron of which the base is parallel to the given plane, the vertex in the point $P$, and the sides parallel to the co-ordinate planes, we shall have

$$\left.\begin{aligned} P &= lP_1 + mT_3 + nT_2, \\ Q &= lT_3 + mP_2 + nT_1, \\ R &= lT_2 + mT_1 + nP_3. \end{aligned}\right\} \ldots\ldots\ldots\ldots (9)$$

In the simple case of a sliding motion for which $u = 0$, $v = f(x)$, $w = 0$, the only forces, besides the pressure $p$, which act on planes parallel to the co-ordinate planes are the two tangential forces $T_3$, the value of which in this case is $-\mu\dfrac{dv}{dx}$. In this case it is easy to show that the axes of extension are, one of them parallel to $Oz$, and the two others in a plane parallel to $xy$, and inclined at angles of $45^0$ to $Ox$. We see also that it is necessary to suppose $\mu$ to be positive, since otherwise the tendency of the forces would be to increase the relative motion of the parts of the fluid, and the equilibrium of the fluid would be unstable.

5. Having found the pressures about the point $P$ on planes parallel to the co-ordinate planes, it will be easy to form the equations of motion. Let $X$, $Y$, $Z$ be the resolved parts, parallel to the axes, of the external force, not including the molecular force; let $\rho$ be the density, $t$ the time. Consider an elementary parallelepiped of the fluid, formed by planes parallel to the co-ordinate planes, and drawn through the point $(x, y, z)$ and the point $(x + \Delta x, y + \Delta y, z + \Delta z)$. The mass of this element will be ultimately $\rho\Delta x\,\Delta y\,\Delta z$, and the moving force parallel to $x$ arising from the external forces will be ultimately $\rho X\Delta x\,\Delta y\,\Delta z$; the effective moving force parallel to $x$ will be ultimately $\rho\dfrac{Du}{Dt}\Delta x\,\Delta y\,\Delta z$, where $D$ is used, as it will be in the rest of this paper,

to denote differentiation in which the independent variables are $t$ and three parameters of the particle considered, (such for instance as its initial co-ordinates,) and not $t, x, y, z$. It is easy also to show that the moving force acting on the element considered arising from the oblique pressures on the faces is ultimately $\left(\frac{dP}{dx}+\frac{dT_3}{dy}+\frac{dT_2}{dz}\right)\Delta x\,\Delta y\,\Delta z$, acting in the negative direction. Hence we have by D'Alembert's principle

$$\rho\left(\frac{Du}{Dt}-X\right)+\frac{dP_1}{dx}+\frac{dT_3}{dy}+\frac{dT_2}{dz}=0,\ \&c.,\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

in which equations we must put for $\frac{Du}{Dt}$ its value $\frac{du}{dt}+u\frac{du}{dx}+v\frac{du}{dy}+w\frac{du}{dz}$, and similarly for $\frac{Dv}{dt}$ and $\frac{Dw}{Dt}$. In considering the general equations of motion it will be needless to write down more than one, since the other two may be at once derived from it by interchanging the requisite quantities. The equations (10), the ordinary equation of continuity, as it is called,

$$\frac{d\rho}{dt}+\frac{d\rho u}{dx}+\frac{d\rho v}{dy}+\frac{d\rho w}{dz}=0,\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

which expresses the condition that there is no generation or destruction of mass in the interior of a fluid, the equation connecting $p$ and $\rho$, or in the case of an incompressible fluid the equivalent equation $\frac{D\rho}{Dt}=0$, and the equation for the propagation of heat, if we choose to take account of that propagation, are the only equations to be satisfied at every point of the interior of the fluid mass.

As it is quite useless to consider cases of the utmost degree of generality, I shall suppose the fluid to be homogeneous, and of a uniform temperature throughout, except in so far as the temperature may be raised by sudden compression in the case of small vibrations. Hence in equations (10) $\mu$ may be supposed to be constant as far as regards the temperature; for, in the case of small vibrations, the terms introduced by supposing it to vary with the temperature would involve the square of the velocity, which is supposed to be neglected. If we suppose $\mu$ to be independent of the pressure also, and substitute in (10) the values of $P_1$ &c. given by (8), the former equations become

$$\rho\left(\frac{Du}{Dt}-X\right)+\frac{dp}{dx}-\mu\left(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\right)-\frac{\mu}{3}\frac{d}{dx}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)=0,\ \&c.\ldots\ldots(12)$$

Let us now consider in what cases it is allowable to suppose $\mu$ to be independent of the pressure. It has been concluded by Dubuat, from his experiments on the motion of water in pipes and canals, that the total retardation of the velocity due to friction is not increased by increasing the pressure. The total retardation depends, partly on the friction of the water against the sides of the pipe or canal, and partly on the mutual friction, or tangential action, of the different portions of the water. Now if these two parts of the whole retardation were separately variable with $p$, it is very unlikely that they should when combined give a result independent of $p$. The amount of the internal friction of the water depends on the value of $\mu$. I shall therefore suppose that for water, and by analogy for other incompressible fluids, $\mu$ is independent of the pressure. On this supposition, we have from equations (11) and (12)

$$\rho\left(\frac{Du}{Dt}-X\right)+\frac{dp}{dx}-\mu\left(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\right)=0,\ \&c.\ldots\ldots\ldots\ldots(13)$$

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0.$$

These equations are applicable to the determination of the motion of water in pipes and canals, to the calculation of the effect of friction on the motions of tides and waves, and such questions.

If the motion is very small, so that we may neglect the square of the velocity, we may put $\frac{Du}{Dt} = \frac{du}{dt}$, &c. in equations (13). The equations thus simplified are applicable to the determination of the motion of a pendulum oscillating in water, or of that of a vessel filled with water and made to oscillate. They are also applicable to the determination of the motion of a pendulum oscillating in air, for in this case we may, with hardly any error, neglect the compressibility of the air.

The case of the small vibrations by which sound is propagated in a fluid, whether a liquid or a gas, is another in which $\frac{d\mu}{dp}$ may be neglected. For in the case of a liquid reasons have been shown for supposing $\mu$ to be independent of $p$, and in the case of a gas we may neglect $\frac{d\mu}{dp}$, if we neglect the small change in the value of $\mu$, arising from the small variation of pressure due to the forces $X, Y, Z$.

6. Besides the equations which must hold good at any point in the interior of the mass, it will be necessary to form also the equations which must be satisfied at its boundaries. Let $M$ be a point in the boundary of the fluid. Let a normal to the surface at $M$, drawn on the outside of the fluid, make with the axes angles whose cosines are $l, m, n$. Let $P', Q', R'$ be the components of the pressure of the fluid about $M$ on the solid or fluid with which it is in contact, these quantities being reckoned positive when the fluid considered presses the solid or fluid outside it in the positive directions of $x, y, z$, supposing $l$, $m$ and $n$ positive. Let $S$ be a very small element of the surface about $M$, which will be ultimately plane, $S'$ a plane parallel and equal to $S$, and directly opposite to it, taken within the fluid. Let the distance between $S$ and $S'$ be supposed to vanish in the limit compared with the breadth of $S$, a supposition which may be made if we neglect the effect of the curvature of the surface at $M$; and let us consider the forces acting on the element of fluid comprised between $S$ and $S'$, and the motion of this element. If we suppose equations (8) to hold good to within an insensible distance from the surface of the fluid, we shall evidently have forces ultimately equal to $PS$, $QS$, $RS$, ($P, Q$ and $R$ being given by equations (9),) acting on the inner side of the element in the positive directions of the axes, and forces ultimately equal to $P'S$, $Q'S$, $R'S$ acting on the outer side in the negative directions. The moving forces arising from the external forces acting on the element, and the effective moving forces will vanish in the limit compared with the forces $PS$, &c.: the same will be true of the pressures acting about the edge of the element, if we neglect capillary attraction, and all forces of the same nature. Hence, taking the limit, we shall have

$$P' = P, \quad Q' = Q, \quad R' = R.$$

The method of proceeding will be different according as the bounding surface considered is a free surface, the surface of a solid, on the surface of separation of two fluids, and it will be necessary to consider these cases separately. Of course the surface of a liquid exposed to the air is really the surface of separation of two fluids, but it may in many cases be regarded as a free surface if we neglect the inertia of the air: it may always be so regarded if we neglect the friction of the air as well as its inertia.

Let us first take the case of a free surface exposed to a pressure $\Pi$, which is supposed to be the same at all points, but may vary with the time; and let $L = 0$ be the equation to the surface. In this case we shall have $P' = l\Pi$, $Q' = m\Pi$, $R' = n\Pi$; and putting for $P, Q, R$ their values given by (9), and for $P_1$ &c. their values given by (8), and observing that in this case $\delta = 0$, we shall have

$$l(\Pi - p) + \mu\left\{2l\frac{du}{dx} + m\left(\frac{du}{dy} + \frac{dv}{dx}\right) + n\left(\frac{du}{dz} + \frac{dw}{dx}\right)\right\} = 0, \text{ \&c.,} \qquad (14)$$

in which equations $l$, $m$, $n$ will have to be replaced by $\frac{dL}{dx}$, $\frac{dL}{dy}$, $\frac{dL}{dz}$, to which they are proportional.

If we choose to take account of capillary attraction, we have only to diminish the pressure $\Pi$ by the quantity $H\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$, where $H$ is a positive constant depending on the nature of the fluid, and $r_1$, $r_2$ are the principal radii of curvature at the point considered, reckoned positive when the fluid is concave outwards. Equations (14) with the ordinary equation

$$\frac{dL}{dt} + u\frac{dL}{dx} + v\frac{dL}{dy} + w\frac{dL}{dz} = 0, \qquad (15)$$

are the conditions to be satisfied for points at the free surface. Equations (14) are for such points what the three equations of motion are for internal points, and (15) is for the former what (11) is for the latter, expressing in fact that there is no generation or destruction of fluid at the free surface.

The equations (14) admit of being differently expressed, in a way which may sometimes be useful. If we suppose the origin to be in the point considered, and the axis of $z$ to be the external normal to the surface, we have $l = m = 0$, $n = 1$, and the equations become

$$\frac{dw}{dx} + \frac{du}{dz} = 0, \quad \frac{dw}{dy} + \frac{dv}{dz} = 0, \quad \Pi - p + 2\mu\frac{dw}{dz} = 0. \qquad (16)$$

The relative velocity parallel to $z$ of a point $(x', y', 0)$ in the free surface, indefinitely near the origin, is $\frac{dw}{dx}x' + \frac{dw}{dy}y'$: hence we see that $\frac{dw}{dx}$, $\frac{dw}{dy}$ are the angular velocities, reckoned from $x$ to $z$ and from $y$ to $z$ respectively, of an element of the free surfaee. Subtracting the linear velocities due to these angular velocities from the relative velocities of the point $(x', y', z')$, and calling the remaining relative velocities $U$, $V$, $W$, we shall have

$$U = \frac{du}{dx}x' + \frac{du}{dy}y' + \left(\frac{du}{dz} + \frac{dw}{dx}\right)z',$$

$$V = \frac{dv}{dx}x' + \frac{dv}{dy}y' + \left(\frac{dv}{dz} + \frac{dw}{dy}\right)z',$$

$$W = \frac{dw}{dz}z'.$$

Hence we see that the first two of equations (16) express the conditions that $\frac{dU}{dz'} = 0$ and $\frac{dV}{dz'} = 0$, which are evidently the conditions to be satisfied in order that there may be no sliding motion in a direction parallel to the free surface. It would be easy to prove that these are the conditions to be satisfied in order that the axis of $z$ may be an axis of extension.

The next case to consider is that of a fluid in contact with a solid. The condition which first occurred to me to assume for this case was, that the film of fluid immediately in contact with the solid did not move relatively to the surface of the solid. I was led to try this condition from the following considerations. According to the hypotheses adopted, if there was a very large relative motion of the fluid particles immediately about any imaginary surface dividing the fluid, the tangential forces called into action would be very large, so that the amount of relative motion would be rapidly diminished. Passing to the limit, we might suppose that if at any instant the velocities altered discontinuously in passing across any imaginary surface, the tangential force called into action would immediately destroy the finite relative motion of particles indefinitely close to each other, so as to render the motion continuous; and from analogy the same might be supposed to be true for the surface of junction of a fluid and solid. But having calculated, according to the conditions which I have mentioned, the discharge of long straight circular pipes and rectangular canals, and compared the resulting formulæ with some of the experiments of Bossut and Dubuat, I found that the formulæ did not at all agree with experiment. I then tried Poisson's conditions in the case of a circular pipe, but with no better success. In fact, it appears from experiment that the tangential force varies nearly as the square of the velocity with which the fluid flows past the surface of a solid, at least when the velocity is not very small. It appears however from experiments on pendulums that the total friction varies as the first power of the velocity, and consequently we may suppose that Poisson's conditions, which include as a particular case those which I first tried, hold good for very small velocities. I proceed therefore to deduce these conditions in a manner conformable with the views explained in this paper.

First, suppose the solid at rest, and let $L = 0$ be the equation to its surface. Let $M'$ be a point within the fluid, at an insensible distance $h$ from $M$. Let $\varpi$ be the pressure which would exist about $M$ if there were no motion of the particles in its neighbourhood, and let $p_{,}$ be the additional normal pressure, and $t_{,}$ the tangential force, due to the relative velocities of the particles, both with respect to one another and with respect to the surface of the solid. If the motion is so slow that the starts take place independently of each other, on the hypothesis of starts, or that the molecules are very nearly in their positions of relative equilibrium, and if we suppose as before that the effects of different relative velocities are superimposed, it is easy to show that $p_{,}$ and $t_{,}$ are linear functions of $u$, $v$, $w$ and their differential coefficients with respect to $x$, $y$, and $z$; $u$, $v$, &c. denoting here the velocities of the fluid about the point $M'$, in the expressions for which however the co-ordinates of $M$ may be used for those of $M'$, since $h$ is neglected. Now the relative velocities about the points $M$ and $M'$ depending on $\frac{du}{dx}$ &c. are comparable with $\frac{du}{dx}h$, while those depending on $u$, $v$ and $w$ are comparable with these quantities, and therefore in considering the action of the fluid on the solid it is only necessary to consider the quantities $u$, $v$ and $w$. Now since, neglecting $h$, the velocity at $M'$ is tangential to the surface at $M$, $u$, $v$, and $w$ are the components of a certain velocity $V$ tangential to the surface. The pressure $p_{,}$ must be zero; for changing the signs of $u$, $v$, and $w$ the circumstances concerned in its production remain the same, whereas its analytical expression changes sign. The tangential force at $M$ will be in the direction of $V$, and proportional to it, and consequently its components along the axes of $x$, $y$, $z$ will be proportional to $u$, $v$, $w$. Reckoning the tangential force positive when, $l$, $m$, and $n$ being positive, the solid is urged in the positive directions of $x$, $y$, $z$, the resolved parts of the tangential force will therefore be $\nu u$, $\nu v$, $\nu w$, where $\nu$ must evidently be positive, since the effect of the forces must be to check the relative motion of the fluid and solid. The normal pressure of the fluid on the solid being equal to $\varpi$, its components will be evidently $l\varpi$, $m\varpi$, $n\varpi$.

Suppose now the solid to be in motion, and let $u'$, $v'$, $w'$ be the resolved parts of the velocity of the point $M$ of the solid, and $\omega'$, $\omega''$, $\omega'''$ the angular velocities of the solid. By hypothesis, the forces by which the pressure at any point differs from the normal pressure due to the action of the molecules supposed to be in a state of relative equilibrium about that point are independent of any velocity of translation or rotation. Supposing then linear and angular velocities equal and opposite to those of the solid impressed both on the solid and on the fluid, the former will be for an instant at rest, and we have only to treat the resulting velocities of the fluid as in the first case.

Hence $P' = l\varpi + \nu(u - u')$, &c.; and in the equations (8) we may employ the actual velocities $u$, $v$, $w$, since the pressures $P$, $Q$, $R$ are independent of any motion of translation and rotation common to the whole fluid. Hence the equations $P' = P$, &c. give us

$$l(\varpi - p) + \nu(u - u') + \mu\left\{2l\left(\frac{du}{dx} - \delta\right) + m\left(\frac{du}{dy} + \frac{dv}{dx}\right) + n\left(\frac{du}{dz} + \frac{dw}{dx}\right)\right\} = 0, \text{ \&c.,} \ldots\ldots(17)$$

which three equations with (15) are those which must be satisfied at the surface of a solid, together with the equation $L = 0$. It will be observed that in the case of a free surface the pressures $P'$, $Q'$, $R'$ are given, whereas in the case of the surface of a solid they are known only by the solution of the problem. But on the other hand the form of the surface of the solid is given, whereas the form of the free surface is known only by the solution of the problem.

Dubuat found by experiment that when the mean velocity of water flowing through a pipe is less than about one inch in a second, the water near the inner surface of the pipe is at rest. If these experiments may be trusted, the conditions to be satisfied in the case of small velocities are those which first occurred to me, and which are included in those just given by supposing $\nu = \infty$.

I have said that when the velocity is not very small the tangential force called into action by the sliding of water over the inner surface of a pipe varies nearly as the square of the velocity. This fact appears to admit of a natural explanation. When a current of water flows past an obstacle, it produces a resistance varying nearly as the square of the velocity. Now even if the inner surface of a pipe is polished we may suppose that little irregularities exist, forming so many obstacles to the current. Each little protuberance will experience a resistance varying nearly as the square of the velocity, from whence there will result a tangential action of the fluid on the surface of the pipe, which will vary nearly as the square of the velocity; and the same will be true of the equal and opposite reaction of the pipe on the fluid. The tangential force due to this cause will be combined with that by which the fluid close to the pipe is kept at rest when the velocity is sufficiently small.

There remains to be considered the case of two fluids having a common surface. Let $u'$, $v'$, $w'$, $\mu'$, $\delta'$ denote the quantities belonging to the second fluid corresponding to $u$, &c. belonging to the first. Together with the two equations $L = 0$ and (15) we shall have in this case the equation derived from (15) by putting $u'$, $v'$, $w'$ for $u$, $v$, $w$; or, which comes to the same, we shall have the two former equations with

$$l(u - u') + m(v - v') + n(w - w') = 0 \ldots\ldots\ldots\ldots\ldots(18)$$

If we consider the principles on which equations (17) were formed to be applicable to the present case, we shall have six more equations to be satisfied, namely (17), and the three equations derived from (17) by interchanging the quantities referring to the two fluids, and changing the signs of $l$, $m$, $n$. These equations give the value of $\varpi$, and leave five equations of condition. If we must suppose $\nu = \infty$, as appears most probable, the six equations above mentioned must be replaced by the six $u' = u$, $v' = v$, $w' = w$, and

$$lp - \mu f(u, v, w) = lp' - \mu' f(u', v', w'), \text{ \&c.,}$$

$f(u, v, w)$ denoting the coefficient of $\mu$ in the first of equations (17). We have here six equations of condition instead of five, but then the equation (18) becomes identical.

7. The most interesting questions connected with this subject require for their solution a knowledge of the conditions which must be satisfied at the surface of a solid in contact with the fluid, which, except perhaps in case of very small motions, are unknown. It may be well however to give some applications of the preceding equations which are independent of these conditions. Let us then in the first place consider in what manner the transmission of sound in a fluid is affected by the tangential action. To take the simplest case, suppose that no forces act on the fluid, so that the pressure and density are constant in the state of

equilibrium, and conceive a series of plane waves to be propagated in the direction of the axis of $x$, so that $u = f(x, t)$, $v = 0$, $w = 0$. Let $p_{\prime}$ be the pressure, and $\rho_{\prime}$ the density of the fluid when it is in equilibrium, and put $p = p_{\prime} + p'$. Then we have from equations (11) and (12), omitting the square of the disturbance,

$$\frac{1}{\rho_{\prime}}\frac{d\rho}{dt} + \frac{du}{dx} = 0, \quad \rho_{\prime}\frac{du}{dt} + \frac{dp'}{dx} - \frac{4}{3}\mu\frac{d^2u}{dx^2} = 0 \text{..........(19)}$$

Let $A\Delta\rho$ be the increment of pressure due to a very small increment $\Delta\rho$ of density, the temperature being unaltered, and let $m$ be the ratio of the specific heat of the fluid when the pressure is constant to its specific heat when the volume is constant; then the relation between $p'$ and $\rho$ will be

$$p' = mA(\rho - \rho_{\prime}) \text{..........................(20)}$$

Eliminating $p'$ and $\rho$ from (19) and (20) we get

$$\frac{d^2u}{dt^2} - mA\frac{d^2u}{dx^2} - \frac{4\mu}{3\rho_{\prime}}\frac{d^3u}{dt\,dx^2} = 0.$$

To obtain a particular solution of this equation, let $u = \phi(t)\cos\frac{2\pi x}{\lambda} + \psi(t)\sin\frac{2\pi x}{\lambda}$. Substituting in the above equation, we see that $\phi(t)$ and $\psi(t)$ must satisfy the same equation, namely,

$$\phi''(t) + \frac{4\pi^2}{\lambda^2}mA\phi(t) + \frac{16\pi^2\mu}{3\lambda^2\rho_{\prime}}\phi'(t) = 0,$$

the integral of which is

$$\phi(t) = \epsilon^{-ct}\left(C\cos\frac{2\pi bt}{\lambda} + C'\sin\frac{2\pi bt}{\lambda}\right),$$

where $c = \frac{8\pi^2\mu}{3\lambda^2\rho_{\prime}}$, $b^2 = mA - \frac{16\pi^2\mu^2}{9\lambda^2\rho_{\prime}^2}$, $C$ and $C'$ being arbitrary constants. Taking the same expression with different arbitrary constants for $\psi(t)$, replacing products of sines and cosines by sums and differences, and combining the resulting sines and cosines two and two, we see that the resulting value of $u$ represents two series of waves propagated in opposite directions. Considering only those waves which are propagated in the positive direction of $x$, we have

$$u = C_1\epsilon^{-ct}\cos\left\{\frac{2\pi}{\lambda}(bt - x) + C_2\right\} \text{..............(21)}$$

We see then that the effect of the tangential force is to make the intensity of the sound diminish as the time increases, and to render the velocity of propagation less than what it would otherwise be. Both effects are greater for high, than for low notes; but the former depends on the first power of $\mu$, while the latter depends only on $\mu^2$. It appears from the experiments of M. Biot, made on empty water pipes in Paris, that the velocity of propagation of sound is sensibly the same whatever be its pitch. Hence it is necessary to suppose that for air $\frac{\mu^2}{\lambda^2\rho_{\prime}^2}$ is insensible compared with $A$ or $\frac{p_{\prime}}{\rho_{\prime}}$. I am not aware of any similar experiments made on water, but the ratio of $\left(\frac{\mu}{\lambda\rho_{\prime}}\right)^2$ to $A$ would probably be insensible for water also. The diminution of intensity as the time increases is, in the case of plane waves, due *entirely* to friction; but as we do not possess any means of measuring the intensity of sound the theory cannot be tested, nor the numerical value of $\mu$ determined, in this way.

The velocity of sound in air, deduced from the note given by a known tube, is sensibly less than that determined by direct observation. Poisson thought that this might be due to the retardation of the air by friction against the sides of the tube. But from the above investigation it seems unlikely that the effect produced by that cause would be sensible.

The equation (21) may be considered as expressing in all cases the effect of friction; for we may represent an arbitrary disturbance of the medium as the aggregate of series of plane waves propagated in all directions.

8. Let us now consider the motion of a mass of uniform inelastic fluid comprised between two cylinders having a common axis, the cylinders revolving uniformly about their axis, and the fluid being supposed to have attained its permanent state of motion. Let the axis of the cylinders be taken for that of $x$, and let $q$ be the actual velocity of any particle, so that $u = -q \sin\theta$, $v = q\cos\theta$, $w = 0$, $r$ and $\theta$ being polar co-ordinates in a plane parallel to $xy$.

Observing that $\dfrac{d^2f}{dx^2} + \dfrac{d^2f}{dy^2} = \dfrac{d^2f}{dr^2} + \dfrac{1}{r}\dfrac{df}{dr} + \dfrac{1}{r^2}\dfrac{d^2f}{d\theta^2}$, where $f$ is any function of $x$ and $y$, and that $\dfrac{dp}{d\theta} = 0$, we have from equations (13), supposing after differentiation that the axis of $x$ coincides with the radius vector of the point considered, and omitting the forces, and the part of the pressure due to them,

$$\frac{dp}{dr} - \rho\frac{q^2}{r} = 0,$$

$$\frac{d^2q}{dr^2} + \frac{1}{r}\frac{dq}{dr} - \frac{q}{r^2} = 0, \quad \dots\dots\dots\dots\dots\dots\dots (22)$$

and the equation of continuity is satisfied identically.

The integral of (22) is $$q = \frac{C}{r} + C'r.$$

If $a$ is the radius of the inner, and $b$ that of the outer cylinder, and if $q_1$, $q_2$ are the velocities of points close to these cylinders respectively, we must have $q = q_1$ when $r = a$, and $q = q_2$ when $r = b$, whence

$$q = \frac{1}{b^2 - a^2}\left\{(bq_1 - aq_2)\frac{ab}{r} + (bq_2 - aq_1)r\right\} \dots\dots\dots\dots\dots\dots (23)$$

If the fluid is infinitely extended, $b = \infty$, and

$$\frac{q}{q_1} = \frac{a}{r}.$$

These cases of motion were considered by Newton, (*Principia*, Lib. II. Prop. 51.) The hypothesis which I have made agrees in this case with his, but he arrives at the result that the velocity is constant, not, that it varies inversely as the distance. This arises from his having taken, as the condition of there being no acceleration or retardation of the motion of an annulus, that the force tending to turn it in one direction must be equal to that tending to turn it in the opposite, whereas the true condition is that the moment of the force tending to turn it one way must be equal to the moment of the force tending to turn it the other. Of course, making this alteration, it is easy to arrive at the above result by Newton's reasoning. The error just mentioned vitiates the result of Prop. 52. It may be shown from the general equations that in this case a permanent motion in annuli is impossible, and that, whatever may be the law of friction between the solid sphere and the fluid. Hence it appears that it is necessary to suppose that the particles move in planes passing through the axis of rotation, while they at the same time move round it. In fact, it is easy to see that from the excess of centrifugal force in the neighbourhood of the equator of the revolving sphere the particles in that part will recede from the sphere, and approach it again in the neighbourhood of the poles, and this circulating motion will be combined with a motion about the axis. If however we leave the centrifugal force out of consideration, as Newton has done, the motion in annuli becomes possible, but the solution is different from Newton's, as might have been expected.

The case of motion considered in this article may perhaps admit of being compared with experiment, without knowing the conditions which must be satisfied at the surface of a solid. A hollow, and a solid cylinder might be so mounted as to admit of being turned with different uniform angular velocities round their common axis, which is supposed to be vertical. If both cylinders are turned, they ought to be turned in opposite directions, if only one, it ought to be the outer one; for if the inner were made to revolve too fast, the fluid near it would have a tendency to fly outwards in consequence of the centrifugal force, and eddies would be produced. As long as the angular velocities are not great, so that the surface of the liquid is very nearly plane, it is not of much importance that the fluid is there terminated; for the conditions which must be satisfied at a free surface are satisfied for any section of the fluid made by a horizontal plane, so long as the motion about that section is supposed to be the same as it would be if the cylinders were infinite. The principal difficulty would probably be to measure accurately the time of revolution, and distance from the axis, of the different annuli. This would probably be best done by observing motes in the fluid. It might be possible also to discover in this way the conditions to be satisfied at the surface of the cylinders; or at least a law might be suggested, which could be afterwards compared more accurately with experiment by means of the discharge of pipes and canals.

If the rotations of the cylinders are in opposite directions, there will be a certain distance from the axis at which the fluid will not revolve at all. Writing $-q_1$ for $q_1$ in equation (23), we have for this distance $\sqrt{\dfrac{ab\,(bq_1 + aq_2)}{bq_2 + aq_1}}$.

9. Although the discharge of a liquid through a long straight pipe or canal, under given circumstances, cannot be calculated without knowing the conditions to be satisfied at the surface of contact of the fluid and solid, it may be well to go a certain way towards the solution.

Let the axis of $z$ be parallel to the generating lines of the pipe or canal, and inclined at an angle $\alpha$ to the horizon; let the plane $yz$ be vertical, and let $y$ and $z$ be measured downwards. The motion being uniform, we shall have $u = 0$, $v = 0$, $w = f(x, y)$, and we have from equations (13)

$$\frac{dp}{dx} = 0, \quad \frac{dp}{dy} = g\rho \cos \alpha, \quad \frac{dp}{dz} = g\rho \sin \alpha + \mu \left(\frac{d^2 w}{dx^2} + \frac{d^2 w}{dy^2}\right).$$

In the case of a canal $\dfrac{dp}{dz} = 0$; and the calculation of the motion in a pipe may always be reduced to that of the motion in the same pipe when $\dfrac{dp}{dz}$ is supposed to be zero, as may be shown by reasoning similar to Dubuat's. Moreover the motion in a canal is a particular case of the motion in a pipe. For consider a pipe for which $\dfrac{dp}{dz} = 0$, and which is divided symmetrically by the plane $xz$. From the symmetry of the motion, it is clear that we must have $\dfrac{dw}{dy} = 0$ when $z = 0$;

but this is precisely the condition which would have to be satisfied if the fluid had a free surface coinciding with the plane $xz$; hence we may suppose the upper half of the fluid removed, without affecting the motion of the rest, and thus we pass to the case of a canal. Hence it is the same thing to determine the motion in a canal, as to determine that in the pipe formed by completing the canal symmetrically with respect to the surface of the fluid.

We have then, to determine the motion, the equation

$$\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{g\rho \sin \alpha}{\mu} = 0.$$

In the case of a rectangular pipe, it would not be difficult to express the value of $w$ at any point in terms of its values at the several points of the perimeter of a section of the pipe. In the case of a cylindrical pipe the solution is extremely easy: for if we take the axis of the pipe for that of $z$, and take polar co-ordinates $r$, $\theta$ in a plane parallel to $xy$, and observe that $\frac{dw}{d\theta} = 0$, since the motion is supposed to be symmetrical with respect to the axis, the above equation becomes

$$\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr} + \frac{g\rho \sin \alpha}{\mu} = 0.$$

Let $a$ be the radius of the pipe, and $U$ the velocity of the fluid close to the surface; then, integrating the above equation, and determining the abitrary constants by the conditions that $w$ shall be finite when $r = 0$, and $w = U$ when $r = a$, we have

$$w = \frac{g\rho \sin \alpha}{4\mu}(a^2 - r^2) + U.$$

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## SECTION II.

*Objections to Lagrange's proof of the theorem that if* udx + vdy + wdz *is an exact differential at any one instant it is always so, the pressure being supposed equal in all directions. Principles of M. Cauchy's proof. A new proof of the theorem. A physical interpretation of the circumstance of the above expression being an exact differential.*

10. THE proof of this theorem given by Lagrange depends on the legitimacy of supposing $u$, $v$ and $w$ capable of expansion according to positive integral powers of $t$, for a sufficiently small finite value of $t$. It is clear that the expansion cannot contain negative powers of $t$, since $u$, $v$ and $w$ are supposed to be finite when $t = 0$; but it may be objected to Lagrange's proof that there are functions of $t$ of which the expansion contains fractional powers of $t$, and that we do not know but that $u$, $v$ and $w$ may be such functions. This objection has been considered by Mr. Power*, who has shown that the theorem is true if we suppose $u$, $v$ and $w$ capable of expansion according to any powers of $t$. Still the proof remains unsatisfactory, in fact inconclusive, for these are functions of $t$, (for instance $e^{-t^{\frac{1}{2}}}$, $t \log t$,) which do not admit of expansion according

* *Cambridge Philosophical Transactions,* Vol. VII. Part 3.