

Compressible Fluids

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Chemical engineering is mostly concerned with incompressible flows in pipes, reactors, mixers, and other process equipment. Gases may be modeled as incompressible fluids in both microscopic and macroscopic calculations as long as the pressure changes are less than about 20% of the mean pressure (Geankoplis, Denn). The friction-factor/Reynolds-number correlation for incompressible fluids is found to apply to compressible fluids in this regime of pressure variation (Perry and Chilton, Denn). Compressible flow is important in selected application, however, including high-speed flow of gasses in pipes, through nozzles, in turbines, and especially in relief valves. We include here a brief discussion of issues related to the flow of compressible fluids; for more information the reader is encouraged to consult the literature.

A compressible fluid is one in which the fluid density changes when it is subjected to high pressure-gradients. For gasses, changes in density are accompanied by changes in temperature, and this complicates considerably the analysis of compressible flow.

The key difference between compressible and incompressible flow is the way that forces are transmitted through the fluid. Consider the flow of water in a straw. When a thirsty child applies suction to one end of a straw submerged in water, the water moves - both the water close to her mouth moves and the water at the far end moves towards the lower-pressure area created in the mouth. Likewise, in a long, completely filled piping system, if a pump is turned on at one end, the water will immediately begin to flow out of the other end of the pipe.

In a compressible fluid, the imposition of a force at one end of a system does not result in an immediate flow throughout the system. Instead, the fluid compresses near where the force was applied; that is, its density increases locally in response to the force. The compressed fluid expands against neighboring fluid particles causing the neighboring fluid itself to compress and setting in motion a wave pulse that travels throughout the system. The pulse of higher density fluid takes some time to travel from the source of the disturbance down through the pipe to the far end of the system.

The wave-pulse mechanism of momentum transfer is a different kind of molecular momentum transfer that needs to be included in the momentum balances. The total stress tensor $\underline{\underline{\Pi}} = p\underline{\underline{I}} + \underline{\underline{\tau}}$ is the quantity that contains a mathematical expression for all the molecular processes that affect stress generation in flow, and it is here that new terms are needed for the microscopic equations of change to properly model compressible flow.

For Newtonian fluids, we have seen that the extra stress tensor $\underline{\underline{\tau}}$ is given by a simple expression that works for all flows:

$$\boxed{\underline{\underline{\tau}} = -\mu \underline{\underline{\dot{\gamma}}}} \qquad \text{Newtonian Constitutive equation for incompressible fluids} \tag{3.185}$$

For compressible fluids, the relationship between stress and deformation is modified to be (Bird et al.)

$$\boxed{\underline{\underline{\tau}} = -\mu \underline{\underline{\dot{\gamma}}} + \left(\frac{2}{3}\mu - \kappa\right)(\nabla \cdot \underline{v})\, I} \qquad \text{Newtonian Constitutive equation for } \textit{compressible} \text{ fluids} \tag{3.186}$$

The parameter κ is the dilatational or bulk viscosity, a coefficient that expresses viscous momentum transport that takes place when density changes take place; κ is zero for ideal, monatomic gases (Bird et al., Denn). The bulk viscosity is only important when very large expansions take place, and it can usually be neglected (Denn). In microscopic momentum balance calculations on compressible fluids, equation 3.186 is used as the stress constitutive equation; the solution methods for velocity and stress profiles are unchanged. Note that for incompressible fluids the continuity equation becomes $\nabla \cdot \underline{v} = 0$, and equation 3.186 reduces to the more familiar expression $\underline{\underline{\tau}} = -\mu \underline{\underline{\dot{\gamma}}}$ for incompressible fluids.

In macroscopic calculations with the mechanical energy balance (MEB), we must make some modifications to allow the MEB to be applicable to compressible fluids. The derivation assumes ideal gas and begins with the mechanical energy balance written over a differential length of straight pipe (see attached derivation from Geankoplis). The final result for pressure/flow rate variation in isothermal flow is given below

$$\boxed{p_2 - p_1 = \frac{4fLG^2}{2D\rho_{ave}} + \frac{G^2}{\rho_{ave}} \ln\left(\frac{p_1}{p_2}\right)} \qquad \text{Isothermal Compressible flow} \tag{3.187}$$

where L is pipe length, and $G = v\rho$.

One of the most striking aspects of this result is that it predicts that there is a maximum velocity at high pressure drops. Taking the derivative of equation 3.187 with respect to p_2 and setting it to zero we calculate that

$$v_{max} = \sqrt{\frac{p_2}{\rho_2}} \tag{3.188}$$

To understand this maximum flow velocity and its implications, we must return again to a discussion of the fundamental mechanism of momentum transfer in compressible fluids.

In steady flows of incompressible fluids, the instantaneous transmission of forces through the fluid allows flow around an obstacle, for example, to rearrange to allow for the smooth passage of the fluid around the obstacle. In incompressible flows, forces are applied and the fluid responds directly to the forces by moving, displacing neighbor particles, and thus establishing the appropriate flow field. In incompressible flows there is no time lag and no difficulty transmitting stress information throughout the flow field.

In compressible fluids, the transmission of forces is not instantaneous, but rather it occurs through the motion of transverse pressure waves. Because the pressure waves take a finite amount of time to travel from one location to another, there are interesting effects that occur when the speed of the flow is close to the speed of propagation of these pressure waves. The propagation velocity of a pressure wave in a compressible medium is called the velocity of sound.

If the pressure variation is not too large the speed of sound in a medium is given by (Tipler)

$$v = \sqrt{\frac{B}{\rho_o}} \tag{3.189}$$

where B is the bulk modulus, defined as the ratio of the change in pressure to the fractional decrease in volume

$$B = \frac{\Delta p}{-\Delta V/V} \tag{3.190}$$

and ρ_o is density. The bulk modulus relates the change in volume to changes in pressure. This is information that is found in the equation of state for a material. For example, for an ideal gas, pressure and volume are related by the ideal gas law.

$$pV = nRT \tag{3.191}$$

If we differentiate the ideal gas law and assume temperature is constant, we obtain

$$pdV + Vdp = 0 \tag{3.192}$$

which may be rearranged to give the bulk modulus for an ideal gas under isothermal conditions:

$$\boxed{B = \frac{dp}{-dV/V} = p} \quad \text{Bulk modulus of an ideal gas at constant T} \tag{3.193}$$

The speed of sound therefore is given by

$$v = \sqrt{\frac{B}{\rho_o}} \tag{3.194}$$

$$= \sqrt{\frac{p}{\rho_o}} = \sqrt{\frac{RT}{M}} \tag{3.195}$$

where we have substituted for p the expression given by the ideal gas law $p = nRT/V = \rho_o RT/M$, and M is the molecular weight of the medium.

This expression for the speed of sound (equation 3.195) is found to be 20% too small when compared with experimental results (Tipler), and the reason is that the passage of sounds through a medium does not occur isothermally. The compressions and decompressions that take place tend to change the temperature of the gas locally (consider the relationship between volume and temperature in the ideal gas law), and because these temperature changes occur quite rapidly, there is no time for much heat transfer to take place. Rather than assume isothermal passage, it is better to assume that the movement of sounds is adiabatic, that is, that no heat transfer occurs.

Application of the first law of thermodynamics ($dQ = dU + dW$), under quasistatic ($dW = pdV$), adiabatic ($dQ = 0$) conditions, results in an expression that relates pressure and volume when a gas undergoes volume changes under adiabatic conditions (Tipler).

$$pV^{\gamma} = \text{constant} \tag{3.196}$$

where $\gamma \equiv C_p/C_v$, the ratio of the heat capacity at constant pressure to the heat capacity at constant volume. We can derive B for the quasi-static, adiabatic case by taking the derivative of equation 3.196 with respect to pressure and rearranging.

$$\frac{d(pV^{\gamma})}{dp} = 0 \tag{3.197}$$

$$p\gamma V^{\gamma-1}\frac{dV}{dp} + V^{\gamma} = 0 \tag{3.198}$$

$$p\gamma dV + V dp = 0 \tag{3.199}$$

$$B = \frac{dp}{-dV/V} = \gamma p \tag{3.200}$$

Substituting this into equation 3.189 for the speed of sound in terms of B yields a more correct expression the speed of sound in an ideal gas.

$$v = \sqrt{\frac{B}{\rho_o}} \tag{3.201}$$

$$= \sqrt{\frac{\gamma p}{\rho_o}} \tag{3.202}$$

$$\boxed{v = \sqrt{\frac{\gamma RT}{M}}} \quad \text{Speed of Sound of an ideal gas (adiabatic)} \tag{3.203}$$

This result makes predictions that are close to experimental observations.

Returning now to the maximum velocity calculated in equation 3.188 for compressible flows we see that the maximum fluid velocity is just the isothermal speed of sound in the medium. Note that v_{max} for adiabatic flow is the adiabatic speed of sound (Geankoplis). In other words, the maximum speed attainable in a pressure-driven compressible flow is the speed at which the pressure waves themselves travel. It is not possible to exceed that speed by adjusting the pressure drop. This has extraordinarily important implications in the design and operation of pressure relief valves, since even a very high pressure drop can only produce relief at the speed of sound, no faster. Proper design of relief valves means including the appropriate number and size of valves so that accidental overpressures can be relieved safely within these constraints.

References

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