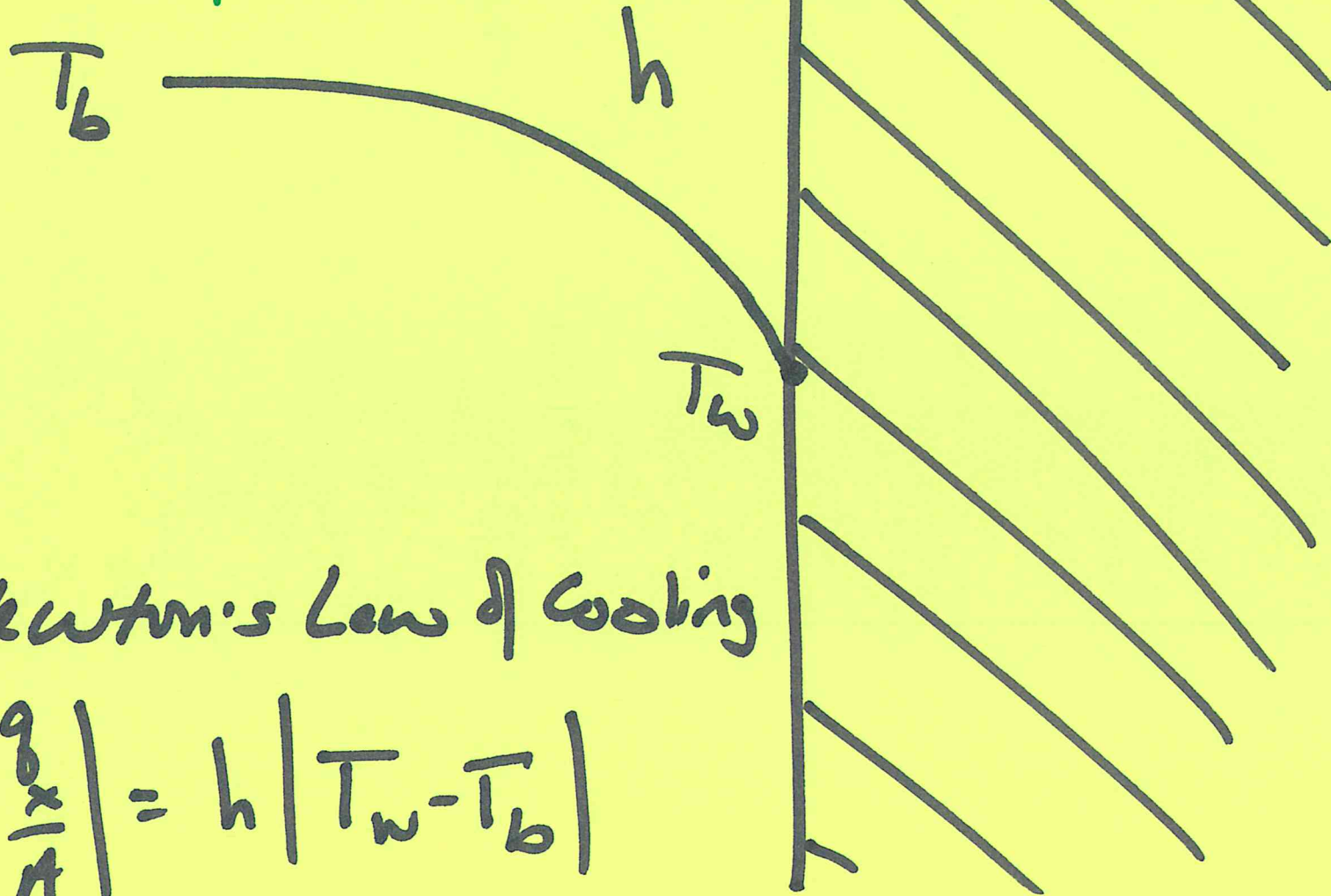


NATURAL CONVECTION (FREE) Heat Xfer (review of forced conv.)

18 NOV 2019

(1)

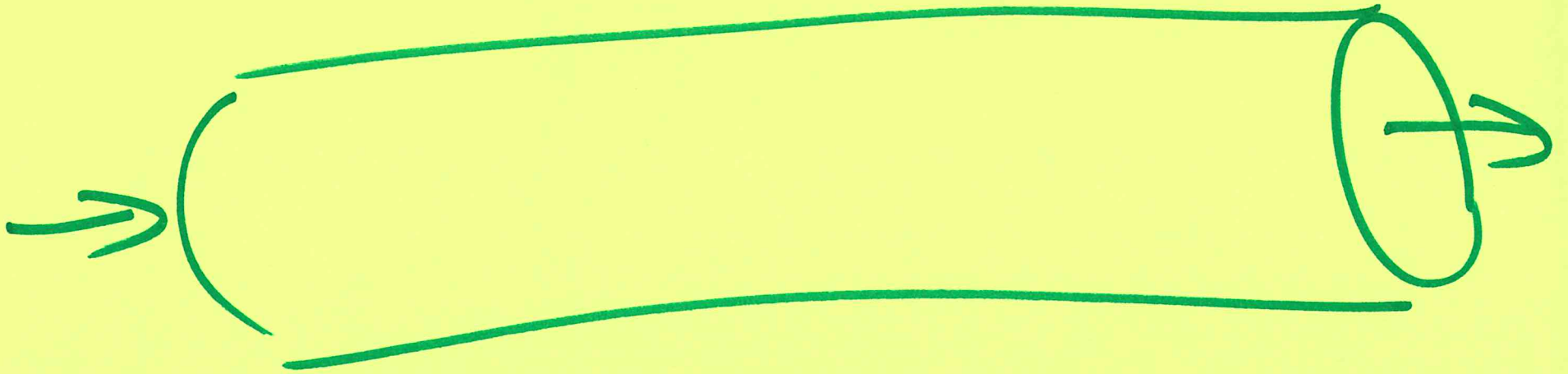


Newton's Law of Cooling

$$\left| \frac{q}{A} \right| = h |T_w - T_b|$$

FORCED CONVECTION: (review)

②



$$\rho \left(\frac{d\mathbf{v}}{dt} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g}$$

$$\mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 2u \end{pmatrix}$$

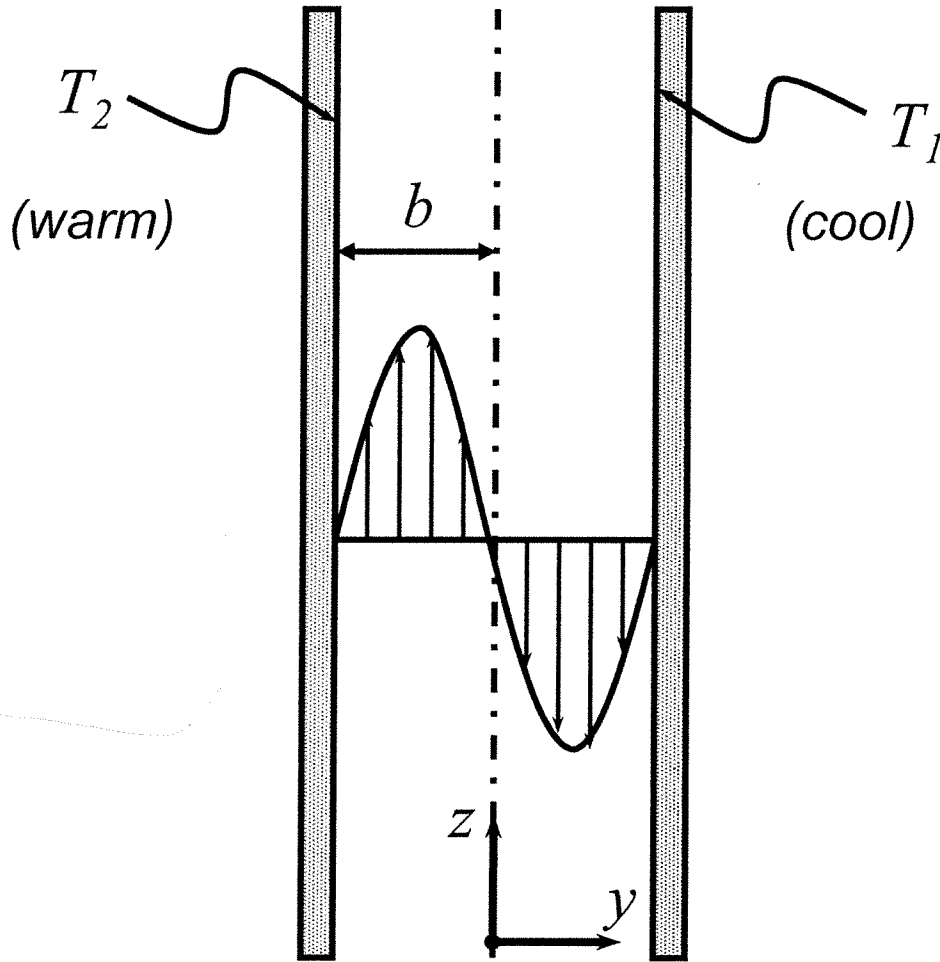
\Rightarrow laminar flow

Now, Natural Convection:

Complex Heat Transfer – Dimensional Analysis—Free Convection

2.5,

Example: Free convection between long parallel plates or heat transfer through double-pane glass windows



$$T_2 > T_1$$

- assumptions:**
- long, wide slit
 - steady state
 - no source terms
 - viscosity constant
 - density varies with T

Calculate: T, v profiles

NAT'L CONV B-TWIN PLATES (3)

The Equation of Continuity and the Equation of Motion in Cartesian, cylindrical, and spherical coordinates

$v_x = 0$
 $v_y = 0$

CM3110 Fall 2011 Faith A. Morrison

Continuity Equation, Cartesian coordinates

$$\frac{\partial \rho}{\partial t} + \left(v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} \right) + \rho \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) = 0$$

Continuity Equation, cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial(\rho r v_r)}{\partial r} + \frac{1}{r} \frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

Continuity Equation, spherical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial(\rho r^2 v_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(\rho v_\theta \sin \theta)}{\partial \theta} + \frac{1}{\sin \theta} \frac{\partial(\rho v_\phi)}{\partial \phi} = 0$$

Equation of Motion for an incompressible fluid, 3 components in Cartesian coordinates

$$\begin{aligned} \rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) &= -\frac{\partial P}{\partial x} + \left(\frac{\partial \bar{\tau}_{xx}}{\partial x} + \frac{\partial \bar{\tau}_{yx}}{\partial y} + \frac{\partial \bar{\tau}_{zx}}{\partial z} \right) + \rho g_x \\ \rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) &= -\frac{\partial P}{\partial y} + \left(\frac{\partial \bar{\tau}_{xy}}{\partial x} + \frac{\partial \bar{\tau}_{yy}}{\partial y} + \frac{\partial \bar{\tau}_{zy}}{\partial z} \right) + \rho g_y \\ \rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) &= -\frac{\partial P}{\partial z} + \left(\frac{\partial \bar{\tau}_{xz}}{\partial x} + \frac{\partial \bar{\tau}_{yz}}{\partial y} + \frac{\partial \bar{\tau}_{zz}}{\partial z} \right) + \rho g_z \end{aligned}$$

Equation of Motion for an incompressible fluid, 3 components in cylindrical coordinates

$$\begin{aligned} \rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) &= -\frac{\partial P}{\partial r} + \left(\frac{1}{r} \frac{\partial(r \bar{\tau}_{rr})}{\partial r} + \frac{1}{r} \frac{\partial \bar{\tau}_{\theta r}}{\partial \theta} - \frac{\bar{\tau}_{\theta\theta}}{r} + \frac{\partial \bar{\tau}_{zr}}{\partial z} \right) + \rho g_r \\ \rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta v_r}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) &= -\frac{1}{r} \frac{\partial P}{\partial \theta} + \left(\frac{1}{r^2} \frac{\partial(r^2 \bar{\tau}_{r\theta})}{\partial r} + \frac{1}{r} \frac{\partial \bar{\tau}_{\theta\theta}}{\partial \theta} + \frac{\partial \bar{\tau}_{z\theta}}{\partial z} + \frac{\bar{\tau}_{\theta r} - \bar{\tau}_{r\theta}}{r} \right) + \rho g_\theta \\ \rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) &= -\frac{\partial P}{\partial z} + \left(\frac{1}{r} \frac{\partial(r \bar{\tau}_{rz})}{\partial r} + \frac{1}{r} \frac{\partial \bar{\tau}_{\theta z}}{\partial \theta} + \frac{\partial \bar{\tau}_{zz}}{\partial z} \right) + \rho g_z \end{aligned}$$

Equation of Motion for an incompressible fluid, 3 components in spherical coordinates

$$\begin{aligned} &\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2}{r} + \frac{v_\phi^2}{r} \right) \\ &= -\frac{\partial P}{\partial r} + \left(\frac{1}{r^2} \frac{\partial(r^2 \bar{\tau}_{rr})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial \bar{\tau}_{\theta r}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \bar{\tau}_{\phi r}}{\partial \phi} - \frac{\bar{\tau}_{\theta\theta}}{r} + \frac{\bar{\tau}_{\phi\phi}}{r} \right) + \rho g_r \\ &\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} + \frac{v_\theta^2 \cot \theta}{r} \right) \\ &= -\frac{1}{r} \frac{\partial P}{\partial \theta} + \left(\frac{1}{r^3} \frac{\partial(r^3 \bar{\tau}_{r\theta})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(\bar{\tau}_{\theta\theta} \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \bar{\tau}_{\phi\theta}}{\partial \phi} + \frac{\bar{\tau}_{\theta r} - \bar{\tau}_{r\theta}}{r} - \frac{\bar{\tau}_{\phi\theta} \cot \theta}{r} \right) + \rho g_\theta \\ &\rho \left(\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{v_\phi v_\theta \cot \theta}{r} \right) \\ &= -\frac{1}{r \sin \theta} \frac{\partial P}{\partial \phi} + \left(\frac{1}{r^3} \frac{\partial(r^3 \bar{\tau}_{r\phi})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(\bar{\tau}_{\theta\phi} \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \bar{\tau}_{\phi\phi}}{\partial \phi} + \frac{\bar{\tau}_{\phi r} - \bar{\tau}_{r\phi}}{r} + \frac{\bar{\tau}_{\phi\theta} \cot \theta}{r} \right) + \rho g_\phi \end{aligned}$$

4

$$u_x = u_y = 0$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation) 3 components in Cartesian coordinates

Stokes

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial P}{\partial x} + \mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) + \rho g_x$$

$$\rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = -\frac{\partial P}{\partial y} + \mu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right) + \rho g_y$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial P}{\partial z} + \mu \left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho g_z$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in cylindrical coordinates

continuity

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} \right) = -\frac{\partial P}{\partial r} + \mu \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial (r v_r)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2} \right) + \rho g_r$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} \right) = -\frac{1}{r} \frac{\partial P}{\partial \theta} + \mu \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial (r v_\theta)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2} \right) + \rho g_\theta$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial P}{\partial z} + \mu \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho g_z$$

dp = rho g

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in spherical coordinates

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} + \frac{v_r v_\theta}{r} + \frac{v_r v_\phi}{r} - \frac{v_\theta^2 + v_\phi^2}{r} \right) = -\frac{\partial P}{\partial r} + \mu \left(\frac{\partial}{\partial r} \left(\frac{1}{r^2} \frac{\partial (r^2 v_r)}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial v_r}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_r}{\partial \phi^2} - \frac{2}{r^2} \frac{\partial}{\sin \theta} \frac{\partial v_\theta}{\partial \theta} - \frac{2}{r^2} \frac{\partial v_\phi}{\sin \theta} \frac{\partial}{\partial \phi} \right) + \rho g_r$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} + \frac{v_\theta v_\phi}{r} - \frac{v_r v_\theta}{r} - \frac{v_\theta^2 \cot \theta}{r} \right) = -\frac{1}{r} \frac{\partial P}{\partial \theta} + \mu \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(\frac{\partial (r^2 v_\theta)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial v_\theta}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_\theta}{\partial \phi^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} - \frac{2 \cot \theta}{r^2 \sin \theta} \frac{\partial v_\phi}{\partial \phi} \right) + \rho g_\theta$$

$$\rho \left(\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{v_\theta v_\phi}{r} \cot \theta \right) = -\frac{1}{r \sin \theta} \frac{\partial P}{\partial \phi} + \mu \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(\frac{\partial (r^2 v_\phi)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial v_\phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_\phi}{\partial \phi^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} - \frac{2 \cot \theta}{r^2 \sin \theta} \frac{\partial v_\theta}{\partial \phi} \right) + \rho g_\phi$$

P(r) = P(r, theta, phi) = rho g

Note: the r -component of the Navier-Stokes equation in spherical coordinates may be simplified by adding $0 = \frac{2}{r} \nabla \cdot \underline{u}$ to the component shown above. This term is zero due to the continuity equation (mass conservation). See Bird et. al.

References:

1. R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena*, 2nd edition, Wiley: NY, 2002.
2. R. B. Bird, R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric Fluids: Volume 1 Fluid Mechanics*, Wiley: NY, 1987.

x-component:

$$\boxed{\frac{\partial p}{\partial x} = 0}$$

y-component:

$$\boxed{\frac{\partial p}{\partial y} = 0}$$

z-component:

$$0 = - \underbrace{\frac{dp}{dz}}_{\bar{p}g} + \mu \frac{d^2 v_z}{dz^2} + \underbrace{\rho g}_{\bar{p} - \bar{p}\beta(T - \bar{T})} (-1)$$

$$0 = \cancel{\bar{p}g} + \mu \frac{d^2 v_z}{dz^2} - g \left(\cancel{\bar{p}} - \bar{p}\beta \left(\frac{T_1 - T_2}{z_b} \right) y \right)$$

(5)

Nat'l Conv. Conf.:

The Equation of Energy for systems with constant k

Microscopic energy balance, constant thermal conductivity; Gibbs notation

$$\nabla \cdot \nabla T = \rho \hat{c}_p \left(\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) = k \nabla^2 T + S_e$$

Microscopic energy balance, constant thermal conductivity; Cartesian coordinates

$$\rho \hat{c}_p \left(\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} \right) = k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + S_e$$

Microscopic energy balance, constant thermal conductivity; cylindrical coordinates

$$\rho \hat{c}_p \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} \right) = k \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right) + S_e$$

no rxn
no uc
no cmt

Microscopic energy balance, constant thermal conductivity; spherical coordinates

$$\rho \hat{c}_p \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial T}{\partial \phi} \right) = k \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial \theta^2} \right) + S_e$$

$$\frac{dT}{dr} = C_1$$

$$T = C_1 r + C_2$$

Reference: F. A. Morrison, "Web Appendix to An Introduction to Fluid Mechanics," Cambridge University Press, New York, 2013 (pages.mtu.edu/~fmorriso/cm310/FMWebAppendixDMicroEBalanceMorrison.pdf). This worksheet is on the web at pages.mtu.edu/~fmorriso/cm310/energy.pdf.

$$\frac{d^2 v_z}{dy^2} = (-g \rho \beta \left(\frac{T_1 - T_2}{2b} \right) y) \frac{L}{\mu} \quad (7)$$

$$\frac{d}{dy} \left(\frac{dv_z}{dy} \right) = \Phi$$

$$\frac{d\Phi}{dy} = \left(\frac{g \rho \beta (T_1 - T_2)}{\mu \frac{2b}{L}} \right) y$$

$$\frac{dv_z}{dy} = \Phi = \left(\downarrow \right) \frac{y^2}{2} + C_1$$

$$v_z = \left(\downarrow \right) \frac{1}{2} \frac{y^3}{3} + C_1 y + C_2$$

BC:

$$y = b$$

$$T = T_1$$

$$y = -b$$

$$T = T_2$$

SOLVE

(see slides)