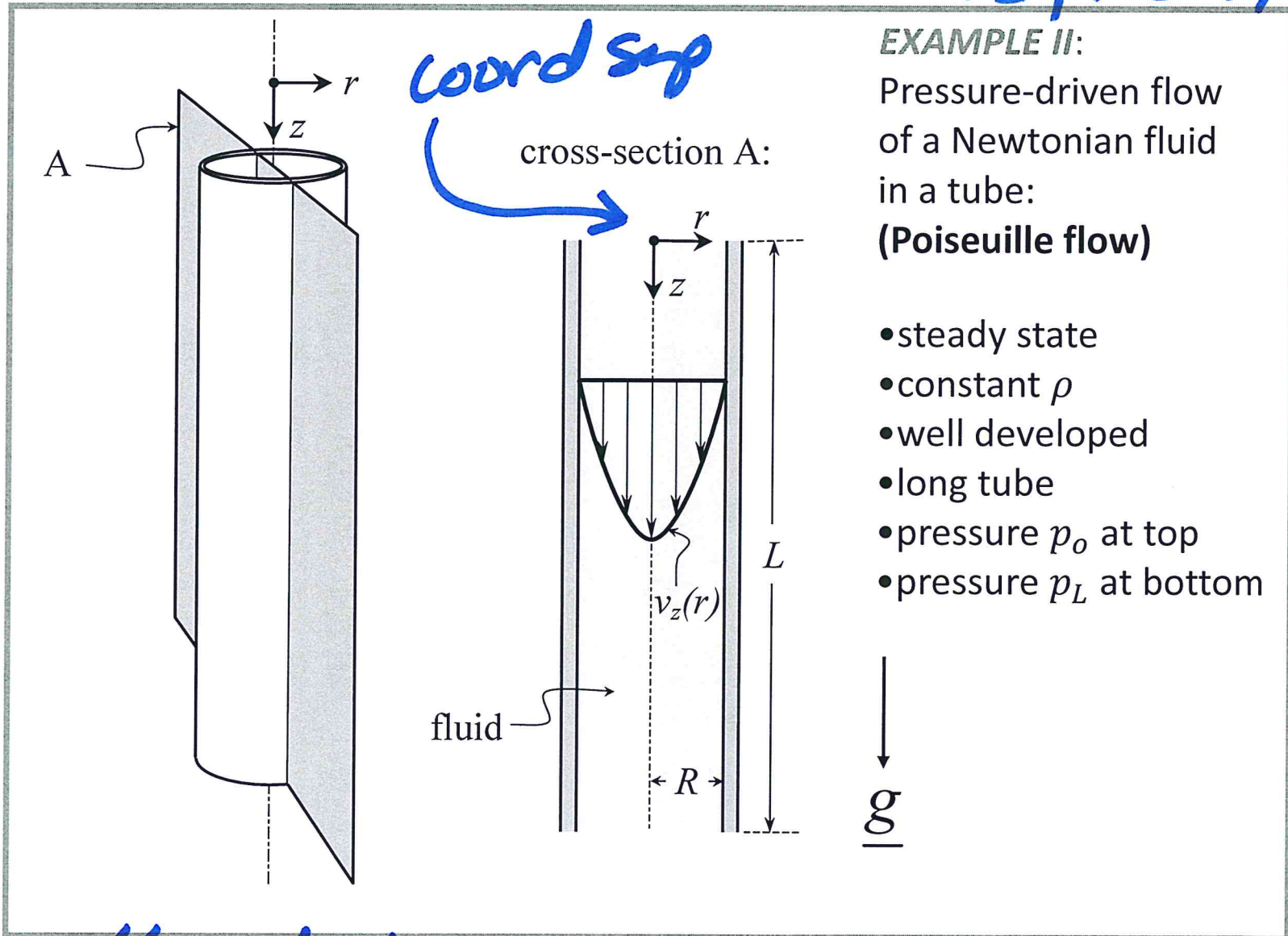


Pressure-driven flow in a tube

①

24 Sept 2019



EXAMPLE II:

Pressure-driven flow of a Newtonian fluid in a tube:

(Poiseuille flow)

- steady state
- constant ρ
- well developed
- long tube
- pressure p_0 at top
- pressure p_L at bottom

How to begin?

① continuity, ② momentum →

3

The Equation of Continuity and the Equation of Motion in Cartesian, cylindrical, and spherical coordinates

CM3110 Fall 2011 Faith A. Morrison

Continuity Equation, Cartesian coordinates

$$\frac{\partial \rho}{\partial t} + \left(v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} \right) + \rho \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) = 0$$

Continuity Equation, cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial(\rho r v_r)}{\partial r} + \frac{1}{r} \frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

Continuity Equation, spherical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial(\rho r^2 v_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(\rho v_\theta \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial(\rho v_\phi \sin \theta)}{\partial \phi} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

Equation of Motion for an incompressible fluid, 3 components in Cartesian coordinates

$$\begin{aligned} \rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) &= -\frac{\partial P}{\partial x} + \left(\frac{\partial \bar{\tau}_{xx}}{\partial x} + \frac{\partial \bar{\tau}_{yx}}{\partial y} + \frac{\partial \bar{\tau}_{zx}}{\partial z} \right) + \rho g_x \\ \rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) &= -\frac{\partial P}{\partial y} + \left(\frac{\partial \bar{\tau}_{xy}}{\partial x} + \frac{\partial \bar{\tau}_{yy}}{\partial y} + \frac{\partial \bar{\tau}_{zy}}{\partial z} \right) + \rho g_y \\ \rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) &= -\frac{\partial P}{\partial z} + \left(\frac{\partial \bar{\tau}_{xz}}{\partial x} + \frac{\partial \bar{\tau}_{yz}}{\partial y} + \frac{\partial \bar{\tau}_{zz}}{\partial z} \right) + \rho g_z \end{aligned}$$

Equation of Motion for an incompressible fluid, 3 components in cylindrical coordinates

$$\begin{aligned} \rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) &= -\frac{\partial P}{\partial r} + \left(\frac{1}{r} \frac{\partial(r \bar{\tau}_{rr})}{\partial r} + \frac{1}{r} \frac{\partial \bar{\tau}_{\theta r}}{\partial \theta} - \frac{\bar{\tau}_{\theta\theta}}{r} + \frac{\partial \bar{\tau}_{zr}}{\partial z} \right) + \rho g_r \\ \rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta v_r}{r} + \frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} \right) &= -\frac{1}{r} \frac{\partial P}{\partial \theta} + \left(\frac{1}{r^2} \frac{\partial(r^2 \bar{\tau}_{r\theta})}{\partial r} + \frac{1}{r} \frac{\partial \bar{\tau}_{\theta\theta}}{\partial \theta} + \frac{\partial \bar{\tau}_{z\theta}}{\partial z} - \frac{\bar{\tau}_{\theta r} - \bar{\tau}_{r\theta}}{r} \right) + \rho g_\theta \\ \rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) &= -\frac{\partial P}{\partial z} + \left(\frac{1}{r} \frac{\partial(r \bar{\tau}_{rz})}{\partial r} + \frac{1}{r} \frac{\partial \bar{\tau}_{\theta z}}{\partial \theta} + \frac{\partial \bar{\tau}_{zz}}{\partial z} \right) + \rho g_z \end{aligned}$$

Equation of Motion for an incompressible fluid, 3 components in spherical coordinates

$$\begin{aligned} &\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2}{r} + \frac{v_\phi^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) \\ &= -\frac{\partial P}{\partial r} + \left(\frac{1}{r^2} \frac{\partial(r^2 \bar{\tau}_{rr})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(\bar{\tau}_{\theta r} \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \bar{\tau}_{\phi r}}{\partial \phi} - \frac{\bar{\tau}_{\theta r} + \bar{\tau}_{r\theta}}{r} + \frac{\partial \bar{\tau}_{zr}}{\partial z} \right) + \rho g_r \\ &= -\frac{1}{r} \frac{\partial P}{\partial \theta} + \left(\frac{1}{r^3} \frac{\partial(r^3 \bar{\tau}_{r\theta})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(\bar{\tau}_{\theta\theta} \sin \theta)}{\partial \theta} + \frac{v_r}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial r} + \frac{v_\theta v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} - \frac{v_\phi^2 \cot \theta}{r} + \frac{\partial \bar{\tau}_{z\theta}}{\partial z} + \frac{\bar{\tau}_{\theta r} \bar{\tau}_{r\theta} - \bar{\tau}_{\phi\theta} \cot \theta}{r} \right) + \rho g_\theta \\ &= -\frac{1}{r \sin \theta} \frac{\partial P}{\partial \phi} + \left(\frac{1}{r^3} \frac{\partial(r^3 \bar{\tau}_{r\phi})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(\bar{\tau}_{\theta\phi} \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \bar{\tau}_{\phi r}}{\partial \phi} + \frac{\bar{\tau}_{\theta r} - \bar{\tau}_{r\theta}}{r} + \frac{\bar{\tau}_{\phi\theta} \cot \theta}{r} \right) + \rho g_\phi \end{aligned}$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation) 3 components in Cartesian coordinates

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial P}{\partial x} + \mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) + \rho g_x$$

$$\rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = -\frac{\partial P}{\partial y} + \mu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right) + \rho g_y$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial P}{\partial z} + \mu \left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho g_z$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in cylindrical coordinates

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} + \frac{\partial v_r}{\partial r} \right) = -\frac{\partial P}{\partial r} + \mu \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial (r v_r)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2} \right) + \rho g_r$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} \right) = -\frac{1}{r} \frac{\partial P}{\partial \theta} + \mu \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial (r v_\theta)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2} \right) + \rho g_\theta$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial P}{\partial z} + \mu \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho g_z$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in spherical coordinates

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2}{r} + \frac{v_\phi^2}{r \sin^2 \theta} \right) = -\frac{\partial P}{\partial r} + \mu \left(\frac{\partial}{\partial r} \left(\frac{1}{r^2} \frac{\partial (r^2 v_r)}{\partial r} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial v_r}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_r}{\partial \phi^2} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial v_\theta}{\partial \theta} - \frac{2}{r^2 \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \rho g_r$$

$v_r = 0$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} + \frac{v_\phi v_\theta}{r} \cot \theta \right) = -\frac{1}{r} \frac{\partial P}{\partial \theta} + \mu \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial v_\theta}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial v_\theta}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_\theta}{\partial \phi^2} \right) + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} - \frac{2 \cot \theta}{r^2 \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \rho g_\theta$$

$v_\theta = 0$

$$\rho \left(\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{v_\phi v_\theta \cot \theta}{r} \right) = -\frac{1}{r \sin \theta} \frac{\partial P}{\partial \phi} + \mu \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial v_\phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial v_\phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_\phi}{\partial \phi^2} \right) + \frac{2}{r^2 \sin \theta} \frac{\partial v_r}{\partial \theta} - \frac{2 \cot \theta}{r^2 \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \rho g_\phi$$

Note: the r -component of the Navier-Stokes equation in spherical coordinates may be simplified by adding $0 = \frac{2}{r} \nabla \cdot \underline{v}$ to the component shown above. This term is zero due to the continuity equation (mass conservation). See Bird et. al.

References:

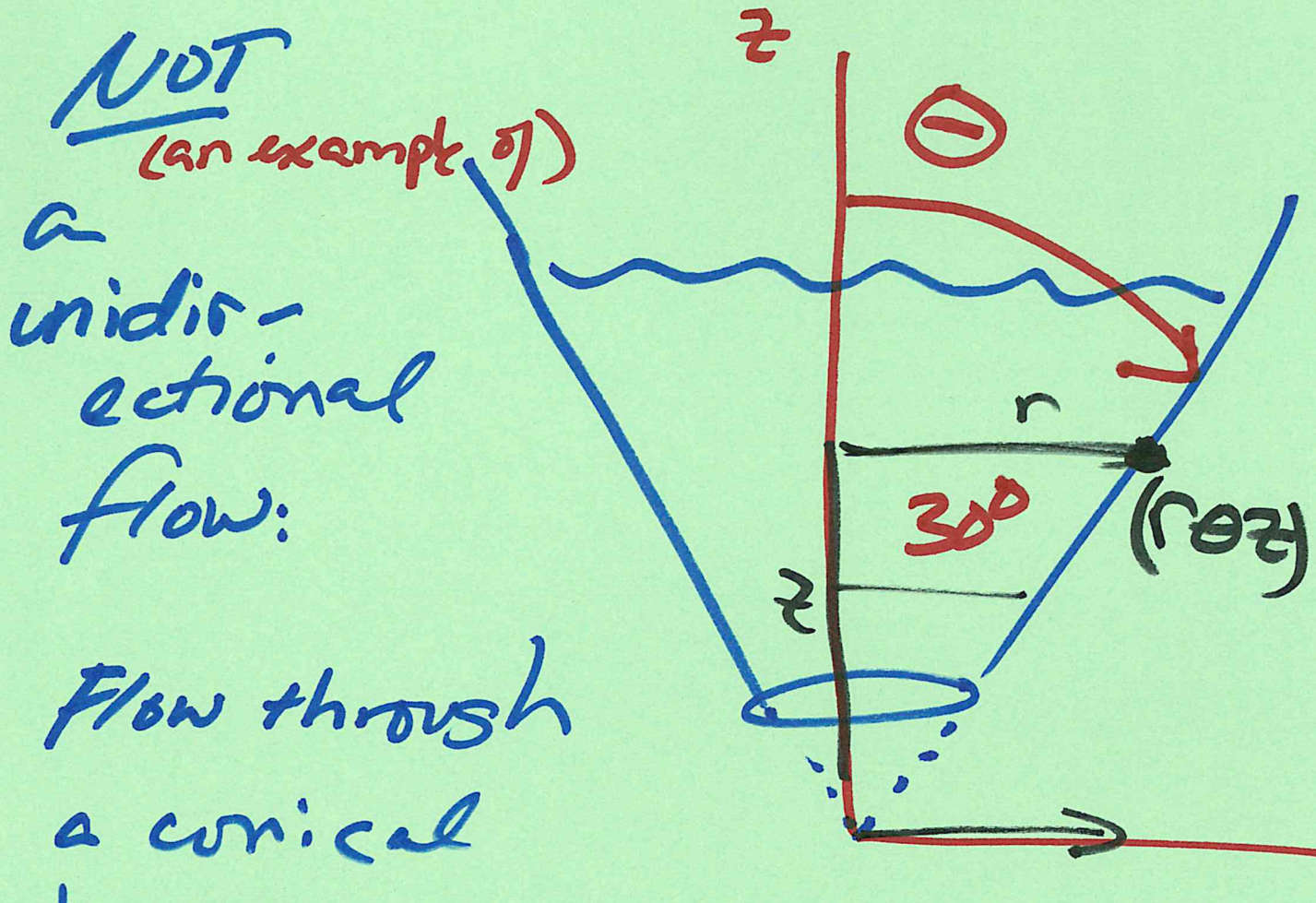
1. R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena*, 2nd edition, Wiley: NY, 2002.
2. R. B. Bird, R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric Fluids: Volume 1 Fluid Mechanics*, Wiley: NY, 1987.

NOT
(an example of)

a
unidirectional
flow:

Flow through
a conical
hopper

(flow not pointing in same direction
everywhere)



B.C.:

$\Theta = 30^\circ$ ← wall

$\underline{V} = 0$

no slip

(good choice of coord. sys)

r-component NS (μ momentum) ⑤

$$\boxed{0 = -\frac{dP}{dr}} \Rightarrow P \neq \text{func of } r$$

θ -component NS

$$(-r) 0 = \cancel{\frac{1}{r}} \frac{\partial P}{\partial \theta} \cancel{(\cancel{r})}$$

$$\boxed{\frac{\partial P}{\partial \theta} = 0} \Rightarrow P \neq \text{function of } \theta$$

z-component of NS

$$0 = -\frac{dP}{dz} + \mu \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{dV_z}{dr} \right) + \rho g$$

$P(z)$ only; $V_z(r)$ only

Separation of variables:

both eqns
to same constant

$$\lambda = \frac{dp}{dz} - \rho g = \mu \frac{L}{r} \frac{d}{dr} \left(r \frac{dv_z}{dr} \right)$$

$f(z)$ only

$=$ $h(r)$ only

$$\lambda = \frac{dp}{dz} - \rho g$$

$$\lambda = \mu \frac{L}{r} \frac{d}{dr} \left(r \frac{dv_z}{dr} \right) \equiv \Phi$$

$$(\lambda + \rho g) = \frac{dp}{dz}$$

$$\frac{d\Phi}{dr} = \left(\frac{\lambda}{\mu} \right) r$$

$$P = (\lambda + \rho g)z + C_1$$

$$\Phi = \left(\frac{\lambda}{\mu} \right) \frac{r^2}{2} + C_2$$

BC: $z = 0 \quad P = P_0$
 $z = L \quad P = P_L$
 SOLVE.

... →

$$\Phi = r \frac{dv_z}{dr} = \left(\frac{\lambda}{2\mu}\right) r^2 + C_2$$

integreat: $\frac{dv_z}{dr} = \left(\frac{\lambda}{2\mu}\right) r + \frac{C_2}{r}$

$$dv_z = \left(\frac{\lambda}{2\mu} r + \frac{C_2}{r}\right) dr$$

$$v_z = \frac{\lambda}{2\mu} \frac{r^2}{2} + C_2 \ln r + C_3$$

BC: $r=0$
 $r=R$

$v_z \neq \ln(0) \Rightarrow C_2 = 0$

$v_z = 0$
no slip

v_z must be finite

Solve:

$$P = P(z)$$

$$V_z = V_z(r) //$$

Then, calculate engineering quantities of interest:

Q (Vol. flow rate)

$\langle V \rangle$ (av. velocity)

F_z (drag on walls)