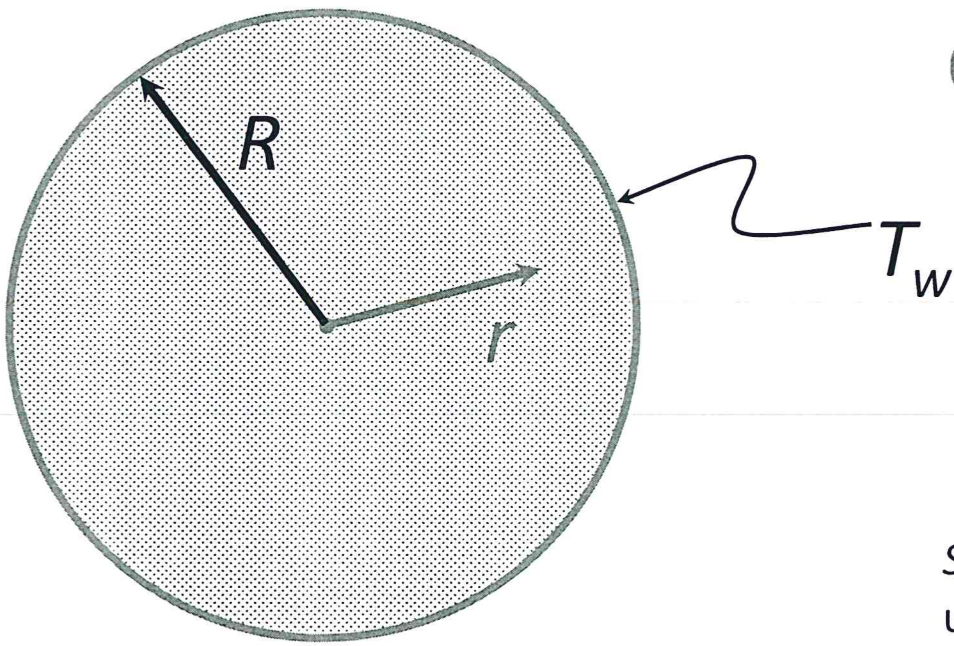
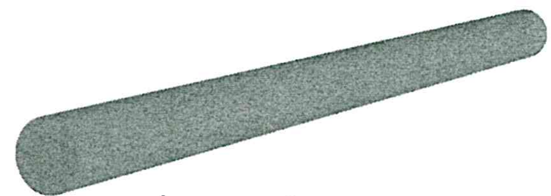
**Example 5: Heat Conduction with Generation**

*What is the steady state temperature profile in a wire if heat is generated uniformly throughout the wire at a rate of $S_e$ W/m$^3$ and the outer radius is held at $T_w$?*




$S_e$ = energy production per unit volume

# The **Equation of Energy** for systems with **constant $k$**

**Microscopic energy balance**, constant thermal conductivity; Gibbs notation

$$\rho\hat{C}_p\left(\frac{\partial T}{\partial t} + \underline{v}\cdot\nabla T\right) = k\nabla^2 T + S_e$$

**Microscopic energy balance**, constant thermal conductivity; Cartesian coordinates

$$\rho\hat{C}_p\left(\frac{\partial T}{\partial t} + v_x\frac{\partial T}{\partial x} + v_y\frac{\partial T}{\partial y} + v_z\frac{\partial T}{\partial z}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + S_e$$

**Microscopic energy balance**, constant thermal conductivity; cylindrical coordinates ⟸ YES

$$\rho\hat{C}_p\left(\cancel{\frac{\partial T}{\partial t}} + \cancel{v_r\frac{\partial T}{\partial r}} + \cancel{\frac{v_\theta}{r}\frac{\partial T}{\partial \theta}} + \cancel{v_z\frac{\partial T}{\partial z}}\right) = k\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}} + \cancel{\frac{\partial^2 T}{\partial z^2}}\right) + S_e$$

SS

LONG

θ-symmetric

**Microscopic energy balance**, constant thermal conductivity; spherical coordinates

$$\rho\hat{C}_p\left(\frac{\partial T}{\partial t} + v_r\frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial T}{\partial \phi}\right)$$
$$= k\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}\right) + S_e$$

$$0 = k\frac{1}{r}\frac{d}{dr}\underbrace{\left(r\frac{dT}{dr}\right)}_{=\Phi} + S_e$$

$$\left(-\frac{S_e}{k}\right)r = \frac{d\Phi}{dr}$$

Reference: F. A. Morrison, "Web Appendix to *An Introduction to Fluid Mechanics*," Cambridge University Press, New York, 2013 (*pages.mtu.edu/~fmorriso/cm310/IFMWebAppendixDMicroEBalanceMorrison.pdf*). This worksheet is on the web at *pages.mtu.edu/~fmorriso/cm310/energy.pdf*.

$$\underbrace{\Phi}_{= r\frac{dT}{dr}} = \left(-\frac{S_e}{k}\right)\frac{r^2}{2} + c_1$$

$$r\frac{dT}{dr} = \left(\frac{-S_e}{2k}\right) r^2 + c_1$$

NOTE: $\boxed{r = 0 \Rightarrow c_1 = 0}$

Proceed:

$$\frac{dT}{dr} = \left(-\frac{S_e}{2k}\right) r$$

note: $0 \le r \le R$ & thus $r = 0$ IS in the solution domain

Integrating:

$$\boxed{T = \left(-\frac{S_e}{2k}\right)\frac{r^2}{2} + C_2}$$

BC: $r = R \quad T = T_w$

$$\Rightarrow \quad T_w = \left(-\frac{S_e}{4k}\right) R^2 + C_2$$

$$\boxed{C_2 = T_w + \frac{S_e R^2}{4k}}$$

$$\boxed{T(r) = \left(\frac{S_e}{4k}\right)\left(R^2 - r^2\right) + T_w}$$

check BC: $r = R^2 \quad T = T_w$

$r = 0 \quad T = \text{finite}$

check units

$$T \,[=]\, (°C) \text{ or } (K) = \frac{\cancel{W}}{\cancel{m^3}}\, \cancel{m^2} \left(\frac{\cancel{m}\,K}{\cancel{W}}\right)$$

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What is the flux at R?

$$\frac{q_r}{A} = -k \frac{dT}{dr}$$

FOURIER'S LAW

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$$\frac{dT}{dr} = \left(\frac{S_e}{4k}\right)(-1)(2r)$$

$$\left.\frac{dT}{dr}\right|_{r=R} = \frac{S_e}{4k}(-2R)$$

$$\left.\frac{q_r}{A}\right|_{r=R} = (-k)\left(\left.\frac{dT}{dr}\right|_{r=R}\right)$$

$$= (\cancel{-k})\left(\frac{S_e}{\cancel{4k}_{\,2}}\right)(\cancel{-2}R)$$

$$\boxed{\left.\frac{q_r}{A}\right|_{r=R} = \frac{S_e R}{2}}$$

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