Equations Summary from Inside Cover of Morrison, 2013

Mechanical Energy Balance

$$\frac{\Delta p}{\rho} + \frac{\Delta \langle v \rangle^2}{2\alpha} + g\Delta z + F_{friction} = -\frac{W_{s,by\,fluid}}{m} \qquad \begin{cases} \alpha_{laminar} = 0.5 \\ \alpha_{turbulent} \approx 1 \end{cases}$$

$$F_{friction} = \left[4f\frac{L}{D} + \sum_{fittings} n_i K_{f,i} \right] \frac{\langle v \rangle^2}{2}$$

Fanning Friction Factor (pipe flow)

$$f = \frac{F_{drag}}{\frac{1}{2}\rho \langle v \rangle^2 (2\pi RL)} = \frac{\Delta pD}{2L\rho \langle v \rangle^2}$$

Drag Coefficient (sphere drop)

$$C_D = \frac{F_{drag}}{\frac{1}{2}\rho v_\infty^2 (\pi R^2)} = \frac{4gD(\rho_{body} - \rho)}{3\rho v_\infty^2}$$

Momentum balance on a CV (Reynolds transport theorem)

$$\frac{d\underline{P}}{dt} + \iint_{CS} (\hat{n} \cdot \underline{v})\rho \underline{v}\, dS = \sum_{on\,CV} \underline{f}$$

Hydrostatic Pressure

$$p_{bottom} = p_{top} + \rho g h$$

Hagen-Poiseuille Equation (steady, laminar tube flow, incompressible)

$$Q = \frac{\pi(p_0 - p_L)R^4}{8\mu L}$$

Prandtl Equation (steady, turbulent tube flow)

$$\frac{1}{\sqrt{f}} = -4.0\log\left(\frac{4.67}{Re\sqrt{f}}\right) + 2.28$$

Stokes-Einstein-Sutherland Equation (steady, slow flow around a sphere)

$$F_{drag} = 6\pi R \mu v_\infty$$

Macroscopic Momentum Balance on a CV

$$\frac{d\underline{P}}{dt} + \sum_{i=1}^{\#streams} \left[\frac{\rho A \cos(\theta)\langle v \rangle^2}{\beta} \hat{v} \right]_{A_i} = \sum_{i=1}^{\#streams} [-pA\hat{n}]_{A_i} + \underline{R} + M_{CV}\underline{g} \qquad \begin{cases} \beta_{laminar} = 0.75 \\ \beta_{turbulent} \approx 1 \end{cases}$$

Navier-Stokes equation (microscopic momentum balance, incompressible, Newtonian fluids)

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v}\right) = -\nabla p + \mu \nabla^2 \underline{v} + \rho \underline{g}$$

Continuity equation (microscopic mass balance, incompressible fluids)

$$\nabla \cdot \underline{v} = 0$$

Total stress tensor

$$\underline{\underline{\Pi}} = -p\underline{\underline{I}} + \underline{\underline{\tilde{\tau}}}$$

$$\begin{pmatrix} \tilde{\Pi}_{11} & \tilde{\Pi}_{12} & \tilde{\Pi}_{13} \\ \tilde{\Pi}_{21} & \tilde{\Pi}_{22} & \tilde{\Pi}_{23} \\ \tilde{\Pi}_{31} & \tilde{\Pi}_{32} & \tilde{\Pi}_{33} \end{pmatrix}_{123} = \begin{pmatrix} \tilde{\tau}_{11} - p & \tilde{\tau}_{12} & \tilde{\tau}_{13} \\ \tilde{\tau}_{21} & \tilde{\tau}_{22} - p & \tilde{\tau}_{23} \\ \tilde{\tau}_{31} & \tilde{\tau}_{32} & \tilde{\tau}_{33} - p \end{pmatrix}_{123}$$

Dynamic pressure

$$\mathcal{P} \equiv p + \rho g h$$

Newtonian constitutive equation

$$\underline{\underline{\tilde{\tau}}} = \mu\left(\nabla \underline{v} + (\nabla \underline{v})^T\right)$$

$$= \mu \begin{pmatrix} 2\frac{\partial v_1}{\partial x_1} & \frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2} & \frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2} & 2\frac{\partial v_2}{\partial x_2} & \frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3} & \frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3} & 2\frac{\partial v_3}{\partial x_3} \end{pmatrix}_{123}$$

Total molecular fluid force on a finite surface $\mathcal{S}$

$$\underline{F} = \iint_S \left[\hat{n} \cdot \underline{\underline{\tilde{\Pi}}}\right]_{at\,surface} dS$$

Stationary fluid

$$\left[\hat{n} \cdot \underline{\underline{\tilde{\Pi}}}\right] = -p\hat{n}$$

Moving fluid

$$\left[\hat{n} \cdot \underline{\underline{\tilde{\Pi}}}\right] = -p\hat{n} + \hat{n} \cdot \underline{\underline{\tilde{\tau}}}$$

Total fluid torque on a finite surface $\mathcal{S}$

$$\underline{T} = \iint_S \left[\underline{R} \times (\hat{n} \cdot \underline{\underline{\tilde{\Pi}}})\right]_{at\,surface} dS$$

Total flow rate out through a finite surface $\mathcal{S}$

$$Q = \dot{V} = \iint_S [\hat{n} \cdot \underline{v}]_{at\,surface} dS$$

Average velocity across a finite surface $\mathcal{S}$

$$\langle v \rangle = \frac{Q}{\mathcal{S}}$$

The equations in F. A. Morrison, *An Introduction to Fluid Mechanics* (Cambridge, 2013) assume the following definitions of the cylindrical and spherical coordinate systems.

Cylindrical Coordinate System: Note that the θ-coordinate swings around the z-axis
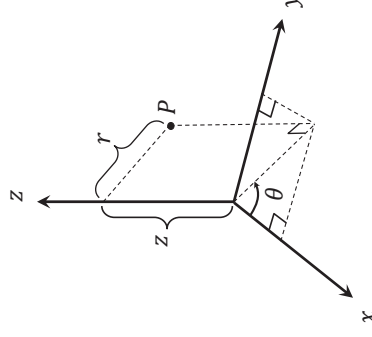
Spherical Coordinate System: Note that the θ-coordinate swings down from the z-axis; this is different from its definition in the cylindrical system above.

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Coordinate system	surface differential dS
Cartesian (top, $\hat{n} = \hat{e}_z$)	$dS = dxdy$
Cartesian (side a, $\hat{n} = \hat{e}_y$)	$dS = dxdz$
Cartesian (side b, $\hat{n} = \hat{e}_x$)	$dS = dydz$
cylindrical (top, $\hat{n} = \hat{e}_z$)	$dS = rdrd\theta$
cylindrical (side, $\hat{n} = \hat{e}_r$)	$dS = Rd\theta dz$
spherical, ($\hat{n} = \hat{e}_r$)	$dS = R^2 \sin\theta d\theta d\phi$

Coordinate system	volume differential dV
Cartesian	$dV = dxdydz$
cylindrical	$dV = rdrd\theta dz$
spherical	$dV = r^2 \sin\theta drd\theta d\phi$

Coordinate system	coordinates	basis vectors
spherical	$x = r\sin\theta\cos\phi$ $y = r\sin\theta\sin\phi$ $z = r\cos\theta$	$\hat{e}_r = (\sin\theta\cos\phi\hat{e}_x) + (\sin\theta\sin\phi\hat{e}_y) + \cos\theta\hat{e}_z$ $\hat{e}_\theta = (\cos\theta\cos\phi)\hat{e}_x + (\cos\theta\sin\phi)\hat{e}_y + (-\sin\theta)\hat{e}_z$ $\hat{e}_\phi = (-\sin\phi)\hat{e}_x + \cos\phi\hat{e}_y$
cylindrical	$x = r\cos\theta$ $y = r\sin\theta$ $z = z$	$\hat{e}_r = \cos\theta\hat{e}_x + \sin\theta\hat{e}_y$ $\hat{e}_\theta = (-\sin\theta)\hat{e}_x + \cos\theta\hat{e}_y$ $\hat{e}_z = \hat{e}_z$

Divergence Theorem $\qquad \iint_S \hat{n}\cdot\underline{F}\,dS \;=\; \iiint_V \nabla\cdot\underline{F}\,dV$

Stokes Theorem $\qquad \oint_C \hat{t}\cdot\underline{F}\,dl \;=\; \iint_S \hat{n}\cdot(\nabla\times\underline{F})\,dS$
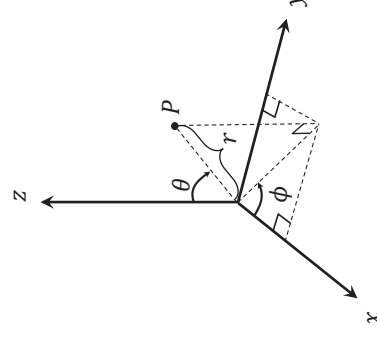
Vector identities:

$$\nabla\cdot\nabla\times\underline{F} = 0 \quad \text{(Divergence of curl} = 0)$$
$$\nabla\times\nabla f = 0 \quad \text{(Curl of gradient} = 0)$$
$$\nabla(fg) = f\nabla g + g\nabla f$$
$$\underline{F}\cdot\nabla\underline{F} = \frac{1}{2}\nabla(\underline{F}^2) - \underline{F}\times(\nabla\times\underline{F})$$
$$\nabla\cdot(f\underline{F}) = f\nabla\cdot\underline{F} + \underline{F}\cdot\nabla f$$
$$\nabla\times\nabla\times\underline{F} = \nabla(\nabla\cdot\underline{F}) - \nabla^2\underline{F}$$
$$\nabla\cdot(\underline{F}\times\underline{G}) = \underline{G}\cdot(\nabla\times\underline{F}) - \underline{F}\cdot(\nabla\times\underline{G})$$

FACTORS FOR UNIT CONVERSIONS

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Quantity	Equivalent Values
Mass	$1 \text{ kg} = 1000 \text{ g} = 0.001 \text{ metric ton} = 2.20462 \text{ lb}_m = 35.27392 \text{ oz}$
	$1 \text{ lb}_m = 16 \text{ oz} = 5 \times 10^{-4} \text{ ton} = 453.593 \text{ g} = 0.453593 \text{ kg}$
Length	$1 \text{ m} = 100 \text{ cm} = 1000 \text{ mm} = 10^6 \text{ microns } (\mu m) = 10^{10} \text{ angstroms (Å)}$
	$= 39.37 \text{ in.} = 3.2808 \text{ ft} = 1.0936 \text{ yd} = 0.0006214 \text{ mile}$
	$1 \text{ ft} = 12 \text{ in.} = 1/3 \text{ yd} = 0.3048 \text{ m} = 30.48 \text{ cm}$
Volume	$1 \text{ m}^3 = 1000 \text{ liters} = 10^6 \text{ cm}^3 = 10^6 \text{ ml}$
	$= 35.3145 \text{ ft}^3 = 220.83 \text{ imperial gallons} = 264.17 \text{ gal}$
	$= 1056.68 \text{ qt}$
	$1 \text{ ft}^3 = 1728 \text{ in}^3 = 7.4805 \text{ gal} = 0.028317 \text{ m}^3 = 28.317 \text{ liters}$
	$= 28\,317 \text{ cm}^3$
Force	$1 \text{ N} = 1 \text{ kg·m/s}^2 = 10^5 \text{ dynes} = 10^5 \text{ g·cm/s}^2 = 0.22481 \text{ lb}_f$
	$1 \text{ lb}_f = 32.174 \text{ lb}_m\text{·ft/s}^2 = 4.4482 \text{ N} = 4.4482 \times 10^5 \text{ dynes}$
Pressure	$1 \text{ atm} = 1.01325 \times 10^5 \text{ N/m}^2 \text{ (Pa)} = 101.325 \text{ kPa} = 1.01325 \text{ bars}$
	$= 1.01325 \times 10^6 \text{ dynes/cm}^2$
	$= 760 \text{ mm Hg at } 0°\text{ C (torr)} = 10.333 \text{ m H}_2\text{O at } 4° \text{ C}$
	$= 14.696 \text{ lb}_f/\text{in}^2 \text{ (psi)} = 33.9 \text{ ft H}_2\text{O at } 4°\text{C}$
	$100 \text{ kPa} = 1 \text{ bar}$
Energy	$1 \text{ J} = 1 \text{ N·m} = 10^7 \text{ ergs} = 10^7 \text{ dyne·cm}$
	$= 2.778 \times 10^{-7} \text{ kW·h} = 0.23901 \text{ cal}$
	$= 0.7376 \text{ ft·lb}_f = 9.47817 \times 10^{-4} \text{ Btu}$
Power	$1 \text{ W} = 1 \text{ J/s} = 0.23885 \text{ cal/s} = 0.7376 \text{ ft·lb}_f/\text{s} = 9.47817 \times 10^{-4} \text{ Btu/s} = 3.4121 \text{ Btu/h}$
	$= 1.341 \times 10^{-3} \text{ hp (horsepower)}$
Viscosity	$1 \text{ Pa·s} = 1 \text{ N·s/m}^2 = 1 \text{ kg/m·s}$
	$= 10 \text{ poise} = 10 \text{ dynes·s/cm}^2 = 10 \text{ g/cm·s}$
	$= 10^3 \text{ cp (centipoise)}$
	$= 0.67197 \text{ lb}_m/\text{ft·s} = 2419.088 \text{ lb}_m/\text{ft·h}$
Density	$1 \text{ kg/m}^3 = 10^{-3} \text{ g/cm}^3$
	$= 0.06243 \text{ lb}_m/\text{ft}^3$
	$10^3 \text{ kg/m}^3 = 1 \text{ g/cm}^3 = 62.428 \text{ lb}_m/\text{ft}^3$
Volumetric Flow	$1 \text{ m}^3/\text{s} = 35.3145 \text{ ft}^3/\text{s} = 15{,}850.2 \text{ gal/min (gpm)}$
	$1 \text{ gpm} = 6.30907 \times 10^{-5} \text{ m}^3/\text{s} = 2.22802 \times 10^{-3} \text{ ft}^3/\text{s} = 3.7854 \text{ liter/min}$
	$1 \text{ liter/min} = 0.26417 \text{ gpm}$

Temperature	$T(^oC) = \frac{5}{9}\left[T(^oF) - 32\right]$
	$T(^oF) = \frac{9}{5}T(^oC) + 32 = 1.8T(^oC) + 32$
Absolute Temperature	$T(K) = T(^oC) + 273.15$
	$T(^oR) = T(^oF) + 459.67$
Temperature Interval (ΔT)	$1 \text{ C}° = 1 \text{ K} = 1.8 \text{ F}° = 1.8 \text{ R}°$
	$1 \text{F}° = 1 \text{ R}° = (5/9) \text{ C}° = (5/9) \text{ K}$

USEFUL QUANTITIES

$$SG = \rho(20°C)/\rho_{water}(4°C)$$

$\rho_{water}(4°C) = 1000 \text{ kg/m}^3 = 62.43 \text{ lb}_m/\text{ft}^3 = 1.000 \text{ g/cm}^3$

$\rho_{water}(25°C) = 997.08 \text{ kg/m}^3 = 62.25 \text{ lb}_m/\text{ft}^3 = 0.99709 \text{ g/cm}^3$

$g = 9.8066 \text{ m/s}^2 = 980.66 \text{ cm/s}^2 = 32.174 \text{ ft/s}^2$

$\mu_{water}(25°C) = 8.937 \times 10^{-4} \text{ Pa·s} = 8.937 \times 10^{-4} \text{ kg/m·s}$

$= 0.8937 \text{ cp} = 0.8937 \times 10^{-2} \text{ g/cm·s} = 6.005 \times 10^{-4} \text{ lb}_m/\text{ft·s}$

Composition of air:

N_2	78.03%
O_2	20.99%
Ar	0.94%
CO_2	0.03%
H_2, He, Ne, Kr, Xe	0.01%
	100.00%

$M_{air} = 29 \text{ g/mol} = 29 \text{ kg/kmol} = 29 \text{ lb}_m/\text{lbmole}$

$\hat{C}_{p,water}(25°C) = 4.182 \text{ kJ/kg K} = 0.9989 \text{ cal/g°C} = 0.9997 \text{ Btu/lb}_m°F$

$R = 8.314 \text{ m}^3\text{·Pa/mol K} = 0.08314 \text{ liter·bar/mol K} = 0.08206 \text{ liter·atm/mol K}$

$= 62.36 \text{ liter·mm Hg/mol K} = 0.7302 \text{ ft}^3\text{·atm/lbmole·}°R$

$= 10.73 \text{ ft}^3\text{·psia/lbmole·}°R$

$= 8.314 \text{ J/mol K}$

$= 1.987 \text{ cal/mol K} = 1.987 \text{ Btu/lbmole·}°R$

Fluid Friction Data Correlations for Examinations

CM3110 Transport Phenomena I
Michigan Technological University
Professor Faith A. Morrison
1 December 2020

I. Data Correlations for Flow through Smooth Pipes

A. All Reynolds numbers: Morrison

The correlation from Morrison (2013) fits the smooth pipe data for all Reynolds numbers; beyond Re = 4000 this correlation follows the Prandtl equation (see Figure 1; Morrison, equation 7.158). This correlation is explicit in f; when flow rate is known, Δp may be found directly; when Δp is known, Q or $\langle v \rangle$ must be solved for iteratively.

Morrison (2013)

$$f = \left(\frac{0.0076 \left(\frac{3170}{Re} \right)^{0.165}}{1 + \left(\frac{3170}{Re} \right)^{7.0}} \right) + \frac{16}{Re}$$

(1)

B. $4,000 \leq Re \leq 1 \times 10^6$: Prandtl

The Prandtl correlation for $f(Re)$ in turbulent flow is not explicit in friction factor and may be solved iteratively except when f is known (Morrison, equation 7.156). When velocity is not known, $Re\sqrt{f}$ may still be calculated, yielding f. The Prandtl correlation is good only for Re > 4,000.

Prandtl
or VonKarman-Nikuradse
(Denn, 1980)

$$\frac{1}{\sqrt{f}} = 4.0 \log\left(Re\sqrt{f} \right) - 0.40$$

(2)

C. $4,000 \leq Re \leq 1 \times 10^6$: A simplified Correlation

For the turbulent regime, an approximate correlation that is much simpler to work with (with a calculator on an exam, for example) is given here and shown in Figure 2 (Morrison, equation 7.157). This is good only for Re > 4,000.

Simplified Turbulent
(White, 1974)

$$f = \frac{1.02}{4} (\log Re)^{-2.5}$$

(3)

Figure 1: Equation 3 captures smooth pipe friction factor as a function of Reynolds number over the entire Reynolds-number range (Morrison, 2013) and is recommended for spreadsheet use. Also shown are Nikuradse's experimental data for flow in smooth pipes (Nikuradse, 1933). Use beyond Re = 10^6 is not recommended; for Re>4000 equation 3 follows the Prandtl equation. (Morrison, 2013, p532)
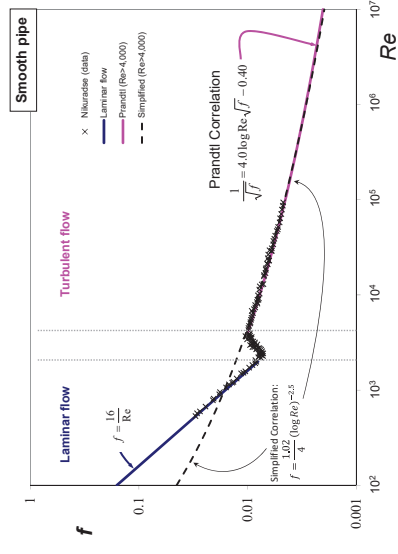
Figure 2: For turbulent flow, the simplified (equation 3) or Prandtl (equation 2) correlations may be used. For work with a calculator, the simplified correlation is perhaps the easiest to work with. (Morrison, 2013, p531)

II. Data Correlations for Flow Around a Sphere

A. All Reynolds Numbers: Morrison

The correlation from Morrison (2013) fits the flow around a sphere for all Reynolds numbers (Figure 3; Morrison equation 8.83); beyond $Re = 10^6$ this correlation follows the curve shown in Figure 3.

Morrison (2013)

$$C_D = \frac{24}{Re} + \frac{2.6\left(\frac{Re}{5.0}\right)}{1+\left(\frac{Re}{5.0}\right)^{1.52}} + \frac{0.411\left(\frac{Re}{263,000}\right)^{-7.94}}{1+\left(\frac{Re}{263,000}\right)^{-8.00}} + \frac{0.25\left(\frac{Re}{10^6}\right)}{1+\left(\frac{Re}{10^6}\right)} \qquad (4)$$

Simplified Correlations

The correlations below (Morrison, 2013; equation 8.82) are simpler relationships more suitable to calculator/exam work.

$Re < 2$
$$C_D = \frac{24}{Re} \qquad (5)$$

$0.1 \le Re \le 1,000$
$$C_D = \frac{24}{Re}\left(1 + 0.14 Re^{0.7}\right) \qquad (6)$$

$1,000 \le Re \le 2.6 \times 10^5$
$$C_D = 0.445 \qquad (7)$$

$2.8 \times 10^5 \le Re \le 10^6$
$$\frac{\log C_D}{\left(\frac{Re}{10^6}\right)} = 4.43 \log Re - 27.3 \qquad (8)$$

sphere

$$C_D = \frac{24}{Re} + \frac{2.6\left(\frac{Re}{5.0}\right)}{1+\left(\frac{Re}{5.0}\right)^{1.52}} + \frac{0.411\left(\frac{Re}{263,000}\right)^{-7.94}}{1+\left(\frac{Re}{263,000}\right)^{-8.00}} + \frac{0.25\left(\frac{Re}{10^6}\right)}{1+\left(\frac{Re}{10^6}\right)}$$
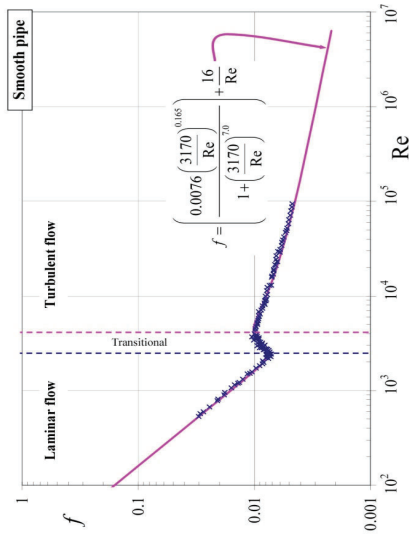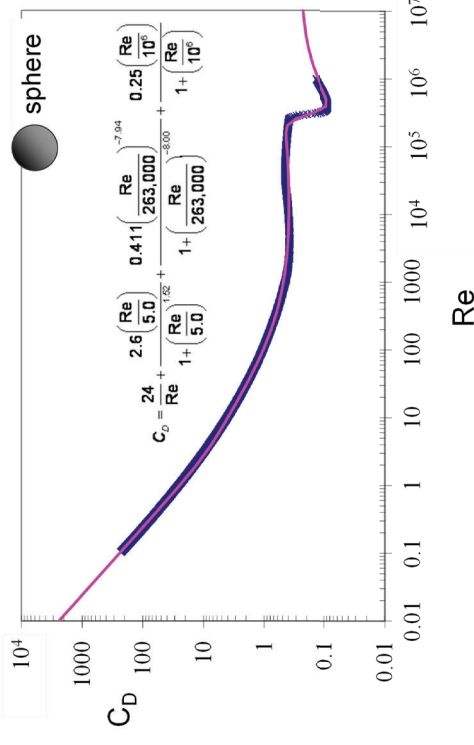
Figure 3: Equation 4 captures flow around a sphere as a function of Reynolds number over the entire Reynolds-number range (Morrison, 2013) and is recommended for spreadsheet use. Also shown are experimental data from White (1974). Use beyond $Re = 10^6$ is not recommended. (Morrison, 2013, p625)

References
M. Denn, Process Fluid Mechanics (Prentice Hall, Englewood Cliffs, NJ, 1980)
F. A. Morrison, *An Introduction to Fluid Mechanics* (Cambridge University Press, New York, 2013).
F. M. White, *Viscous Fluid Flow* (McGraw-Hill, Inc.: New York, 1974).

The Equation of Continuity and the Equation of Motion in Cartesian, cylindrical, and spherical coordinates

CM3110 Fall 2011 Faith A. Morrison

Continuity Equation, Cartesian coordinates

$$\frac{\partial \rho}{\partial t} + \left(v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} \right) + \rho \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) = 0$$

Continuity Equation, cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial (\rho r v_r)}{\partial r} + \frac{1}{r} \frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{\partial (\rho v_z)}{\partial z} = 0$$

Continuity Equation, spherical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial (\rho r^2 v_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (\rho v_\theta \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial (\rho v_\phi)}{\partial \phi} = 0$$

Equation of Motion for an incompressible fluid, 3 components in Cartesian coordinates

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial P}{\partial x} + \left(\frac{\partial \tilde{\tau}_{xx}}{\partial x} + \frac{\partial \tilde{\tau}_{yx}}{\partial y} + \frac{\partial \tilde{\tau}_{zx}}{\partial z} \right) + \rho g_x$$

$$\rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = -\frac{\partial P}{\partial y} + \left(\frac{\partial \tilde{\tau}_{xy}}{\partial x} + \frac{\partial \tilde{\tau}_{yy}}{\partial y} + \frac{\partial \tilde{\tau}_{zy}}{\partial z} \right) + \rho g_y$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial P}{\partial z} + \left(\frac{\partial \tilde{\tau}_{xz}}{\partial x} + \frac{\partial \tilde{\tau}_{yz}}{\partial y} + \frac{\partial \tilde{\tau}_{zz}}{\partial z} \right) + \rho g_z$$

Equation of Motion for an incompressible fluid, 3 components in cylindrical coordinates

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) = -\frac{\partial P}{\partial r} + \left(\frac{1}{r} \frac{\partial (r \tilde{\tau}_{rr})}{\partial r} + \frac{1}{r} \frac{\partial \tilde{\tau}_{\theta r}}{\partial \theta} - \frac{\tilde{\tau}_{\theta\theta}}{r} + \frac{\partial \tilde{\tau}_{zr}}{\partial z} \right) + \rho g_r$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) = -\frac{1}{r} \frac{\partial P}{\partial \theta} + \left(\frac{1}{r^2} \frac{\partial (r^2 \tilde{\tau}_{r\theta})}{\partial r} + \frac{1}{r} \frac{\partial \tilde{\tau}_{\theta\theta}}{\partial \theta} + \frac{\partial \tilde{\tau}_{z\theta}}{\partial z} \right) + \rho g_\theta$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial P}{\partial z} + \left(\frac{1}{r} \frac{\partial (r \tilde{\tau}_{rz})}{\partial r} + \frac{1}{r} \frac{\partial \tilde{\tau}_{\theta z}}{\partial \theta} + \frac{\partial \tilde{\tau}_{zz}}{\partial z} \right) + \rho g_z$$

Equation of Motion for an incompressible fluid, 3 components in spherical coordinates

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r} \right)$$
$$= -\frac{\partial P}{\partial r} + \left(\frac{1}{r^2} \frac{\partial (r^2 \tilde{\tau}_{rr})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (\tilde{\tau}_{\theta r} \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \tilde{\tau}_{\phi r}}{\partial \phi} - \frac{\tilde{\tau}_{\theta\theta} + \tilde{\tau}_{\phi\phi}}{r} \right) + \rho g_r$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2 \cot \theta}{r} \right)$$
$$= -\frac{1}{r} \frac{\partial P}{\partial \theta} + \left(\frac{1}{r^3} \frac{\partial (r^3 \tilde{\tau}_{r\theta})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (\tilde{\tau}_{\theta\theta} \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \tilde{\tau}_{\phi\theta}}{\partial \phi} + \frac{\tilde{\tau}_{\theta r} - \tilde{\tau}_{r\theta}}{r} - \frac{\tilde{\tau}_{\phi\phi} \cot \theta}{r} \right) + \rho g_\theta$$

$$\rho \left(\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{v_\phi v_\theta \cot \theta}{r} \right)$$
$$= -\frac{1}{r \sin \theta} \frac{\partial P}{\partial \phi} + \left(\frac{1}{r^3} \frac{\partial (r^3 \tilde{\tau}_{r\phi})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (\tilde{\tau}_{\theta\phi} \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \tilde{\tau}_{\phi\phi}}{\partial \phi} + \frac{\tilde{\tau}_{\phi r} - \tilde{\tau}_{r\phi}}{r} + \frac{\tilde{\tau}_{\phi\theta} \cot \theta}{r} \right) + \rho g_\phi$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation) 3 components in Cartesian coordinates

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial P}{\partial x} + \mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) + \rho g_x$$

$$\rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = -\frac{\partial P}{\partial y} + \mu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right) + \rho g_y$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial P}{\partial z} + \mu \left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho g_z$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in cylindrical coordinates

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) = -\frac{\partial P}{\partial r} + \mu \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial (r v_r)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2} \right) + \rho g_r$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) = -\frac{1}{r} \frac{\partial P}{\partial \theta} + \mu \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial (r v_\theta)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2} \right) + \rho g_\theta$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial P}{\partial z} + \mu \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho g_z$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in spherical coordinates

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r} \right)$$
$$= -\frac{\partial P}{\partial r} + \mu \left(\frac{\partial}{\partial r} \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 v_r \right) \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial v_r}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_r}{\partial \phi^2} \right.$$
$$\left. - \frac{2}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (v_\theta \sin \theta) - \frac{2}{r^2 \sin \theta} \frac{\partial v_\phi}{\partial \phi} \right) + \rho g_r$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2 \cot \theta}{r} \right)$$
$$= -\frac{1}{r} \frac{\partial P}{\partial \theta} + \mu \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial v_\theta}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (v_\theta \sin \theta) \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_\theta}{\partial \phi^2} \right.$$
$$\left. + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} - \frac{2 \cot \theta}{r^2 \sin \theta} \frac{\partial v_\phi}{\partial \phi} \right) + \rho g_\theta$$

$$\rho \left(\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{v_\phi v_\theta \cot \theta}{r} \right)$$
$$= -\frac{1}{r \sin \theta} \frac{\partial P}{\partial \phi} + \mu \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial v_\phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (v_\phi \sin \theta) \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_\phi}{\partial \phi^2} \right.$$
$$\left. + \frac{2}{r^2 \sin \theta} \frac{\partial v_r}{\partial \phi} + \frac{2 \cot \theta}{r^2 \sin \theta} \frac{\partial v_\theta}{\partial \phi} \right) + \rho g_\phi$$

Note: the r-component of the Navier-Stokes equation in spherical coordinates may be simplified by adding $0 = \frac{2}{r} \nabla \cdot \underline{v}$ to the component shown above. This term is zero due to the continuity equation (mass conservation). See Bird et. al.

References:

1. R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena*, 2nd edition, Wiley: NY, 2002.

2. R. B. Bird, R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric Fluids: Volume 1 Fluid Mechanics*, Wiley: NY, 1987.

The **Equation of Energy** in Cartesian, cylindrical, and spherical coordinates for

Newtonian fluids of constant density, with source term S. Source could be electrical energy due to current flow, chemical energy, etc. Two cases are presented: the general case where thermal conductivity may be a function of temperature (vector flux $\underline{q} = q/A$ appears in the equations); and the more usual case, where thermal conductivity is constant.

Fall 2013 Faith A. Morrison, Michigan Technological University

Microscopic energy balance, in terms of flux; Gibbs notation

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + \underline{v} \cdot \nabla T\right) = -\nabla \cdot \underline{\tilde{q}} + S$$

Microscopic energy balance, in terms of flux; Cartesian coordinates

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z}\right) = -\left(\frac{\partial \tilde{q}_x}{\partial x} + \frac{\partial \tilde{q}_y}{\partial y} + \frac{\partial \tilde{q}_z}{\partial z}\right) + S$$

Microscopic energy balance, in terms of flux; cylindrical coordinates

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z}\right) = -\left(\frac{1}{r}\frac{\partial (r \tilde{q}_r)}{\partial r} + \frac{1}{r}\frac{\partial \tilde{q}_\theta}{\partial \theta} + \frac{\partial \tilde{q}_z}{\partial z}\right) + S$$

Microscopic energy balance, in terms of flux spherical coordinates

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + \frac{v_\phi}{r \sin\theta} \frac{\partial T}{\partial \phi}\right) = -\left(\frac{1}{r^2}\frac{\partial (r^2 \tilde{q}_r)}{\partial r} + \frac{1}{r \sin\theta}\frac{\partial (\tilde{q}_\theta \sin\theta)}{\partial \theta} + \frac{1}{r \sin\theta}\frac{\partial \tilde{q}_\phi}{\partial \phi}\right) + S$$

Fourier's law of heat conduction, Gibbs notation: $\underline{\tilde{q}} = -k\nabla T$

Fourier's law of heat conduction, Cartesian coordinates:
$$\begin{pmatrix}\tilde{q}_x \\ \tilde{q}_y \\ \tilde{q}_z\end{pmatrix}_{xyz} = \begin{pmatrix}-k\frac{\partial T}{\partial x} \\ -k\frac{\partial T}{\partial y} \\ -k\frac{\partial T}{\partial z}\end{pmatrix}_{xyz}$$

Fourier's law of heat conduction, cylindrical coordinates:
$$\begin{pmatrix}\tilde{q}_r \\ \tilde{q}_\theta \\ \tilde{q}_z\end{pmatrix}_{xyz} = \begin{pmatrix}-k\frac{\partial T}{\partial r} \\ -\frac{k}{r}\frac{\partial T}{\partial \theta} \\ -k\frac{\partial T}{\partial z}\end{pmatrix}_{r\theta z}$$

Fourier's law of heat conduction, spherical coordinates:
$$\begin{pmatrix}\tilde{q}_r \\ \tilde{q}_\theta \\ \tilde{q}_\phi\end{pmatrix}_{xyz} = \begin{pmatrix}-k\frac{\partial T}{\partial r} \\ -\frac{k}{r}\frac{\partial T}{\partial \theta} \\ -\frac{k}{r\sin\theta}\frac{\partial T}{\partial \phi}\end{pmatrix}_{r\theta\phi}$$

The **Equation of Energy** for systems with **constant k**

Microscopic energy balance, constant thermal conductivity; Gibbs notation

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + \underline{v} \cdot \nabla T\right) = k\nabla^2 T + S$$

Microscopic energy balance, constant thermal conductivity; Cartesian coordinates

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + S$$

Microscopic energy balance, constant thermal conductivity; cylindrical coordinates

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z}\right) = k\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}\right) + S$$

Microscopic energy balance, constant thermal conductivity; spherical coordinates

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + \frac{v_\phi}{r \sin\theta} \frac{\partial T}{\partial \phi}\right)$$
$$= k\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}\right) + S$$

A.2-11 Heat-Transfer Properties of Liquid Water, SI Units

T (°C)	T (K)	ρ (kg/m³)	c_p (kJ/kg·K)	$\mu \times 10^3$ (Pa·s, or kg/m·s)	k (W/m·K)	N_{Pr}	$\beta \times 10^4$ (1/K)	$(g\beta\rho^2/\mu^2) \times 10^{-8}$ (1/K·m³)
0	273.2	999.6	4.229	1.786	0.5694	13.3	−0.630	
15.6	288.8	998.0	4.187	1.131	0.5884	8.07	1.44	10.93
26.7	299.9	996.4	4.183	0.860	0.6109	5.89	2.34	30.70
37.8	311.0	994.7	4.183	0.682	0.6283	4.51	3.24	68.0
65.6	338.8	981.9	4.187	0.432	0.6629	2.72	5.04	256.2
93.3	366.5	962.7	4.229	0.3066	0.6802	1.91	6.66	642
121.1	394.3	943.5	4.271	0.2381	0.6836	1.49	8.46	1300
148.9	422.1	917.9	4.312	0.1935	0.6836	1.22	10.08	2231
204.4	477.6	858.6	4.522	0.1384	0.6611	0.950	14.04	5308
260.0	533.2	784.9	4.982	0.1042	0.6040	0.859	19.8	11 030
315.6	588.8	679.2	6.322	0.0862	0.5071	1.07	31.5	19 260

A.2-11 Heat-Transfer Properties of Liquid Water, English Units

T (°F)	ρ $\left(\frac{lb_m}{ft^3}\right)$	c_p $\left(\frac{btu}{lb_m \cdot °F}\right)$	$\mu \times 10^3$ $\left(\frac{lb_m}{ft \cdot s}\right)$	k $\left(\frac{btu}{h \cdot ft \cdot °F}\right)$	N_{Pr}	$\beta \times 10^4$ (1/°R)	$(g\beta\rho^2/\mu^2) \times 10^{-6}$ (1/°R·ft³)
32	62.4	1.01	1.20	0.329	13.3	−0.350	
60	62.3	1.00	0.760	0.340	8.07	0.800	17.2
80	62.2	0.999	0.578	0.353	5.89	1.30	48.3
100	62.1	0.999	0.458	0.363	4.51	1.80	107
150	61.3	1.00	0.290	0.383	2.72	2.80	403
200	60.1	1.01	0.206	0.393	1.91	3.70	1010
250	58.9	1.02	0.160	0.395	1.49	4.70	2045
300	57.3	1.03	0.130	0.395	1.22	5.60	3510
400	53.6	1.08	0.0930	0.382	0.950	7.80	8350
500	49.0	1.19	0.0700	0.349	0.859	11.0	17 350
600	42.4	1.51	0.0579	0.293	1.07	17.5	30 300

Geankoplis, 4th edition

NOTE: Equate the label to the provided quantity in the supplied units. For example, for water at $0°C$:

$$\mu \times 10^3 = 1.786 \; Pa \; s$$

$$\mu = 1.786 \times 10^{-3} Pa \; s$$

A.3-3 Physical Properties of Air at 101.325 kPa (1 Atm Abs), SI Units

T (°C)	T (K)	ρ (kg/m³)	c_p (kJ/kg·K)	$\mu \times 10^5$ (Pa·s, or kg/m·s)	k (W/m·K)	N_{Pr}	$\beta \times 10^3$ (1/K)	$g\beta\rho^2/\mu^2$ (1/K·m³)
−17.8	255.4	1.379	1.0048	1.62	0.02250	0.720	3.92	2.79×10^8
0	273.2	1.293	1.0048	1.72	0.02423	0.715	3.65	2.04×10^8
10.0	283.2	1.246	1.0048	1.78	0.02492	0.713	3.53	1.72×10^8
37.8	311.0	1.137	1.0048	1.90	0.02700	0.705	3.22	1.12×10^8
65.6	338.8	1.043	1.0090	2.03	0.02925	0.702	2.95	0.775×10^8
93.3	366.5	0.964	1.0090	2.15	0.03115	0.694	2.74	0.534×10^8
121.1	394.3	0.895	1.0132	2.27	0.03323	0.692	2.54	0.386×10^8
148.9	422.1	0.838	1.0174	2.37	0.03531	0.689	2.38	0.289×10^8
176.7	449.9	0.785	1.0216	2.50	0.03721	0.687	2.21	0.214×10^8
204.4	477.6	0.740	1.0258	2.60	0.03894	0.686	2.09	0.168×10^8
232.2	505.4	0.700	1.0300	2.71	0.04084	0.684	1.98	0.130×10^8
260.0	533.2	0.662	1.0341	2.80	0.04258	0.680	1.87	0.104×10^8

A.3-3 Physical Properties of Air at 101.325 kPa (1 Atm Abs), English Units

T (°F)	ρ $\left(\dfrac{lb_m}{ft^3}\right)$	c_p $\left(\dfrac{btu}{lb_m \cdot °F}\right)$	μ (centipoise)	k $\left(\dfrac{btu}{h \cdot ft \cdot °F}\right)$	N_{Pr}	$\beta \times 10^3$ (1/°R)	$g\beta\rho^2/\mu^2$ (1/°R·ft³)
0	0.0861	0.240	0.0162	0.0130	0.720	2.18	4.39×10^6
32	0.0807	0.240	0.0172	0.0140	0.715	2.03	3.21×10^6
50	0.0778	0.240	0.0178	0.0144	0.713	1.96	2.70×10^6
100	0.0710	0.240	0.0190	0.0156	0.705	1.79	1.76×10^6
150	0.0651	0.241	0.0203	0.0169	0.702	1.64	1.22×10^6
200	0.0602	0.241	0.0215	0.0180	0.694	1.52	0.840×10^6
250	0.0559	0.242	0.0227	0.0192	0.692	1.41	0.607×10^6
300	0.0523	0.243	0.0237	0.0204	0.689	1.32	0.454×10^6
350	0.0490	0.244	0.0250	0.0215	0.687	1.23	0.336×10^6
400	0.0462	0.245	0.0260	0.0225	0.686	1.16	0.264×10^6
450	0.0437	0.246	0.0271	0.0236	0.674	1.10	0.204×10^6
500	0.0413	0.247	0.0280	0.0246	0.680	1.04	0.163×10^6

Geankoplis, 4th edition

NOTE: Equate the label to the provided quantity in the supplied units. For example, for <u>air</u> at $0°C$:

$$\mu \times 10^5 = 1.72 \ Pa \ s$$

$$\mu = 1.72 \times 10^{-5} Pa \ s$$

Heat Transfer Data Correlations for Examinations

CM3110 Transport Phenomena I
Michigan Technological University
Professor Faith A. Morrison
1 December 2020

I. Forced Convection Through Pipes

In forced convection, we determined from dimensional analysis that the Nusselt number is a function of at most Re, Pr, L/D, and viscosity ratio.

Prandtl number
(fluid properties)
$$\text{Pr} = \frac{\hat{c}_p \mu}{k} \qquad (1)$$

In pipe flow with heat transfer taking place, the fluid enters at bulk fluid temperature T_{bi} and exits at T_{bo}. T_w is the temperature of the wall. For Nu data correlations in forced convection through pipes, all fluid material properties except $\mu_w = \mu(T_w)$ are evaluated at the mean bulk temperature. The mean bulk temperature is given by

Mean bulk temperature
$$\overline{T}_b = \frac{T_{bi} + T_{bo}}{2} \qquad (2)$$

A. Laminar Flow in Pipes

Sieder and Tate's correlation (Geankoplis, p260) for laminar flow is

Laminar flow
$$\text{Nu}_a = \frac{h_a D}{k} = 1.86 \left(\text{RePr}\frac{D}{L}\right)^{\frac{1}{3}} \left(\frac{\mu_b}{\mu_w}\right)^{0.14} \qquad (3)$$

$$q = h_a A \Delta T_a \qquad (4)$$

Arithmetic mean driving force
$$\Delta T_a = \frac{(T_w - T_{bi}) + (T_w - T_{bo})}{2} \qquad (5)$$

B. Turbulent Flow in Pipes

Sieder and Tate's correlation (Geankoplis, p261) for turbulent flow is

Turbulent flow
$$\text{Nu}_{lm} = \frac{h_{lm} D}{k} = 0.027 \text{Re}^{0.8}\text{Pr}^{\frac{1}{3}} \left(\frac{\mu_b}{\mu_w}\right)^{0.14} \qquad (6)$$

$$q = h_{lm} A \Delta T_{lm} \qquad (7)$$

Log mean driving force
$$\Delta T_{lm} = \frac{\Delta T_{w-bi} - \Delta T_{w-bo}}{ln\left(\frac{\Delta T_{w-bi}}{\Delta T_{w-bo}}\right)} \qquad (8)$$

II. Forced Convection Around the Outside of a Cylinder

In heat transfer taking place between a fluid at bulk temperature T_b flowing perpendicular to a cylinder with wall temperature T_w, the material properties are evaluated at the film temperature.

Film temperature
$$T_f = \frac{T_w + T_b}{2} \qquad (9)$$

The data correlation for Nusselt number in this case is

Outside Cylinder
$$\text{Nu} = \frac{hD}{k} = C\text{Re}^m \text{Pr}^{\frac{1}{3}} \qquad (10)$$

Wall-bulk driving force
$$q = hA(T_w - T_b) \qquad (11)$$

The values of C and m depend on the Reynolds number (Geankoplis, Table 4.6-1, p272). These values are valid for Pr > 0.6.

Re	m	C
$1 - 4$	0.330	0.989
$4 - 40$	0.385	0.911
$40 - 4,000$	0.466	0.683
$4,000 - 4 \times 10^4$	0.618	0.193
$4 \times 10^4 - 2.5 \times 10^5$	0.805	0.0266

III. Natural Convection from Various Geometries

Natural convection heat transfer coefficients from various surfaces have been found by dimensional analysis and experimentally to correlate as follows:

Natural convection (various geometries)

$$\text{Nu} = \frac{hL}{k} = a(\text{Gr Pr})^m \qquad (12)$$

Grashof number

$$\text{Gr} \equiv \frac{L^3 \rho^2 g \beta \Delta T}{\mu^2} \qquad (13)$$

The values for a and m depend on the geometry; values may be found in Geankoplis in Table 4.7-1 (p278, shown below). Table 4.7-2 (p280, next pages) provides simplified versions of the correlations specialized to common fluids (air, water, organic liquids).

TABLE 4.7-1. *Constants for Use with Eq. (4.7-4) for Natural Convection*

Physical Geometry	$N_{Gr}N_{Pr}$	a	m	Ref.
Vertical planes and cylinders [vertical height $L < 1$ m (3 ft)]				
	$<10^4$	1.36	$\frac{1}{5}$	(P3)
	10^4–10^9	0.59	$\frac{1}{4}$	(M1)
	$>10^9$	0.13	$\frac{1}{3}$	(M1)
Horizontal cylinders [diameter D used for L and $D < 0.20$ m (0.66 ft)]				
	$<10^{-5}$	0.49	0	(P3)
	10^{-5}–10^{-3}	0.71	$\frac{1}{25}$	(P3)
	10^{-3}–1	1.09	$\frac{1}{10}$	(P3)
	1–10^4	1.09	$\frac{1}{5}$	(P3)
	10^4–10^9	0.53	$\frac{1}{4}$	(M1)
	$>10^9$	0.13	$\frac{1}{3}$	(P3)
Horizontal plates				
Upper surface of heated plates or lower surface of cooled plates	10^5–2×10^7	0.54	$\frac{1}{4}$	(M1)
	2×10^7–3×10^{10}	0.14	$\frac{1}{3}$	(M1)
Lower surface of heated plates or upper surface of cooled plates	10^5–10^{11}	0.58	$\frac{1}{5}$	(F1)

TABLE 4.7-2. *Simplified Equations for Natural Convection from Various Surfaces*

Physical Geometry	$N_{Gr}N_{Pr}$	Equation		Ref.
		$h = btu/h \cdot ft^2 \cdot °F$ $L = ft, \Delta T = °F$ $D = ft$	$h = W/m^2 \cdot K$ $L = m, \Delta T = K$ $D = m$	
Air at 101.32 kPa (1 atm) abs pressure				
Vertical planes and cylinders	10^4–10^9	$h = 0.28(\Delta T/L)^{1/4}$	$h = 1.37(\Delta T/L)^{1/4}$	(P1)
	$>10^9$	$h = 0.18(\Delta T)^{1/3}$	$h = 1.24\,\Delta T^{1/3}$	(P1)
Horizontal cylinders	10^3–10^9	$h = 0.27(\Delta T/D)^{1/4}$	$h = 1.32(\Delta T/D)^{1/4}$	(M1)
	$>10^9$	$h = 0.18(\Delta T)^{1/3}$	$h = 1.24\,\Delta T^{1/3}$	(M1)
Horizontal plates				
Heated plate facing upward or cooled plate facing downward	10^5–2×10^7	$h = 0.27(\Delta T/L)^{1/4}$	$h = 1.32(\Delta T/L)^{1/4}$	(M1)
	2×10^7–3×10^{10}	$h = 0.22(\Delta T)^{1/3}$	$h = 1.52\,\Delta T^{1/3}$	(M1)
Heated plate facing downward or cooled plate facing upward	3×10^5–3×10^{10}	$h = 0.12(\Delta T/L)^{1/4}$	$h = 0.59(\Delta T/L)^{1/4}$	(M1)
Water at 70°F (294 K)				
Vertical planes and cylinders	10^4–10^9	$h = 26(\Delta T/L)^{1/4}$	$h = 127(\Delta T/L)^{1/4}$	(P1)
Organic liquids at 70°F (294 K)				
Vertical planes and cylinders	10^4–10^9	$h = 12(\Delta T/L)^{1/4}$	$h = 59(\Delta T/L)^{1/4}$	(P1)

Reference: C. J. Geankoplis, Transport Processes and Separation Process Principles, 4th Edition, Prentice Hall, 2003.

Table 1: Emissivity ε of solids (300K)

Material	ε
Aluminum foil	0.04
Asbestos board	0.96
Polished brass	0.03
Cast iron, turned and heated	0.60-0.70
Concrete	0.85
Ice, smooth	0.966
Ice, rough	0.985
Plaster	0.98
Roofing paper	0.91
Sand	0.76
Steel, Oxidized	0.79
Wrought Iron	0.94

Stephan-Boltzman Constant:

$$\sigma = 0.1712 \times 10^{-8}\, \frac{BTU}{h\, ft^2 R^4}$$

$$\sigma = 5.676 \times 10^{-8}\, \frac{W}{m^2 K^4}$$

Reference: Engineering Toolbox, www.engineeringtoolbox.com/emissivity-coefficients-d_447.html

Mechanism	$h, \frac{BTU}{hr\ ft^{2\,o}F}$	$h, \frac{W}{m2K}$
Condensing steam	1000-5000	5700-28,000
Condensing organics	200-500	1100-2800
Boiling liquids	300-5000	1700-28,000
Moving water	50-3000	280-17,000
Moving hydrocarbons	10-300	55-1700
Still air	0.5-4	2.8-23
Moving air	2-10	11.3-55

Reference: C. J. Geankoplis, Magnitude of Some Heat-Transfer Coefficients, page 241

Typical values of the convection heat transfer coefficient. From Incropera et al., Fundamentals of Heat and Mass Transfer, 6th edition, Wiley, 2007.

Process		$h\left(\frac{W}{m^2K}\right)$
Free convection	Gases	2-25
	Liquids	50-1000
Forced convection	Gases	25-250
	Liquids	100-20,000
Convection with phase change	Boiling or condensation	2500-10^5

Slide 1

Unsteady Macroscopic Energy Balance

$$accumulation = input - output$$

Q_{in} = Heat **in**to the chosen macroscopic control volume

$$\frac{d}{dt}\left(U_{sys} + E_{k,sys} + E_{p,sys}\right) = -\Delta H - \Delta E_k - \Delta E_p + Q_{in} + W_{s,on}$$

$Q_{in} = \sum_i q_{in,i}$ comes from a variety of sources:

- Thermal conduction: $q_{in} = -kA\frac{dT}{dx}$
- Convection heat xfer: $q_{in} = hA(T_b - T)$
- Radiation: $q_{in} = \varepsilon\sigma A\left(T_{surroundings}^4 - T_{surface}^4\right)$
- Electric current: $q_{in} = I^2 R_{elec}L$
- Chemical Reaction: $q_{in} = S_{rxn}V_{sys}$

$$S[=]\frac{\text{energy}}{\text{time volume}}$$

Slide 2

Unsteady Macroscopic Energy Balance

$$\frac{d}{dt}\left(U_{sys} + E_{k,sys} + E_{p,sys}\right)$$
$$= -\Delta H - \Delta E_k - \Delta E_p + Q_{in} + W_{s,on}$$

$Q_{in} = \sum_i q_{in,i}$ **comes from a variety of sources:**

- **Thermal conduction:** $q_{in} = -kA\frac{dT}{dx}$

  e.g. device held by bracket; a solid phase that extends through boundaries of control volume

- **Convection heat xfer:** $q_{in} = hA(T_b - T)$

  e.g. device dropped in stirred liquid; forced air stream flows past, natural convection occurs outside system; phase change at boundary

- **Radiation:** $q_{in} = \varepsilon\sigma A\left(T_{surroundings}^4 - T_{surface}^4\right)$

  e.g. device at high temp. exposed to a gas/vacuum; hot enough to produce nat. conv.=possibly hot enough for radiation

  S-B constant: $\sigma = 5.676 \times 10^{-8}\frac{W}{m^2K^4}$

- **Electric current:** $q_{in} = I^2 R_{elec}L$

  e.g. if electric current is flowing within the device/control volume/ system

- **Chemical Reaction:** $q_{in} = S_{rxn}V_{sys}$

  e.g. if a homogeneous reaction is taking place <u>throughout</u> the device/ control volume/system

1D Heat Transfer: Unsteady State Heat Conduction in a Semi-Infinite Slab

$$1 - Y = \left(\frac{T - T_0}{T_1 - T_0}\right)$$

$$\frac{h\sqrt{\alpha t}}{k}$$

$t < 0$
$T = T_o$

$t < 0$
$T = T_o$

$t \geq 0$
$T = T_1$

$t > 0$
$T = T(x,t)$
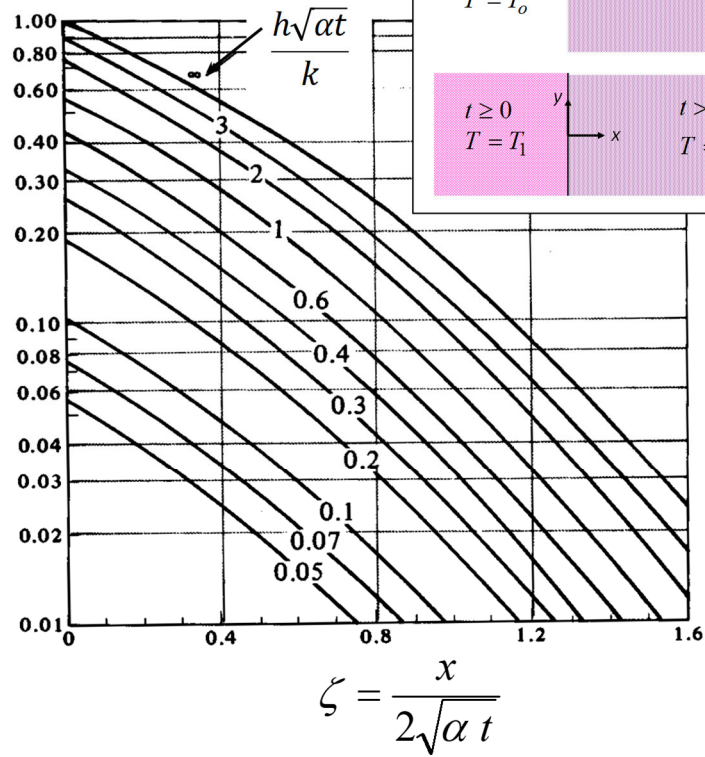
Geankoplis 4th ed.,
Figure 5.3-3, page 364

$$\zeta = \frac{x}{2\sqrt{\alpha t}}$$

- -

Heisler chart (sphere)

$$\text{label} = \frac{k}{hR} = \frac{1}{\text{Bi}}$$

$$= \frac{k}{h x_1}$$

$Y_0, \dfrac{T_1 - T}{T_1 - T_0}$

$$X = \frac{\alpha t}{x_1^2}$$

Initially:

$t < t_0$
$T = T_0$

Suddenly:

$t \geq t_0$
$T = T(t)$
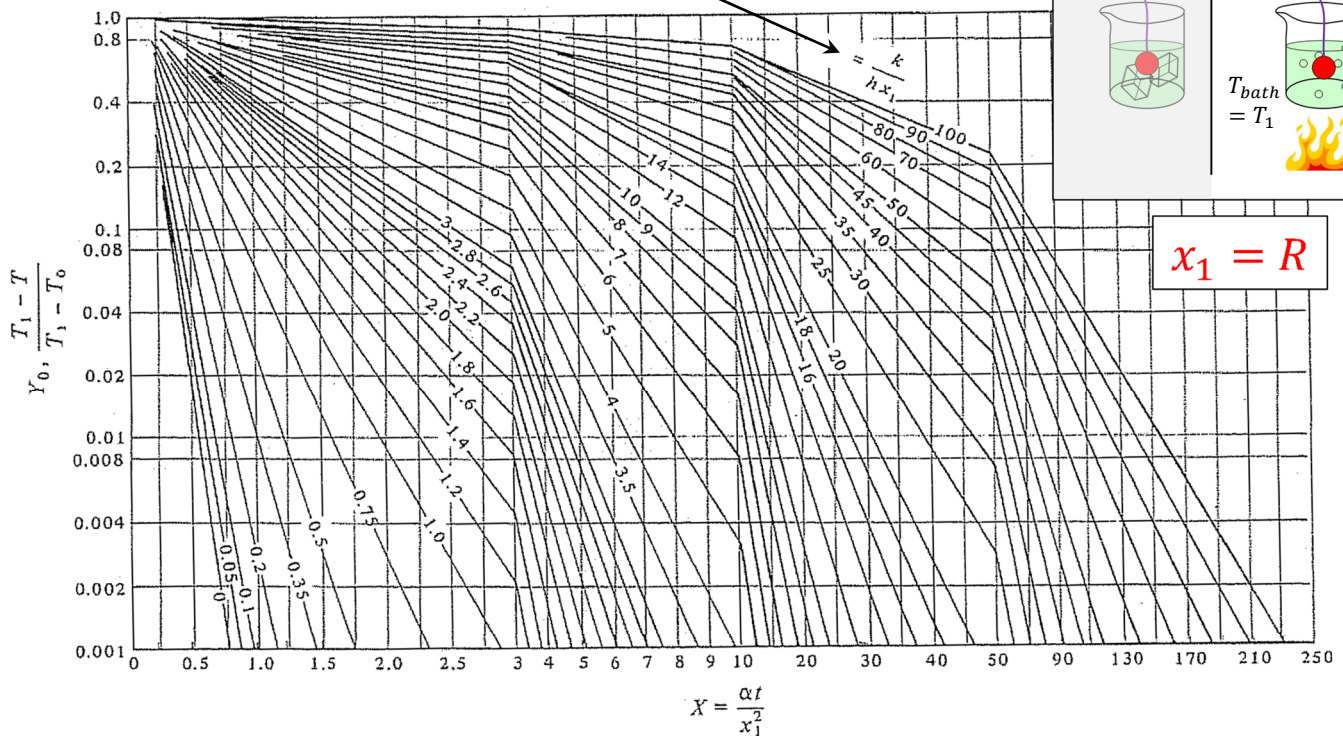
$T_{bath} = T_1$

$$x_1 = R$$

FIGURE 5.3-10. *Chart for determining the temperature at the center of a sphere for unsteady-state heat conduction.* [From H. P. Heisler, Trans. A.S.M.E., 69, 227 (1947). With permission.] *From Geankoplis, 4th edition, page 374*
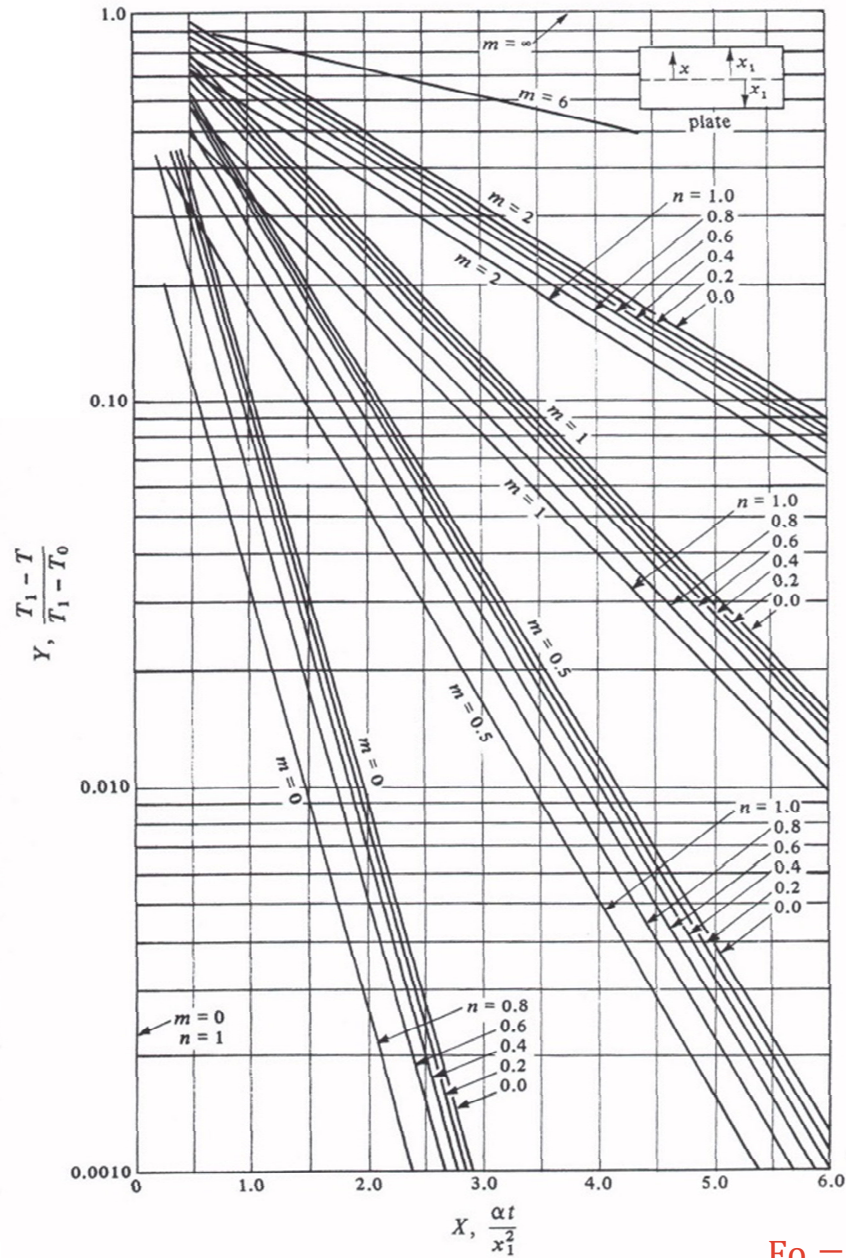
© Faith A. Morrison, Michigan Tech U.

FIGURE 5.3-5. *Unsteady-state heat conduction in a large flat plate.* [From H. P. Gurney and J. Lurie, Ind. Eng. Chem., **15**, 1170 (1923).]
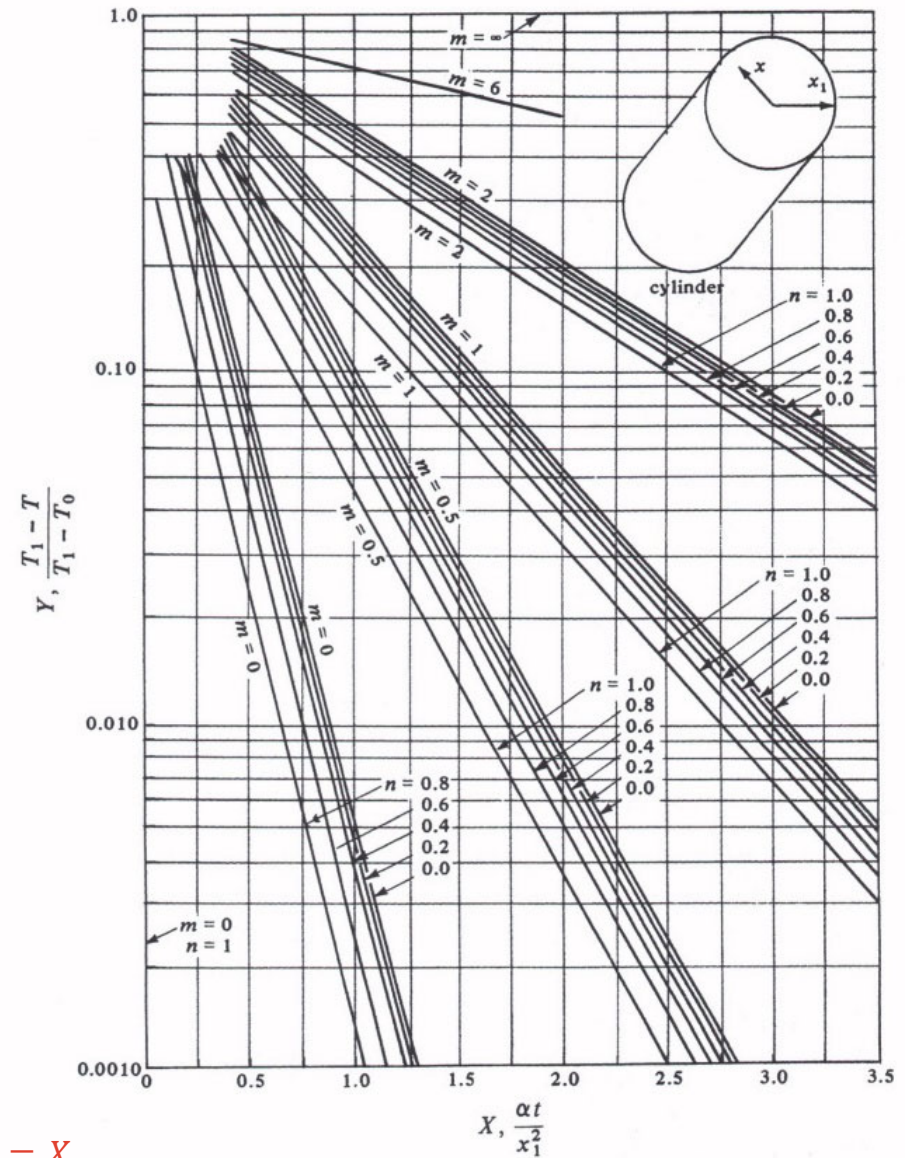
$$\text{Fo} = \frac{\alpha t}{x_1^2} = X$$

FIGURE 5.3-7. *Unsteady-state heat conduction in a long cylinder.* [From H. P. Gurney and J. Lurie, Ind. Eng. Chem., **15**, 1170 (1923).]

A.2-9 Properties of Saturated Steam and Water (Steam Table), SI Units

Temperature (°C)	Vapor Pressure (kPa)	Specific Volume (m³/kg)		Enthalpy (kJ/kg)		Entropy (kJ/kg · K)	
		Liquid	Sat'd Vapor	Liquid	Sat'd Vapor	Liquid	Sat'd Vapor
0.01	0.6113	0.0010002	206.136	0.00	2501.4	0.0000	9.1562
3	0.7577	0.0010001	168.132	12.57	2506.9	0.0457	9.0773
6	0.9349	0.0010001	137.734	25.20	2512.4	0.0912	9.0003
9	1.1477	0.0010003	113.386	37.80	2517.9	0.1362	8.9253
12	1.4022	0.0010005	93.784	50.41	2523.4	0.1806	8.8524
15	1.7051	0.0010009	77.926	62.99	2528.9	0.2245	8.7814
18	2.0640	0.0010014	65.038	75.58	2534.4	0.2679	8.7123
21	2.487	0.0010020	54.514	88.14	2539.9	0.3109	8.6450
24	2.985	0.0010027	45.883	100.70	2545.4	0.3534	8.5794
25	3.169	0.0010029	43.360	104.89	2547.2	0.3674	8.5580
27	3.567	0.0010035	38.774	113.25	2550.8	0.3954	8.5156
30	4.246	0.0010043	32.894	125.79	2556.3	0.4369	8.4533
33	5.034	0.0010053	28.011	138.33	2561.7	0.4781	8.3927
36	5.947	0.0010063	23.940	150.86	2567.1	0.5188	8.3336
40	7.384	0.0010078	19.523	167.57	2574.3	0.5725	8.2570
45	9.593	0.0010099	15.258	188.45	2583.2	0.6387	8.1648
50	12.349	0.0010121	12.032	209.33	2592.1	0.7038	8.0763
55	15.758	0.0010146	9.568	230.23	2600.9	0.7679	7.9913
60	19.940	0.0010172	7.671	251.13	2609.6	0.8312	7.9096
65	25.03	0.0010199	6.197	272.06	2618.3	0.8935	7.8310
70	31.19	0.0010228	5.042	292.98	2626.8	0.9549	7.7553
75	38.58	0.0010259	4.131	313.93	2635.3	1.0155	7.6824
80	47.39	0.0010291	3.407	334.91	2643.7	1.0753	7.6122
85	57.83	0.0010325	2.828	355.90	2651.9	1.1343	7.5445
90	70.14	0.0010360	2.361	376.92	2660.1	1.1925	7.4791
95	84.55	0.0010397	1.9819	397.96	2668.1	1.2500	7.4159
100	101.35	0.0010435	1.6729	419.04	2676.1	1.3069	7.3549
105	120.82	0.0010475	1.4194	440.15	2683.8	1.3630	7.2958
110	143.27	0.0010516	1.2102	461.30	2691.5	1.4185	7.2387
115	169.06	0.0010559	1.0366	482.48	2699.0	1.4734	7.1833
120	198.53	0.0010603	0.8919	503.71	2706.3	1.5276	7.1296
125	232.1	0.0010649	0.7706	524.99	2713.5	1.5813	7.0775
130	270.1	0.0010697	0.6685	546.31	2720.5	1.6344	7.0269
135	313.0	0.0010746	0.5822	567.69	2727.3	1.6870	6.9777
140	316.3	0.0010797	0.5089	589.13	2733.9	1.7391	6.9299
145	415.4	0.0010850	0.4463	610.63	2740.3	1.7907	6.8833
150	475.8	0.0010905	0.3928	632.20	2746.5	1.8418	6.8379
155	543.1	0.0010961	0.3468	653.84	2752.4	1.8925	6.7935
160	617.8	0.0011020	0.3071	675.55	2758.1	1.9427	6.7502
165	700.5	0.0011080	0.2727	697.34	2763.5	1.9925	6.7078
170	791.7	0.0011143	0.2428	719.21	2768.7	2.0419	6.6663
175	892.0	0.0011207	0.2168	741.17	2773.6	2.0909	6.6256
180	1002.1	0.0011274	0.19405	763.22	2778.2	2.1396	6.5857
190	1254.4	0.0011414	0.15654	807.62	2786.4	2.2359	6.5079
200	1553.8	0.0011565	0.12736	852.45	2793.2	2.3309	6.4323
225	2548	0.0011992	0.07849	966.78	2803.3	2.5639	6.2503
250	3973	0.0012512	0.05013	1085.36	2801.5	2.7927	6.0730
275	5942	0.0013168	0.03279	1210.07	2785.0	3.0208	5.8938
300	8581	0.0010436	0.02167	1344.0	2749.0	3.2534	5.7045

Source: Abridged from J. H. Keenan, F. G. Keyes, P. G. Hill, and J. G. Moore, *Steam Tables—Metric Units.* New York: John Wiley & Sons, Inc., 1969. Reprinted by permission of John Wiley & Sons, Inc.

Typo in value of α_{Cu} corrected, 24Feb2019.

Appendix H

Physical Properties of Solids

Material	ρ (lb$_m$/ft³) (68°F)	ρ (kg/m³) (293 K)	c_p (Btu/lb$_m$°F) (293 K)	c_p (J/kg·1K) ×10⁻² (293K)	α (ft²/h) (68°F)	α (m²/s)·10⁵ (293k)	k (Btu/h ft°F) °F (68)	k (212)	k (572)	(W/m·K) K (293)	(373)	(573)
Metals												
Aluminum	168.6	2,701.1	0.224	9.383	3.55	9.16	132	133	133	229	229	230
Copper	555	8,890	0.092	3.854	3.98	11.27	223	219	213	386	379	369
Gold	1206	19,320	0.031	1.299	4.52	11.66	169	170	172	293	294	298
Iron	492	7,880	0.122	5.110	0.83	2.14	42.3	39	31.6	73.2	68	54
Lead	708	11,300	0.030	1.257	0.80	2.06	20.3	19.3	17.2	35.1	33.4	29.8
Magnesium	109	1,750	0.248	10.39	3.68	9.50	99.5	96.8	91.4	172	168	158
Nickel	556	8,910	0.111	4.560	0.87	2.24	53.7	47.7	36.9	93.0	82.6	63.9
Platinum	1340	21,500	0.032	1.340	0.09	0.23	40.5	41.9	43.5	70.1	72.5	75.3
Silver	656	10,500	0.057	2.388	6.42	16.57	240	237	209	415	410	362
Tin	450	7,210	0.051	2.136	1.57	4.05	36	34	—	62	59	—
Tungsten	1206	19,320	0.032	1.340	2.44	6.30	94	87	77	160	150	130
Uranium	1167	18,700	0.027	1.131	0.53	1.37	16.9	17.2	19.6	29.3	29.8	33.9
Zinc	446	7,150	0.094	3.937	1.55	4.00	65	63	58	110	110	100
Alloys												
Aluminum 2024	173	2,770	0.230	9.634	1.76	4.54	70.2			122		
Brass (70% Cu, 30% Ni)	532	8,520	0.091	3.812	1.27	3.28	61.8	73.9	85.3	107	128	148
Constantan (60% Cu, 40% Ni)	557	8,920	0.098	4.105	0.24	0.62	13.1	15.4		22.7	26.7	
Iron, cast	455	7,920	0.100	4.189	0.65	1.68	29.6	26.8		51.2	46.4	
Nichrome V	530	8,490	0.106	4.440	0.12	0.31	7.06	7.99	9.94	12.2	13.8	17.2
Stainless steel	488	7,820	0.110	4.608	0.17	0.44	9.4	10.0	13	16	17.3	23
Steel, mild (1% C)	488	7,820	0.113	4.733	0.45	1.16	24.8	24.8	22.9	42.9	42.9	39.0
Nonmetals												
Asbestos	36	580	0.25	10.5			0.092	0.11	0.125	0.159	0.190	0.21
Brick (fire clay)	144	2,310	0.22	9.22				0.65			1.13	
Brick (masonry)	106	1,670	0.20	8.38			0.38			0.66		
Brick (chrome)	188	3,010	0.20	8.38				0.67			1.16	
Concrete	144	2,310	0.21	8.80			0.70			1.21		
Corkboard	10	160	0.4	17			0.025			0.043		
Diatomaceous earth, powdered	14	220	0.2	8.4			0.03			0.05		
Glass, window	170	2,720	0.2	8.4			0.45			0.78		
Glass, Pyrex	140	2,240	0.2	8.4			0.63	0.67	0.84	1.09	1.16	1.45
Kaolin firebrick	19	300							0.052			0.09
85% Magnesia	17	270					0.038	0.041		0.066	0.071	
Sandy loam, 4% H$_2$O	104	1,670	0.4	17			0.54			0.94		
Sandy loam, 10% H$_2$O	121	1,940					1.08			1.87		
Rock wool	10	160	0.2	8.4			0.023	0.033		0.040	0.057	
Wood, oak ⊥ to grain	51	820	0.57	23.9			0.12			0.21		
Wood, oak ∥ to grain	51	820	0.57	23.9			0.23			0.40		

TABLE 6.2-1. *Diffusion Coefficients of Gases at 101.32 kPa Pressure*

System	Temperature °C	K	Diffusivity $[(m^2/s)10^4$ or $cm^2/s]$	Ref.
Air–NH$_3$	0	273	0.198	(W1)
Air–H$_2$O	0	273	0.220	(N2)
	25	298	0.260	(L1)
	42	315	0.288	(M1)
Air–CO$_2$	3	276	0.142	(H1)
	44	317	0.177	
Air–H$_2$	0	273	0.611	(N2)
Air–C$_2$H$_5$OH	25	298	0.135	(M1)
	42	315	0.145	
Air–CH$_3$COOH	0	273	0.106	(N2)
Air–*n*-hexane	21	294	0.080	(C1)
Air–benzene	25	298	0.0962	(L1)
Air–toluene	25.9	298.9	0.086	(G1)
Air–*n*-butanol	0	273	0.0703	(N2)
	25.9	298.9	0.087	
H$_2$–CH$_4$	25	298	0.726	(C2)
H$_2$–N$_2$	25	298	0.784	(B1)
	85	358	1.052	
H$_2$–benzene	38.1	311.1	0.404	(H2)
H$_2$–Ar	22.4	295.4	0.83	(W2)
H$_2$–NH$_3$	25	298	0.783	(B1)
H$_2$–SO$_2$	50	323	0.61	(S1)
H$_2$–C$_2$H$_5$OH	67	340	0.586	(T1)
He–Ar	25	298	0.729	(S2)
He–*n*-butanol	150	423	0.587	(S2)
He–air	44	317	0.765	(H1)
He–CH$_4$	25	298	0.675	(C2)
He–N$_2$	25	298	0.687	(S2)
He–O$_2$	25	298	0.729	(S2)
Ar–CH$_4$	25	298	0.202	(C2)
CO$_2$–N$_2$	25	298	0.167	(W3)
CO$_2$–O$_2$	20	293	0.153	(W4)
N$_2$–*n*-butane	25	298	0.0960	(B2)
H$_2$O–CO$_2$	34.3	307.3	0.202	(S3)
CO–N$_2$	100	373	0.318	(A1)
CH$_3$Cl–SO$_2$	30	303	0.0693	(C3)
(C$_2$H$_5$)$_2$O–NH$_3$	26.5	299.5	0.1078	(S4)

From Geankoplis, 4th Edition, Prentice Hall, 2003

TABLE 6.3-1. *Diffusion Coefficients for Dilute Liquid Solutions*

Solute	Solvent	Temperature °C	K	Diffusivity $[(m^2/s)10^9$ or $(cm^2/s)10^5]$	Ref.
NH_3	Water	12	285	1.64	(N2)
		15	288	1.77	
O_2	Water	18	291	1.98	(N2)
		25	298	2.41	(V1)
CO_2	Water	25	298	2.00	(V1)
H_2	Water	25	298	4.8	(V1)
Methyl alcohol	Water	15	288	1.26	(J1)
Ethyl alcohol	Water	10	283	0.84	(J1)
		25	298	1.24	(J1)
n-Propyl alcohol	Water	15	288	0.87	(J1)
Formic acid	Water	25	298	1.52	(B4)
Acetic acid	Water	9.7	282.7	0.769	(B4)
		25	298	1.26	(B4)
Propionic acid	Water	25	298	1.01	(B4)
HCl (9 g mol/liter)	Water	10	283	3.3	(N2)
(2.5 g mol/liter)		10	283	2.5	(N2)
Benzoic acid	Water	25	298	1.21	(C4)
Acetone	Water	25	298	1.28	(A2)
Acetic acid	Benzene	25	298	2.09	(C5)
Urea	Ethanol	12	285	0.54	(N2)
Water	Ethanol	25	298	1.13	(H4)
KCl	Water	25	298	1.870	(P2)
KCl	Ethylene glycol	25	298	0.119	(P2)

TABLE 6.4-1. *Diffusion Coefficients for Dilute Biological Solutes in Aqueous Solution*

Solute	Temperature °C	K	Diffusivity (m^2/s)	Molecular Weight	Ref.
Urea	20	293	1.20×10^{-9}	60.1	(N2)
	25	298	1.378×10^{-9}		(G5)
Glycerol	20	293	0.825×10^{-9}	92.1	(G3)
Glycine	25	298	1.055×10^{-9}	75.1	(L3)
Sodium caprylate	25	298	8.78×10^{-10}	166.2	(G6)
Bovine serum albumin	25	298	6.81×10^{-11}	67 500	(C6)
Urease	25	298	4.01×10^{-11}	482 700	(C7)
	20	293	3.46×10^{-11}		(S6)
Soybean protein	20	293	2.91×10^{-11}	361 800	(S6)
Lipoxidase	20	293	5.59×10^{-11}	97 440	(S6)
Fibrinogen, human	20	293	1.98×10^{-11}	339 700	(S6)
Human serum albumin	20	293	5.93×10^{-11}	72 300	(S6)
γ-Globulin, human	20	293	4.00×10^{-11}	153 100	(S6)
Creatinine	37	310	1.08×10^{-9}	113.1	(C8)
Sucrose	37	310	0.697×10^{-9}	342.3	(C8)
	20	293	0.460×10^{-9}		(P3)

ATOMIC WEIGHTS AND NUMBERS

Atomic weights apply to naturally occurring isotopic compositions and are based on an atomic mass of $^{12}C = 12$

Element	Symbol	Atomic Number	Atomic Weight	Element	Symbol	Atomic Number	Atomic Weight
Actinium	Ac	89		Iridium	Ir	77	192.2
Aluminum	Al	13	26.9815	Iron	Fe	26	55.847
Americium	Am	95		Krypton	Kr	36	83.80
Antimony	Sb	51	121.75	Lanthanum	La	57	138.91
Argon	Ar	18	39.948	Lawrencium	Lr	103	
Arsenic	As	33	74.9216	Lead	Pb	82	207.19
Astatine	At	85		Lithium	Li	3	6.939
Barium	Ba	56	137.34	Lutetium	Lu	71	174.97
Berkelium	Bk	97		Magnesium	Mg	12	24.312
Beryllium	Be	4	9.0122	Manganese	Mn	25	54.9380
Bismuth	Bi	83	208.980	Mendelevium	Md	101	
Boron	B	5	10.811	Mercury	Hg	80	200.59
Bromine	Br	35	79.904	Molybdenum	Mo	42	95.94
Cadmium	Cd	48	112.40	Neodymium	Nd	60	144.24
Calcium	Ca	20	40.08	Neon	Ne	10	20.183
Californium	Cf	98		Neptunium	Np	93	
Carbon	C	6	12.01115	Nickel	Ni	28	58.71
Cerium	Ce	58	140.12	Niobium	Nb	41	92.906
Cesium	Cs	55	132.905	Nitrogen	N	7	14.0067
Chlorine	Cl	17	35.453	Nobelium	No	102	
Chromium	Cr	24	51.996	Osmium	Os	75	190.2
Cobalt	Co	27	58.9332	Oxygen	O	8	15.9994
Copper	Cu	29	63.546	Palladium	Pd	46	106.4
Curium	Cm	96		Phosphorus	P	15	30.9738
Dysprosium	Dy	66	162.50	Platinum	Pt	78	195.09
Einsteinium	Es	99		Plutonium	Pu	94	
Erbium	Er	68	167.26	Polonium	Po	84	
Europium	Eu	63	151.96	Potassium	K	19	39.102
Fermium	Fm	100		Praseodymium	Pr	59	140.907
Fluorine	F	9	18.9984	Promethium	Pm	61	
Francium	Fr	87		Protactinium	Pa	91	
Gadolinium	Gd	64	157.25	Radium	Ra	88	
Gallium	Ga	31	69.72	Radon	Rn	86	
Germanium	Ge	32	72.59	Rhenium	Re	75	186.2
Gold	Au	79	196.967	Rhodium	Rh	45	102.905
Hafnium	Hf	72	178.49	Rubidium	Rb	37	84.57
Helium	He	2	4.0026	Ruthenium	Ru	44	101.07
Holmium	Ho	67	164.930	Samarium	Sm	62	150.35
Hydrogen	H	1	1.00797	Scandium	Sc	21	44.956
Indium	In	49	114.82	Selenium	Se	34	78.96
Iodine	I	53	126.9044	Silicon	Si	14	28.086
Silver	Ag	47	107.868	Tin	Sn	50	118.69
Sodium	Na	11	22.9898	Titanium	Ti	22	47.90
Strontium	Sr	38	87.62	Tungsten	W	74	183.85
Sulfur	S	16	32.064	Uranium	U	92	238.03
Tantalum	Ta	73	180.948	Vanadium	V	23	50.942
Technetium	Tc	43	—	Xenon	Xe	54	131.30
Tellurium	Te	52	127.60	Ytterbium	Yb	70	173.04
Terbium	Tb	65	158.924	Yttrium	Y	39	88.905
Thallium	Tl	81	204.37	Zinc	Zn	30	65.37
Thorium	Th	90	232.038	Zirconium	Zr	40	91.22
Thulium	Tm	69	168.934				

FRONT OF PURPLE

The **Equation of Species Mass Balance** in Cartesian, cylindrical, and spherical coordinates for binary mixtures of A and B. Two cases are presented: the general case, where the mass flux with respect to mass-average velocity ($\underline{j}_A$) appears (p. 1), and the more usual case (p. 2), where the diffusion coefficient is constant and Fick's law has been incorporated.

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Microscopic species mass balance, in terms of mass flux; Gibbs notation

$$\rho\left(\frac{\partial \omega_A}{\partial t} + \underline{v}\cdot\nabla\omega_A\right) = -\nabla\cdot\underline{j}_A + r_A$$

Microscopic species mass balance, in terms of mass flux; Cartesian coordinates

$$\rho\left(\frac{\partial \omega_A}{\partial t} + v_x\frac{\partial \omega_A}{\partial x} + v_y\frac{\partial \omega_A}{\partial y} + v_z\frac{\partial \omega_A}{\partial z}\right) = -\left(\frac{\partial j_{A,x}}{\partial x} + \frac{\partial j_{A,y}}{\partial y} + \frac{\partial j_{A,z}}{\partial z}\right) + r_A$$

Microscopic species mass balance, in terms of mass flux; cylindrical coordinates

$$\rho\left(\frac{\partial \omega_A}{\partial t} + v_r\frac{\partial \omega_A}{\partial r} + \frac{v_\theta}{r}\frac{\partial \omega_A}{\partial \theta} + v_z\frac{\partial \omega_A}{\partial z}\right) = -\left(\frac{1}{r}\frac{\partial (rj_{A,r})}{\partial r} + \frac{1}{r}\frac{\partial j_{A,\theta}}{\partial \theta} + \frac{\partial j_{A,z}}{\partial z}\right) + r_A$$

Microscopic species mass balance, in terms of mass flux; spherical coordinates

$$\rho\left(\frac{\partial \omega_A}{\partial t} + v_r\frac{\partial \omega_A}{\partial r} + \frac{v_\theta}{r}\frac{\partial \omega_A}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial \omega_A}{\partial \phi}\right) = -\left(\frac{1}{r^2}\frac{\partial (r^2 j_{A,r})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (j_{A,\theta}\sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial j_{A,\phi}}{\partial \phi}\right) + r_A$$

Fick's law of diffusion, Gibbs notation: $\underline{j}_A = -\rho D_{AB}\nabla\omega_A$

$$= \rho\omega_A(\underline{v}_A - \underline{v})$$

Fick's law of diffusion, Cartesian coordinates:
$$\begin{pmatrix} j_{A,x} \\ j_{A,y} \\ j_{A,z} \end{pmatrix}_{xyz} = \begin{pmatrix} -\rho D_{AB}\frac{\partial \omega_A}{\partial x} \\ -\rho D_{AB}\frac{\partial \omega_A}{\partial y} \\ -\rho D_{AB}\frac{\partial \omega_A}{\partial z} \end{pmatrix}_{xyz}$$

Fick's law of diffusion, cylindrical coordinates:
$$\begin{pmatrix} j_{A,r} \\ j_{A,\theta} \\ j_{A,z} \end{pmatrix}_{r\theta z} = \begin{pmatrix} -\rho D_{AB}\frac{\partial \omega_A}{\partial r} \\ -\frac{\rho D_{AB}}{r}\frac{\partial \omega_A}{\partial \theta} \\ -\rho D_{AB}\frac{\partial \omega_A}{\partial z} \end{pmatrix}_{r\theta z}$$

Fick's law of diffusion, spherical coordinates:
$$\begin{pmatrix} j_{A,r} \\ j_{A,\theta} \\ j_{A,\phi} \end{pmatrix}_{r\theta\phi} = \begin{pmatrix} -\rho D_{AB}\frac{\partial \omega_A}{\partial r} \\ -\frac{\rho D_{AB}}{r}\frac{\partial \omega_A}{\partial \theta} \\ -\frac{\rho D_{AB}}{r\sin\theta}\frac{\partial \omega_A}{\partial \phi} \end{pmatrix}_{r\theta\phi}$$

The **Equation of Species Mass Balance, constant ρD_{AB}.** For binary systems, and Fick's law has been incorporated. Good for dilute liquid solutions at constant temperature and pressure.

Microscopic species mass balance, constant thermal conductivity; Gibbs notation

$$\rho\left(\frac{\partial \omega_A}{\partial t} + \underline{v} \cdot \nabla \omega_A\right) = \rho D_{AB}\nabla^2\omega_A + r_A$$

Microscopic species mass balance, constant thermal conductivity; Cartesian coordinates

$$\rho\left(\frac{\partial \omega_A}{\partial t} + v_x\frac{\partial \omega_A}{\partial x} + v_y\frac{\partial \omega_A}{\partial y} + v_z\frac{\partial \omega_A}{\partial z}\right) = \rho D_{AB}\left(\frac{\partial^2 \omega_A}{\partial x^2} + \frac{\partial^2 \omega_A}{\partial y^2} + \frac{\partial^2 \omega_A}{\partial z^2}\right) + r_A$$

Microscopic species mass balance, constant thermal conductivity; cylindrical coordinates

$$\rho\left(\frac{\partial \omega_A}{\partial t} + v_r\frac{\partial \omega_A}{\partial r} + \frac{v_\theta}{r}\frac{\partial \omega_A}{\partial \theta} + v_z\frac{\partial \omega_A}{\partial z}\right) = \rho D_{AB}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \omega_A}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \omega_A}{\partial \theta^2} + \frac{\partial^2 \omega_A}{\partial z^2}\right) + r_A$$

Microscopic species mass balance, constant thermal conductivity; spherical coordinates

$$\rho\left(\frac{\partial \omega_A}{\partial t} + v_r\frac{\partial \omega_A}{\partial r} + \frac{v_\theta}{r}\frac{\partial \omega_A}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial \omega_A}{\partial \phi}\right)$$
$$= \rho D_{AB}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \omega_A}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \omega_A}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 \omega_A}{\partial \phi^2}\right) + r_A$$

In terms of Diffusivity, D_{AB}

$$cx_A = c_A = \frac{1}{M_A}(\rho_A) = \frac{1}{M_A}(\rho\omega_A) \qquad \left(\text{units: } c[=]\frac{mol\ mix}{vol\ soln}; \rho[=]\frac{mass\ mix}{vol\ soln}; c_A[=]\frac{mol\ A}{vol\ soln}; \rho_A[=]\frac{mass\ A}{vol\ soln}\right)$$

$\underline{J_A} \equiv$ mass flux of species A relative to a mixture's mass average velocity, $\underline{v}$ $\qquad$ (units: $\underline{J_A}[=]\frac{mass\ A}{area\cdot time}$)

$\qquad = \rho_A(\underline{v_A} - \underline{v})$

$\underline{J_A} + \underline{J_B} = 0$, i.e. these fluxes are measured relative to the mixture's center of mass

$\underline{n_A} \equiv \rho_A\underline{v_A} = \underline{J_A} + \rho_A\underline{v} =$ combined mass flux relative to stationary coordinates

$\underline{n_A} + \underline{n_B} = \rho\underline{v}$

$\underline{v_A} \equiv$ velocity of species A in a mixture, i.e. average velocity of all molecules of species A within a small volume

$\underline{v} = \omega_A\underline{v_A} + \omega_B\underline{v_B} \equiv$ mass average velocity; same velocity as in the microscopic momentum and energy balances

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Reference: R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena*, 2nd edition, Wiley, 2002.
(p. 515, 584)

The **Equation of Species Mass Balance in Terms of** <mark>**Combined Molar**</mark> **quantities** in Cartesian, cylindrical, and spherical coordinates for binary mixtures of A and B.

The general case, where the combined molar flux with respect to molar velocity ($\underline{N}_A$), is given on page 1.

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<div style="text-align:center">**In terms of total molar flux, $\underline{N}_A$**</div>

Microscopic species mass balance, in terms of molar flux; Gibbs notation

$$\frac{\partial c_A}{\partial t} = -\nabla \cdot \underline{N}_A + R_A$$

Microscopic species mass balance, in terms of combined molar flux; Cartesian coordinates

$$\frac{\partial c_A}{\partial t} = -\left(\frac{\partial N_{A,x}}{\partial x} + \frac{\partial N_{A,y}}{\partial y} + \frac{\partial N_{A,z}}{\partial z}\right) + R_A$$

Microscopic species mass balance, in terms of combined molar flux; cylindrical coordinates

$$\frac{\partial c_A}{\partial t} = -\left(\frac{1}{r}\frac{\partial (r N_{A,r})}{\partial r} + \frac{1}{r}\frac{\partial N_{A,\theta}}{\partial \theta} + \frac{\partial N_{A,z}}{\partial z}\right) + R_A$$

Microscopic species mass balance, in terms of combined molar flux; spherical coordinates

$$\frac{\partial c_A}{\partial t} = -\left(\frac{1}{r^2}\frac{\partial (r^2 N_{A,r})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (N_{A,\theta}\sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial N_{A,\phi}}{\partial \phi}\right) + R_A$$

Fick's law of diffusion, Gibbs notation: $\underline{N}_A = x_A(\underline{N}_A + \underline{N}_B) - cD_{AB}\nabla x_A$

$$= c_A \underline{v}^* - cD_{AB}\nabla x_A$$

Fick's law of diffusion, Cartesian coordinates:
$$\begin{pmatrix} N_{A,x} \\ N_{A,y} \\ N_{A,z} \end{pmatrix}_{xyz} = \begin{pmatrix} x_A(N_{A,x} + N_{B,x}) - cD_{AB}\frac{\partial x_A}{\partial x} \\ x_A(N_{A,y} + N_{B,y}) - cD_{AB}\frac{\partial x_A}{\partial y} \\ x_A(N_{A,z} + N_{B,z}) - cD_{AB}\frac{\partial x_A}{\partial z} \end{pmatrix}_{xyz}$$

Fick's law of diffusion, cylindrical coordinates:
$$\begin{pmatrix} N_{A,r} \\ N_{A,\theta} \\ N_{A,z} \end{pmatrix}_{r\theta z} = \begin{pmatrix} x_A(N_{A,r} + N_{B,r}) - cD_{AB}\frac{\partial x_A}{\partial r} \\ x_A(N_{A,\theta} + N_{B,\theta}) - \frac{cD_{AB}}{r}\frac{\partial x_A}{\partial \theta} \\ x_A(N_{A,z} + N_{B,z}) - cD_{AB}\frac{\partial x_A}{\partial z} \end{pmatrix}_{r\theta z}$$

Fick's law of diffusion, spherical coordinates:
$$\begin{pmatrix} N_{A,r} \\ N_{A,\theta} \\ N_{A,\phi} \end{pmatrix}_{r\theta\phi} = \begin{pmatrix} x_A(N_{A,r} + N_{B,r}) - cD_{AB}\frac{\partial x_A}{\partial r} \\ x_A(N_{A,\theta} + N_{B,\theta}) - \frac{cD_{AB}}{r}\frac{\partial x_A}{\partial \theta} \\ x_A(N_{A,\phi} + N_{B,\phi}) - \frac{cD_{AB}}{r\sin\theta}\frac{\partial x_A}{\partial \phi} \end{pmatrix}_{r\theta\phi}$$

NOTES:

- If component A has no sink, $\underline{N}_A = 0$.
- If A diffuses through stagnant B, $\underline{N}_B = 0$.
- If a binary mixture of A and B are undergoing steady equimolar counterdiffusion, $\underline{N}_A = -\underline{N}_B$.
- If, for example, two moles of A diffuse to a surface at which a rapid, irreversible reaction coverts it to one mole of B, then at steady state $-0.5\underline{N}_A = \underline{N}_B$.

$$cx_A = c_A = \frac{1}{M_A}(\rho_A) = \frac{1}{M_A}(\rho\omega_A) \qquad \left(\text{units: } c[=]\frac{mol\ mix}{vol\ soln}; \rho[=]\frac{mass\ mix}{vol\ soln}; c_A[=]\frac{mol\ A}{vol\ soln}; \rho_A[=]\frac{mass\ A}{vol\ soln}\right)$$

$\underline{J}_A^* \equiv$ molar flux relative to a mixture's molar average velocity, $\underline{v}^*$ $\qquad \left(\text{units: } \underline{J}_A^*[=]\frac{mole\ A}{area\cdot time}\right)$

$\quad = c_A(\underline{v}_A - \underline{v}^*)$

$\underline{J}_A^* + \underline{J}_B^* = 0$

$\underline{N}_A \equiv c_A\underline{v}_A = \underline{J}_A^* + c_A\underline{v}^* =$ combined molar flux relative to stationary coordinates

$\underline{N}_A + \underline{N}_B = c\underline{v}^*$

$\underline{v}_A \equiv$ velocity of species A in a mixture, i.e. average velocity of all molecules of species A within a small volume

$\underline{v}^* = x_A\underline{v}_A + x_B\underline{v}_B \equiv$ molar average velocity

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Reference: R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena*, 2nd edition, Wiley, 2002.

(p. 515, 584)

The **Equation of Species Mass Balance in Terms of <mark>Molar</mark> quantities** in Cartesian, cylindrical, and spherical coordinates for binary mixtures of A and B. Two cases are presented: the general case, where the molar flux with respect to molar velocity ($\underline{J}_A^*$) appears (p. 1), and the more usual case (p. 2), where the diffusion coefficient is constant and Fick's law has been incorporated.

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In terms of molar flux, $\underline{J}_A^*$

Microscopic species mass balance, in terms of molar flux; Gibbs notation

$$c\left(\frac{\partial x_A}{\partial t} + \underline{v}^* \cdot \nabla x_A\right) = -\nabla \cdot \underline{J}_A^* + (x_B R_A - x_A R_B)$$

Microscopic species mass balance, in terms of molar flux; Cartesian coordinates

$$c\left(\frac{\partial x_A}{\partial t} + v_x^*\frac{\partial x_A}{\partial x} + v_y^*\frac{\partial x_A}{\partial y} + v_z^*\frac{\partial x_A}{\partial z}\right) = -\left(\frac{\partial J_{A,x}^*}{\partial x} + \frac{\partial J_{A,y}^*}{\partial y} + \frac{\partial J_{A,z}^*}{\partial z}\right) + (x_B R_A - x_A R_B)$$

Microscopic species mass balance, in terms of molar flux; cylindrical coordinates

$$c\left(\frac{\partial x_A}{\partial t} + v_r^*\frac{\partial x_A}{\partial r} + \frac{v_\theta^*}{r}\frac{\partial x_A}{\partial \theta} + v_z^*\frac{\partial x_A}{\partial z}\right) = -\left(\frac{1}{r}\frac{\partial (rJ_{A,r}^*)}{\partial r} + \frac{1}{r}\frac{\partial J_{A,\theta}^*}{\partial \theta} + \frac{\partial J_{A,z}^*}{\partial z}\right) + (x_B R_A - x_A R_B)$$

Microscopic species mass balance, in terms of molar flux; spherical coordinates

$$c\left(\frac{\partial x_A}{\partial t} + v_r^*\frac{\partial x_A}{\partial r} + \frac{v_\theta^*}{r}\frac{\partial x_A}{\partial \theta} + \frac{v_\phi^*}{r\sin\theta}\frac{\partial x_A}{\partial \phi}\right) = -\left(\frac{1}{r^2}\frac{\partial (r^2 J_{A,r}^*)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (J_{A,\theta}^*\sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial J_{A,\phi}^*}{\partial \phi}\right) + (x_B R_A - x_A R_B)$$

Fick's law of diffusion, Gibbs notation: $\underline{J}_A^* = -cD_{AB}\nabla x_A$ WRF 24-16

$$= cx_A(\underline{v}_A - \underline{v}^*)$$

Fick's law of diffusion, Cartesian coordinates:
$$\begin{pmatrix} J_{A,x}^* \\ J_{A,y}^* \\ J_{A,z}^* \end{pmatrix}_{xyz} = \begin{pmatrix} -cD_{AB}\dfrac{\partial x_A}{\partial x} \\ -cD_{AB}\dfrac{\partial x_A}{\partial y} \\ -cD_{AB}\dfrac{\partial x_A}{\partial z} \end{pmatrix}_{xyz}$$

Fick's law of diffusion, cylindrical coordinates:
$$\begin{pmatrix} J_{A,r}^* \\ J_{A,\theta}^* \\ J_{A,z}^* \end{pmatrix}_{r\theta z} = \begin{pmatrix} -cD_{AB}\dfrac{\partial x_A}{\partial r} \\ -\dfrac{cD_{AB}}{r}\dfrac{\partial x_A}{\partial \theta} \\ -cD_{AB}\dfrac{\partial x_A}{\partial z} \end{pmatrix}_{r\theta z}$$

Fick's law of diffusion, spherical coordinates:
$$\begin{pmatrix} J_{A,r}^* \\ J_{A,\theta}^* \\ J_{A,\phi}^* \end{pmatrix}_{r\theta\phi} = \begin{pmatrix} -cD_{AB}\dfrac{\partial x_A}{\partial r} \\ -\dfrac{cD_{AB}}{r}\dfrac{\partial x_A}{\partial \theta} \\ -\dfrac{cD_{AB}}{r\sin\theta}\dfrac{\partial x_A}{\partial \phi} \end{pmatrix}_{r\theta\phi}$$

The Equation of Species Mass Balance in Terms of <mark>Molar</mark> Quantities, constant cD_{AB}.

For binary systems, and Fick's law has been incorporated. Good for low density gases at constant temperature and pressure.

Microscopic species mass balance, constant thermal conductivity; Gibbs notation

$$c\left(\frac{\partial x_A}{\partial t} + \underline{v}^* \cdot \nabla x_A\right) = cD_{AB}\nabla^2 x_A + (x_B R_A - x_A R_B)$$

Microscopic species mass balance, constant thermal conductivity; Cartesian coordinates

$$c\left(\frac{\partial x_A}{\partial t} + v_x^*\frac{\partial x_A}{\partial x} + v_y^*\frac{\partial x_A}{\partial y} + v_z^*\frac{\partial x_A}{\partial z}\right) = cD_{AB}\left(\frac{\partial^2 x_A}{\partial x^2} + \frac{\partial^2 x_A}{\partial y^2} + \frac{\partial^2 x_A}{\partial z^2}\right) + (x_B R_A - x_A R_B)$$

Microscopic species mass balance, constant thermal conductivity; cylindrical coordinates

$$c\left(\frac{\partial x_A}{\partial t} + v_r^*\frac{\partial x_A}{\partial r} + \frac{v_\theta^*}{r}\frac{\partial x_A}{\partial \theta} + v_z^*\frac{\partial x_A}{\partial z}\right) = cD_{AB}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial x_A}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 x_A}{\partial \theta^2} + \frac{\partial^2 x_A}{\partial z^2}\right) + (x_B R_A - x_A R_B)$$

Microscopic species mass balance, constant thermal conductivity; spherical coordinates

$$c\left(\frac{\partial x_A}{\partial t} + v_r^*\frac{\partial x_A}{\partial r} + \frac{v_\theta^*}{r}\frac{\partial x_A}{\partial \theta} + \frac{v_\phi^*}{r\sin\theta}\frac{\partial x_A}{\partial \phi}\right)$$
$$= cD_{AB}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial x_A}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial x_A}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 x_A}{\partial \phi^2}\right) + (x_B R_A - x_A R_B)$$

In terms of Diffusivity, D_{AB}

$cx_A = c_A = \frac{1}{M_A}(\rho_A) = \frac{1}{M_A}(\rho\omega_A)$ $\left(\text{units: } c[=]\frac{mol\ mix}{vol\ soln}; \rho[=]\frac{mass\ mix}{vol\ soln}; c_A[=]\frac{mol\ A}{vol\ soln}; \rho_A[=]\frac{mass\ A}{vol\ soln}\right)$

$\underline{J}_A^* \equiv$ molar flux relative to a mixture's molar average velocity, $\underline{v}^*$ $\left(\text{units: } \underline{J}_A^*[=]\frac{mole}{area\cdot time}\right)$

$\quad = c_A(\underline{v}_A - \underline{v}^*)$

$\underline{J}_A^* + \underline{J}_B^* = 0$

$\underline{N}_A \equiv c_A\underline{v}_A = \underline{J}_A^* + c_A\underline{v}^* =$ combined molar flux relative to stationary coordinates

$\underline{N}_A + \underline{N}_B = c\underline{v}^*$

$\underline{v}_A \equiv$ velocity of species A in a mixture, i.e. average velocity of all molecules of species A within a small volume
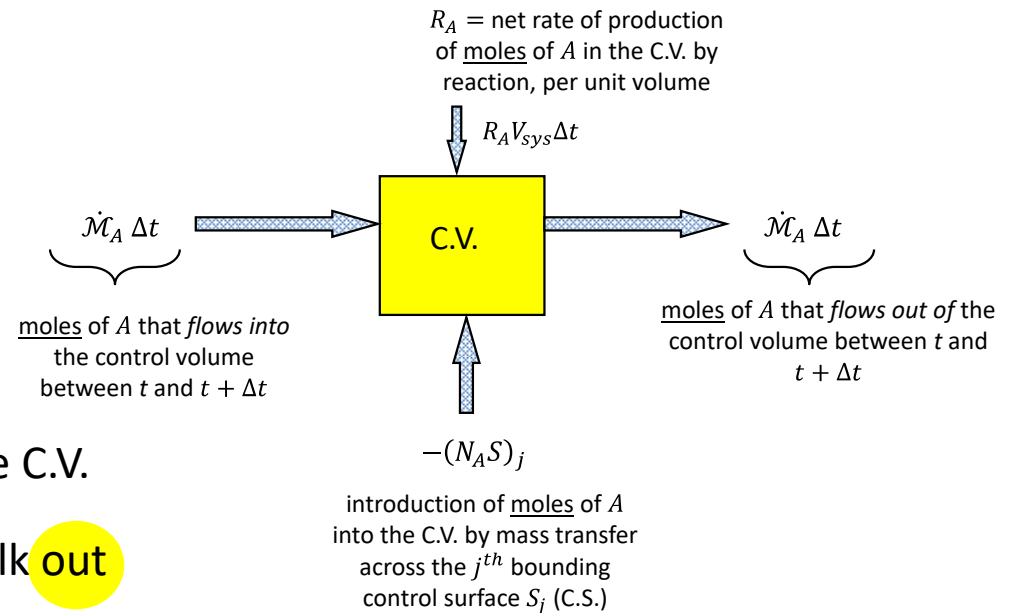
$\underline{v}^* = x_A\underline{v}_A + x_B\underline{v}_B \equiv$ molar average velocity

Reference: R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena*, 2nd edition, Wiley, 2002.   (p. 515, 584)

Recurring Modeling Assumptions in Diffusion ("Classics")

- Near a liquid-gas interface, the region in the gas near the liquid is a film where slow diffusion takes place

- The vapor near the liquid-gas interface is often saturated (Raoult's law, $x_A = p_A^*/p$)

- If component A has no sink, flux $\underline{N}_A = 0$

- If A diffuses through stagnant B, $\underline{N}_B = 0$

- If A is dilute in B, we can neglect the convection term, yielding $N_{Az} = J_{Az}^*$; also liquid concentration c will be constant in the dilute case

- Because diffusion is slow, we can make a quasi-steady-state assumption

- If, for example, two moles of A diffuse to a surface at which a rapid, irreversible reaction coverts it to one mole of B, then at steady state $-0.5\underline{N}_A = \underline{N}_B$.

- Homogeneous reactions appear in the mass balance; heterogeneous reactions appear in the boundary conditions and relate fluxes

- If a binary mixture of A and B are undergoing steady equimolar counter diffusion, $\underline{N}_A = -\underline{N}_B$

- Penetration model is a viable perspective for some homogeneous reaction systems (e.g. gas absorption)

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MOLES

accumulation = net flow in + production + introduction

$$\frac{d}{dt}\left(\mathcal{M}_{A,sys}\right) = -\Delta\dot{\mathcal{M}}_A + R_A V_{sys} - \sum_j (N_A S)_j$$

R_A = net rate of production of moles of A in the C.V. by reaction, per unit volume

$R_A V_{sys} \Delta t$

$\dot{\mathcal{M}}_A \Delta t$

C.V.

$\dot{\mathcal{M}}_A \Delta t$

moles of A that *flows into* the control volume between t and $t + \Delta t$

moles of A that *flows out of* the control volume between t and $t + \Delta t$

$-(N_A S)_j$

introduction of moles of A into the C.V. by mass transfer across the jth bounding control surface S_j (C.S.)

$\mathcal{M}_{A,sys} = c_A V_{sys}$ = total moles of A in the C.V.

$\Delta\dot{\mathcal{M}}_A = \sum_{j,outs}\dot{\mathcal{M}}_{A,j} - \sum_{j,ins}\dot{\mathcal{M}}_{A,j}$ = bulk out

R_A = net rate of production of moles of A in the C.V. by reaction, per unit volume

V_{sys} = system volume

$N_{A_j} = K\Delta c_{df}$ = molar flux of A out through the jth C.S.

$S_{sys} = \sum_j S_j$

Δ is "out"- "in"
C.S. = control surface
C.V. = control volume

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Liquid-phase-units
Film Linear driving
force model:

$$N_A \equiv k_x(x_{A,i} - x_{A,b})$$
$$N_A \equiv k_{cL}(c_{AL,i} - c_{AL,b})$$

Gas-phase-units:
Film Linear driving force
model:

$$N_A \equiv k_p(p_{A,b} - p_{A,i})$$
$$N_A \equiv k_{cG}(c_{AG,b} - c_{A,i})$$
$$N_A \equiv k_y(y_{A,b} - y_{A,i})$$

Let's take these tools out for a spin!

Liquid-phase-units
Overall Linear driving
force model:

$$N_A \equiv K_x(x_A^*(\ \ ) - x_{A,b})$$
$$N_A \equiv K_{cL}(c_{AL}^*(\ \ ) - c_{AL,b})$$

$$(\ \ ) = p_{A,b} \ or \ c_{A,b} \ or \ y_{A,b}$$

Gas-phase-units:
Overall Linear driving
force model:

$$N_A \equiv K_p\left(p_{A,b} - p_A^*(\ \ )\right)$$
$$N_A \equiv K_{cG}\left(c_{AG,b} - c_{AG}^*(\ \ )\right)$$
$$N_A \equiv K_y\left(y_{A,b} - y_A^*(\ \ )\right)$$

$$(\ \ ) = x_{A,b} \ or \ c_{AL,b}$$

$$K_x = \cfrac{1}{\cfrac{1}{k_x} + \cfrac{1}{m''k_y}}\ ,$$

$$K_y = \cfrac{1}{\cfrac{1}{k_y} + \cfrac{m'}{k_x}}\ , \text{etc.}$$

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Dimensional Analysis

These numbers tell us about the relative importance of the terms they precede in the governing equations.

Dimensionless numbers from the
Equations of Change *(microscopic balances)*

momentum

Non-dimensional Navier-Stokes Equation

$$\left(\frac{\partial v_z^*}{\partial t^*} + \underline{v}^* \cdot \nabla^* v_z^*\right) = -\frac{\partial P^*}{\partial z^*} + \frac{1}{\text{Re}}(\nabla^{*2} v_z^*) + \frac{1}{\text{Fr}} g^*$$

Re — Reynolds
Fr — Froude

energy

Non-dimensional Energy Equation

$$\left(\frac{\partial T^*}{\partial t^*} + \underline{v}^* \cdot \nabla^* T^*\right) = \frac{1}{\text{RePr}}(\nabla^{*2} T^*) + S^*$$

Pe — Péclet$_h$ = RePr
Pr — Prandtl

mass

Non-dimensional Continuity Equation (species A)

$$\left(\frac{\partial x_A^*}{\partial t^*} + \underline{v}^* \cdot \nabla^* x_A^*\right) = \frac{1}{\text{ReSc}}(\nabla^{*2} x_A^*)$$

Pe — Péclet$_m$ = ReSc
Sc — Schmidt

Oops! This is dimensionless $\underline{v}$, NOT molar average velocity; sorry!

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ref: BSL1, p581, 644

Dimensionless Numbers

Dimensionless numbers from the
Equations of Change

$$\text{Re} - \text{Reynolds} = \frac{\rho V D}{\mu} = \frac{V D}{\nu}$$

$$\text{Fr} - \text{Froude} = \frac{V^2}{gD}$$

$$\text{Pe} - \text{Péclet}_h = \text{RePr} = \frac{\hat{C}_p \rho V D}{k} = \frac{V D}{\alpha}$$

$$\text{Pe} - \text{Péclet}_m = \text{ReSc} = \frac{V D}{\mathcal{D}_{AB}}$$

These numbers tell us about the **relative importance of the terms** they precede in the microscopic balances (*scenario properties*).

$$\text{Pr} - \text{Prandtl} = \frac{\hat{C}_p \mu}{k} = \frac{\nu}{\alpha}$$

$$\text{Sc} - \text{Schmidt} = \text{LePr} = \frac{\mu}{\rho \mathcal{D}_{AB}} = \frac{\nu}{\mathcal{D}_{AB}}$$

$$\text{Le} - \text{Lewis} = \frac{\alpha}{\mathcal{D}_{AB}}$$

These numbers compare the **magnitudes of the diffusive transport coefficients** $\nu, \alpha, \mathcal{D}_{AB}$ (*material properties*).

Transport coefficients

Dimensional Analysis

These numbers are defined to help us build **transport data correlations** based on the fewest number of grouped (dimensionless) variables (*scenario property*).

Dimensionless numbers from the **Engineering Quantities of Interest**

momentum

Dimensionless Force on the Wall (Drag)

$$f = \frac{1}{\pi}\frac{D}{L}\frac{1}{\text{Re}} \int_0^{2\pi}\int_0^{L/D} \left(\frac{\partial v_z^*}{\partial r^*}\right)\Bigg|_{r^*=\frac{1}{2}} d\theta dz^*$$

(Fanning)

f — Friction Factor

$\dfrac{L}{D}$ — Aspect Ratio

$$f = \frac{\mathcal{F}_{drag}}{\left(\frac{1}{2}\rho V^2\right) A_c}$$

energy

Newton's Law of Cooling

$$Nu = \frac{1}{2\pi L/D} \int_0^{2\pi}\int_0^{L/D} -\frac{\partial T^*}{\partial r^*}\Bigg|_{r^*=1/2} dz^* d\theta$$

Nu — Nusselt

$\dfrac{L}{D}$ — Aspect Ratio

$$\text{St}_h = \frac{h}{\rho V \hat{C}_p} = \frac{\text{Nu}}{\text{RePr}}$$

$$\text{Nu} = \frac{hD}{k}$$

mass xfer

Dimensionless Mass Transfer Coefficient

$$Sh = \frac{1}{2\pi}\frac{D}{L} \int_0^{2\pi}\int_0^{L/D} \left(-\frac{\partial x_A^*}{\partial r^*}\right)\Bigg|_{r^*=\frac{1}{2}} d\theta dz^*$$

Sh — Sherwood

$\dfrac{L}{D}$ — Aspect Ratio

$$\text{St}_m = \frac{k_c}{V} = \frac{\text{Sh}}{\text{ReSc}}$$

$$\text{Sh} = \frac{k_c D}{\mathcal{D}_{AB}}$$

St — Stanton

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Dimensionless Numbers

$\text{Re} - \text{Reynolds} = \dfrac{\rho V D}{\mu} = \dfrac{V D}{\nu}$

$\text{Fr} - \text{Froude} = \dfrac{V^2}{gD}$

$\text{Pe} - \text{Péclet}_h = \text{RePr} = \dfrac{\hat{C}_p \rho V D}{k} = \dfrac{V D}{\alpha}$

$\text{Pe} - \text{Péclet}_m = \text{ReSc} = \dfrac{V D}{\mathcal{D}_{AB}}$

These numbers from the governing equations tell us about the relative importance of the terms they precede in the microscopic balances (**scenario** properties).

$\text{Pr} - \text{Prandtl} = \dfrac{\hat{C}_p \mu}{k} = \dfrac{\nu}{\alpha}$

$\text{Sc} - \text{Schmidt} = \text{LePr} = \dfrac{\mu}{\rho \mathcal{D}_{AB}} = \dfrac{\nu}{\mathcal{D}_{AB}}$

$\text{Le} - \text{Lewis} = \dfrac{\alpha}{\mathcal{D}_{AB}}$

These numbers compare the magnitudes of the diffusive transport coefficients $\nu, \alpha, \mathcal{D}_{AB}$ (**material** properties).

$f - \text{Friction Factor} = \dfrac{\mathcal{F}_{drag}}{\left(\frac{1}{2}\rho V^2\right)A_c}$

$\text{Nu} - \text{Nusselt} = \dfrac{hD}{k}$

$\text{Sh} - \text{Sherwood} = \dfrac{k_c D}{\mathcal{D}_{AB}}$

These numbers are defined to help us build transport data correlations based on the fewest number of grouped (dimensionless) variables (**scenario** properties).

$\text{St}_h = \text{Nu}/\text{Pe}_h, \text{St}_m = \text{Sh}/\text{Pe}_m - \text{Stanton}$

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