CM3120: Module 2

Unsteady State Heat Transfer

I. Introduction
II. Unsteady Microscopic Energy Balance—(slash and burn)
III. Unsteady Macroscopic Energy Balance
IV. Dimensional Analysis (unsteady)—Biot number, Fourier number
V. Low Biot number solutions—Lumped parameter analysis
VI. Short Cut Solutions—(initial temperature T_0; finite h), Gurney and Lurie charts (as a function of position, $m = \frac{1}{\mathrm{Bi}}$, and Fo); Heissler charts (center point only, as a function of $m = 1/\mathrm{Bi}$, and Fo)
VII. Full Analytical Solutions (stretch)

 1

CM3120: Module 2

Module 2 Lecture II

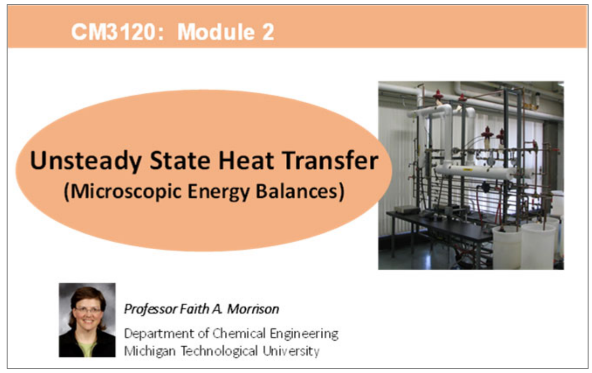
Unsteady State Heat Transfer

(Microscopic Energy Balances)

Professor Faith A. Morrison
Department of Chemical Engineering
Michigan Technological University

www.chem.mtu.edu/~fmorriso/cm3120/cm3120.html

 2



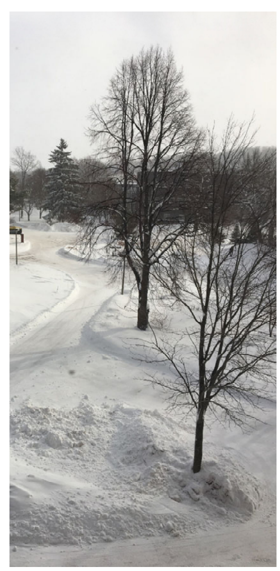
We model the dynamics of unsteady state heat transfer because there are very practical problems that we can solve with such models.

 3

Example:
When will my pipes freeze?

The temperature has been 35°F for a while now, sufficient to chill the ground to this temperature for many tens of feet below the surface. Suddenly the temperature drops to -20°F. How long will it take for freezing temperatures (32°F) to reach my pipes, which are 8 *ft* under ground?

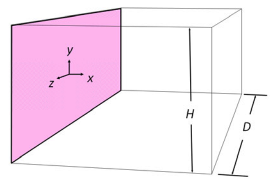
1/30/19



 4

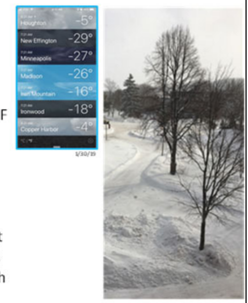
Unsteady State Heat Transfer: Dimensional Analysis

Develop a model:



Example:
When will my pipes freeze?

The temperature has been 35ºF for a while now, sufficient to chill the ground to this temperature for many tens of feet below the surface. Suddenly the temperature drops to -20ºF. How long will it take for freezing temperatures (32ºF) to reach my pipes, which are 8 *ft* under ground?



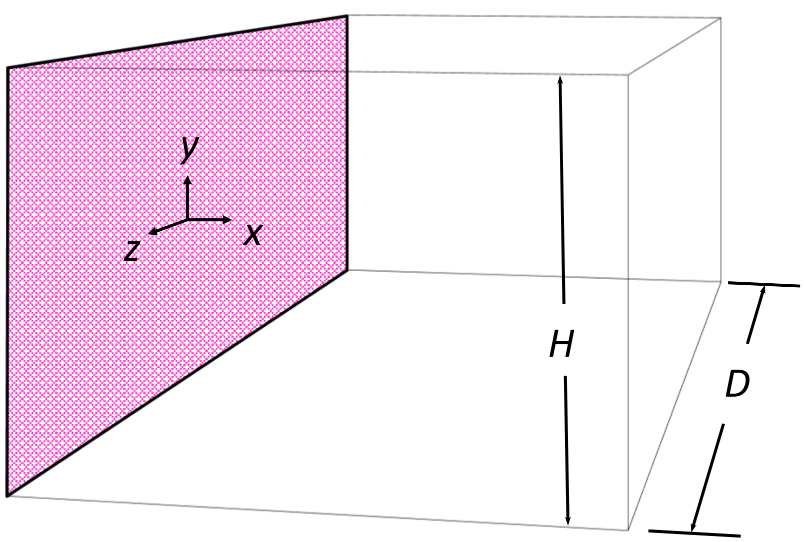
***Example 1:* Unsteady Heat Conduction in a Semi-infinite solid**

A very long, very wide, very tall slab is initially at a temperature T_0. At time $t = 0$, the left face of the slab is <u>exposed to a vigorously mixed gas</u> at temperature T_1. What is the time-dependent temperature profile in the slab?

5

Unsteady State Heat Transfer

Example: Unsteady Heat Conduction in a Semi-infinite solid



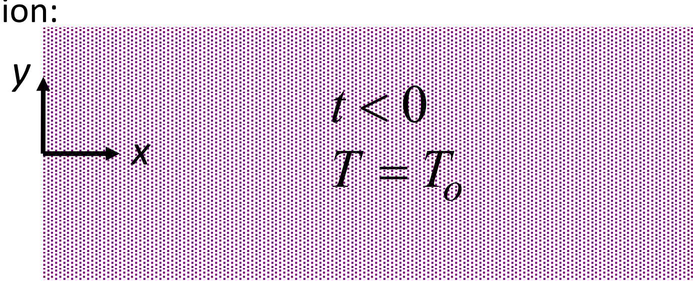
H, D, very large

6

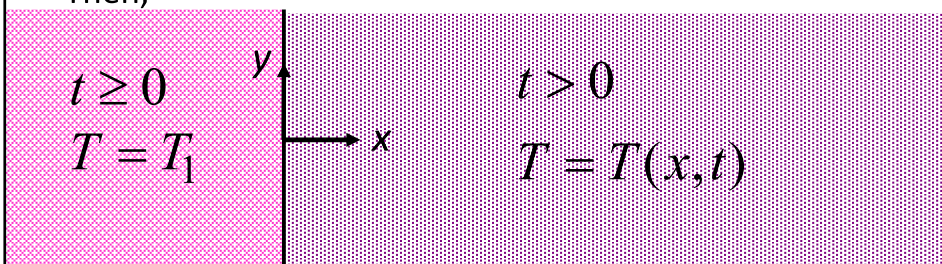
1D Heat Transfer: Unsteady State

Initial Condition:

$$t < 0$$
$$T = T_o$$



Then,

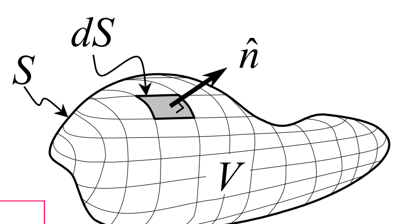


1D Heat Transfer: Unsteady State

General Energy Transport Equation

(microscopic energy balance)

As for the derivation of the microscopic momentum balance, the microscopic energy balance is derived on an arbitrary volume, *V*, enclosed by a surface, *S*.



Gibbs notation:

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + \underline{v} \cdot \nabla T \right) = k \nabla^2 T + S$$

see handout for component notation

1D Heat Transfer: Unsteady State

General Energy Transport Equation

(microscopic energy balance)

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + \underline{v} \cdot \nabla T \right) = k \nabla^2 T + S$$

- $\frac{\partial T}{\partial t}$: rate of change
- $\underline{v} \cdot \nabla T$: convection
- $k \nabla^2 T$: conduction (all directions)
- S: source (energy generated per unit volume per time)
- $\underline{v}$: velocity must satisfy equation of motion, equation of continuity

see handout for component notation

Equation of energy for Newtonian fluids of constant density, ρ, and thermal conductivity, k, with source term (source could be viscous dissipation, electrical energy, chemical energy, etc., with units of energy/(volume time)).

Source: R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Processes*, Wiley, NY, 1960, page 319.

www.chem.mtu.edu/~fmorriso/cm310/energy2013.pdf

thermal diffusivity $\alpha \equiv \frac{k}{\rho \hat{C}_p}$

Gibbs notation (vector notation)

$$\left(\frac{\partial T}{\partial t} + \underline{v} \cdot \nabla T \right) = \frac{k}{\rho \hat{C}_p} \nabla^2 T + \frac{S}{\rho \hat{C}_p}$$

Cartesian (xyz) coordinates:

$$\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} = \frac{k}{\rho \hat{C}_p} \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{S}{\rho \hat{C}_p}$$

Cylindrical (rθz) coordinates:

$$\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} = \frac{k}{\rho \hat{C}_p} \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{S}{\rho \hat{C}_p}$$

Spherical (rθϕ) coordinates:

$$\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + \frac{v_\phi}{r \sin\theta} \frac{\partial T}{\partial \phi} = \frac{k}{\rho \hat{C}_p} \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2}$$

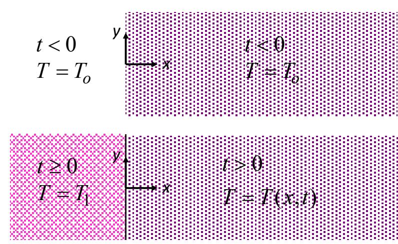
1D Heat Transfer: Unsteady State

Example 1: **Unsteady Heat Conduction in a Semi-infinite solid**

A very long, very wide, very tall slab is initially at a temperature T_0. At time $t = 0$, the left face of the slab is exposed to a vigorously mixed gas at temperature T_1. What is the time-dependent temperature profile in the slab?

Newton's law of cooling BC's:

$$|q_x| = hA\left|T_{bulk} - T_{surface}\right|$$



11

1D Heat Transfer: Unsteady State

Microscopic Energy Equation in Cartesian Coordinates

$$\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} = \frac{k}{\rho \hat{C}_p}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{S}{\rho \hat{C}_p}$$

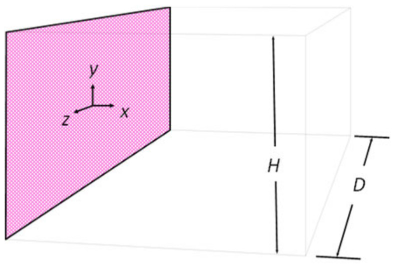
$$\alpha \equiv \frac{k}{\rho \hat{C}_p} =$$ thermal diffusivity

what are the boundary conditions? initial conditions?

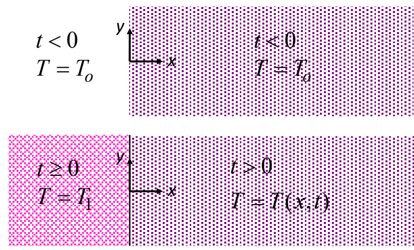
12

1D Heat Transfer: Unsteady State

Example: Unsteady Heat Conduction in a Semi-infinite solid



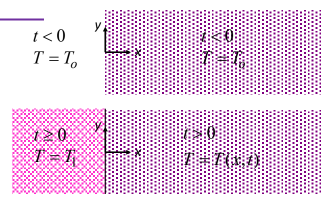
Initial Condition:



You try.

1D Heat Transfer: Unsteady State

Unsteady State Heat Conduction in a Semi-Infinite Slab



$$\frac{\partial T}{\partial t} = \frac{k}{\rho \hat{C}_p}\left(\frac{\partial^2 T}{\partial x^2}\right) = \alpha\left(\frac{\partial^2 T}{\partial x^2}\right)$$

thermal diffusivity $\alpha \equiv \frac{k}{\rho \hat{C}_p}$

Initial condition: $t = 0 \quad T = T_0 \quad \forall\ x$

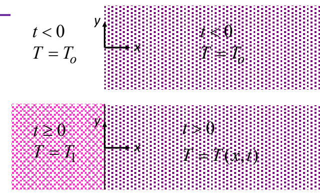
Boundary conditions:

$$x = 0 \quad \frac{q_x}{A} = -k\frac{dT}{dx} = h(T_1 - T) \quad t > 0$$

$$x = \infty \quad T = T_0 \quad \forall\ t$$

1D Heat Transfer: Unsteady State

Unsteady State Heat Conduction in a Semi-Infinite Slab



$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2}\right)$$

thermal diffusivity

$$\alpha \equiv \frac{k}{\rho \hat{C}_p}$$

Initial condition: $t = 0 \quad T = T_0 \quad \forall\ x$ "for all x"

Boundary conditions:

$$x = 0 \qquad \frac{q_x}{A} = -k\frac{dT}{dx} = h(T_1 - T) \qquad t > 0$$

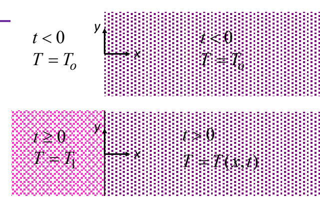
$$x = \infty \qquad T = T_0 \qquad \forall\ t$$

"for all t"

15

1D Heat Transfer: Unsteady State

Unsteady State Heat Conduction in a Semi-Infinite Slab



$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2}\right)$$

The solution of the PDE is obtained by combination of variables.

Initial condition: $t = 0 \quad T = T_0 \quad \forall\ x$

Boundary conditions:

$$x = 0 \qquad \frac{q_x}{A} = -k\frac{dT}{dx} = h(T_1 - T) \qquad t > 0$$

$$x = \infty \qquad T = T_0 \qquad \forall\ t$$

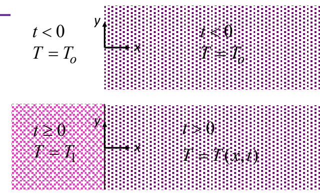
$$\beta \equiv \frac{h\sqrt{\alpha\, t}}{k} \qquad \zeta \equiv \frac{x}{2\sqrt{\alpha\, t}}$$

See text WRF 6th ed p284 or BSL1 p353for the solution with temp BCs

16

1D Heat Transfer: Unsteady State

Unsteady State Heat Conduction in a Semi-Infinite Slab



$$\frac{\partial T}{\partial t} = \alpha \left(\frac{\partial^2 T}{\partial x^2} \right)$$

The solution of the PDE is obtained by combination of variables.

Initial condition: $t = 0 \quad T = T_0 \quad \forall\ x$

Boundary conditions:

$$x = 0 \qquad \frac{q_x}{A} = -k\frac{dT}{dx} = h(T_1 - T) \qquad t > 0$$

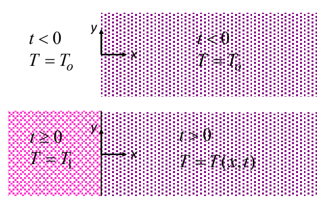
$$x = \infty \qquad T = T_0 \qquad \forall\ t$$

$$\beta \equiv \frac{h\sqrt{\alpha\, t}}{k} \qquad \zeta \equiv \frac{x}{2\sqrt{\alpha\, t}}$$

See text WRF 6th ed p284 or BSL1 p353for the solution with temp BCs

17

Unsteady State Heat Conduction in a Semi-Infinite Slab



Solution:

$$\frac{T - T_0}{T_1 - T_0} = \operatorname{erfc}\zeta - e^{\beta(2\zeta+\beta)}\operatorname{erfc}(\zeta+\beta)$$

$$\beta \equiv \frac{h\sqrt{\alpha\, t}}{k} \qquad \zeta \equiv \frac{x}{2\sqrt{\alpha\, t}}$$

$$Y \equiv \frac{(T_1 - T)}{(T_1 - T_0)}$$

$$1 - Y = \frac{(T - T_0)}{(T_1 - T_0)}$$

complementary error function of y *(a standard function in Excel)*

$$\operatorname{erfc}(y) \equiv 1 - \operatorname{erf}(y)$$

error function of y

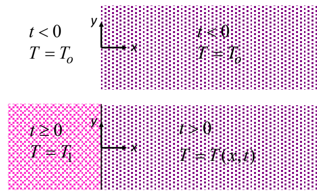
$$\operatorname{erf}(y) \equiv \frac{2}{\sqrt{\pi}} \int_0^y e^{-(y')^2}\, dy'$$

- **Geankoplis 4th ed., eqn 5.3-7, page 363**
- **WRF, eqn 18-21, page 286**

thermal diffusivity $\alpha \equiv \frac{k}{\rho \hat{C}_p}$

18

Unsteady State Heat Conduction in a Semi-Infinite Slab



Solution:

$$\frac{T - T_0}{T_1 - T_0} = \operatorname{erfc}\zeta - e^{\beta(2\zeta+\beta)}\operatorname{erfc}(\zeta+\beta)$$

$$\beta \equiv \frac{h\sqrt{\alpha\, t}}{k} \qquad \zeta \equiv \frac{x}{2\sqrt{\alpha\, t}}$$

To make this solution easier to use, we can plot it.

complementary error function of y

$$\operatorname{erfc}(y) \equiv 1 - \operatorname{erf}(y)$$

error function of y

$$\operatorname{erf}(y) \equiv \frac{2}{\sqrt{\pi}}\int_0^y e^{-(y')^2}\, dy'$$

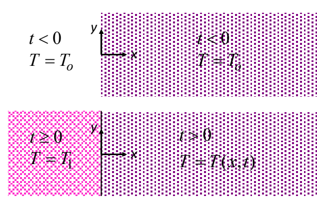
$$Y \equiv \frac{(T_1 - T)}{(T_1 - T_0)}$$

$$1 - Y = \frac{(T - T_0)}{(T_1 - T_0)}$$

thermal diffusivity $\alpha \equiv \frac{k}{\rho \hat{C}_p}$

19

Unsteady State Heat Conduction in a Semi-Infinite Slab



This:

$$\frac{T - T_0}{T_1 - T_0} = \operatorname{erfc}\zeta - e^{\beta(2\zeta+\beta)}\operatorname{erfc}(\zeta+\beta)$$

Versus this: $\zeta \equiv \frac{x}{2\sqrt{\alpha\, t}}$

To make this solution easier to use, we can plot it.

At various values of this:

$$\beta \equiv \frac{h\sqrt{\alpha\, t}}{k}$$

$$Y \equiv \frac{(T_1 - T)}{(T_1 - T_0)}$$

$$1 - Y = \frac{(T - T_0)}{(T_1 - T_0)}$$

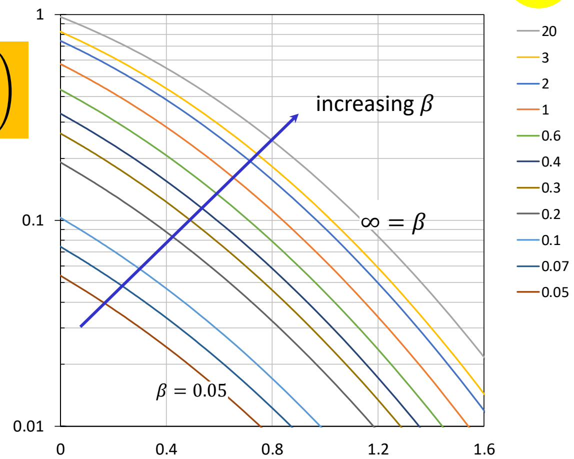
thermal diffusivity $\alpha \equiv \frac{k}{\rho \hat{C}_p}$

20

Unsteady State Heat Conduction in a Semi-Infinite Slab

$$\beta \equiv \frac{h\sqrt{\alpha\, t}}{k}$$

$$1 - Y = \left(\frac{T - T_0}{T_1 - T_0}\right)$$



$$Y \equiv \frac{(T_1 - T)}{(T_1 - T_0)}$$
$$1 - Y = \frac{(T - T_0)}{(T_1 - T_0)}$$

$$\zeta = \frac{x}{2\sqrt{\alpha\, t}}$$

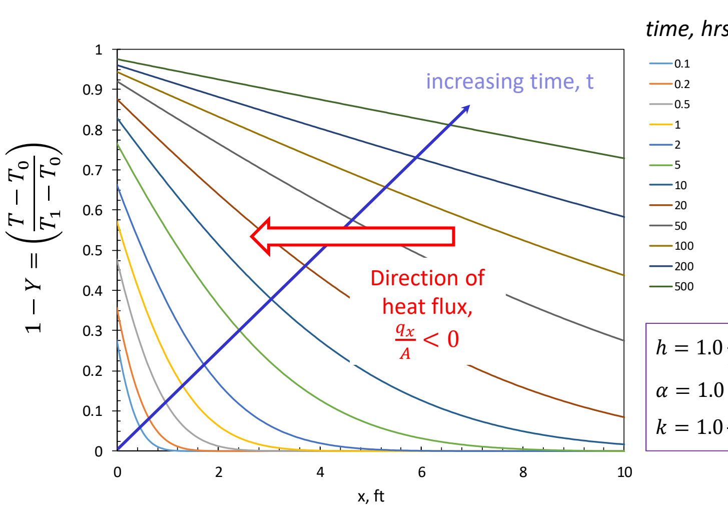
Plot design after Geankoplis 4th ed., Figure 5.3-3, page 364

21

1D Heat Transfer: Unsteady State Heat Conduction in a Semi-Infinite Slab

With modern tools, we can plot the solution directly (evaluated in Excel)

Unsteady State Heat Conduction in a Semi-Infinite Slab



$$h = 1.0\,\frac{btu}{h\,ft^2\,F}$$
$$\alpha = 1.0\,\frac{ft^2}{h}$$
$$k = 1.0\,\frac{btu}{h}\,ft\,F$$

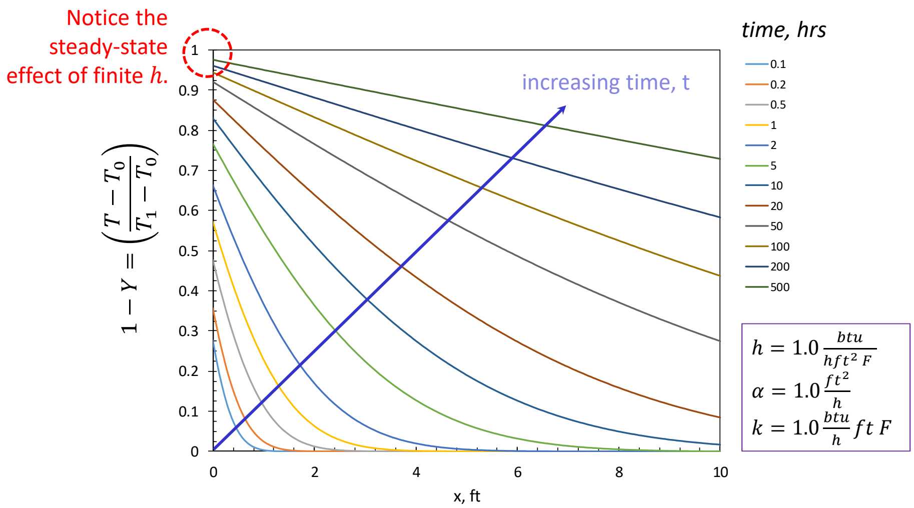
thermal diffusivity $\alpha \equiv \frac{k}{\rho \hat{C}_p}$

22

1D Heat Transfer: Unsteady State Heat Conduction in a Semi-Infinite Slab

With modern tools, we can plot the solution directly (evaluated in Excel)

Unsteady State Heat Conduction in a Semi-Infinite Slab



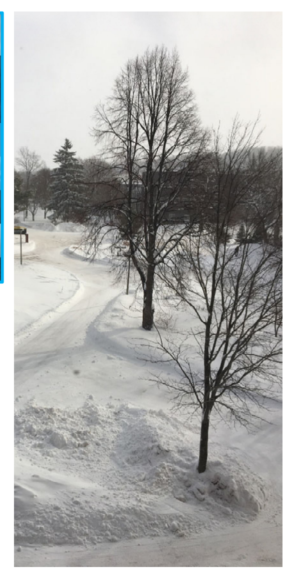
thermal diffusivity $\alpha \equiv \frac{k}{\rho \hat{C}_p}$

 23

Example:
When will my pipes freeze?

The temperature has been 35°F for a while now, sufficient to chill the ground to this temperature for many tens of feet below the surface. Suddenly the temperature drops to -20°F. How long will it take for freezing temperatures (32°F) to reach my pipes, which are 8 *ft* under ground?

1/30/19



 24

1D Heat Transfer: Unsteady State Heat Conduction in a Semi-Infinite Slab

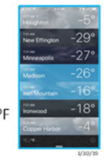
We need the appropriate physical property data for the soil.

$$h = 2.0\,\frac{BTU}{h\ ft^2\ {}^oF}$$

$$\alpha_{soil} = 0.018\,\frac{ft^2}{h}$$

$$k_{soil} = 0.5\,\frac{BTU}{h\ ft\ {}^oF}$$

Example: When will my pipes freeze?



The temperature has been 35ºF for a while now, sufficient to chill the ground to this temperature for many tens of feet below the surface. Suddenly the temperature drops to -20ºF. How long will it take for freezing temperatures (32ºF) to reach my pipes, which are 8 *ft* under ground?

thermal diffusivity $\alpha \equiv \frac{k}{\rho \hat{C}_p}$

Geankoplis 4th ed.

Example: When will my pipes freeze?

1D Heat Transfer: Unsteady State Heat Conduction in a Semi-Infinite Slab

$$\frac{T - T_0}{T_1 - T_0} = \operatorname{erfc}\zeta - e^{\beta(2\zeta+\beta)}\operatorname{erfc}(\zeta + \beta)$$

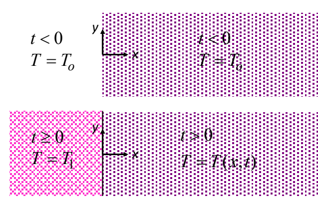
$$\zeta \equiv \frac{x}{2\sqrt{\alpha\, t}}$$

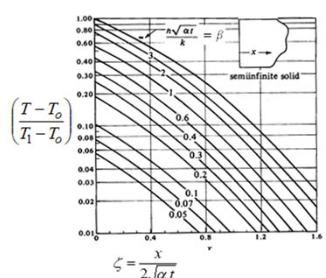
$$\beta \equiv \frac{h\sqrt{\alpha\, t}}{k}$$

Both ζ and β depend on time

$T_0 = ?$
$T_1 = ?$
$T = ?$

$$\frac{T - T_0}{T_1 - T_0} = ?$$





$$Y \equiv \frac{(T_1 - T)}{(T_1 - T_0)}$$

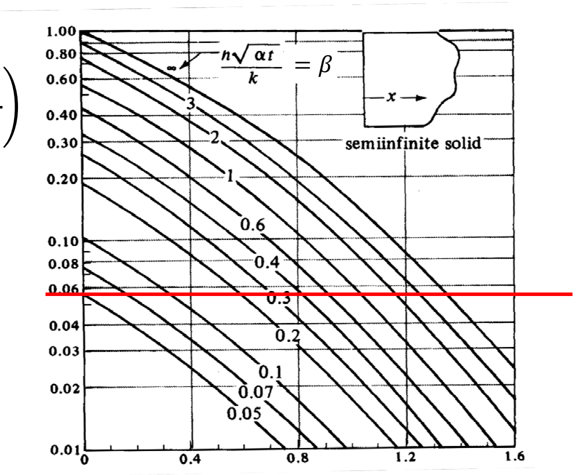
$$1 - Y = \frac{(T - T_0)}{(T_1 - T_0)}$$

Example: When will my pipes freeze?

1D Heat Transfer: Unsteady State Heat Conduction in a Semi-Infinite Slab

$$1 - Y = \left(\frac{T - T_0}{T_1 - T_0}\right)$$

You try.



$$\zeta = \frac{x}{2\sqrt{\alpha\, t}}$$

Geankoplis 4th ed., Figure 5.3-3, page 364

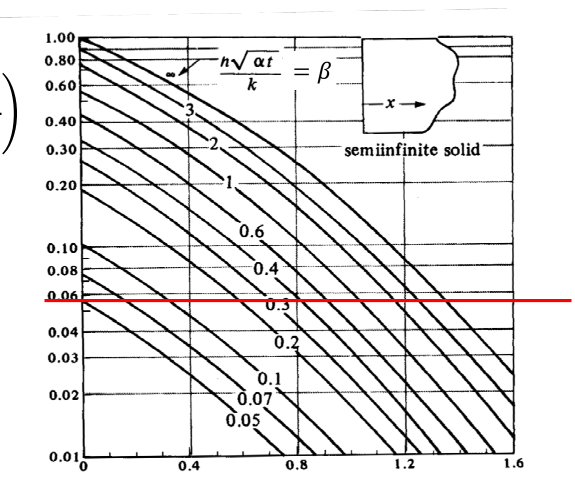
Example: When will my pipes freeze?

1D Heat Transfer: Unsteady State Heat Conduction in a Semi-Infinite Slab

$$1 - Y = \left(\frac{T - T_0}{T_1 - T_0}\right)$$

Solution:

Guess large β)
(Interative solution)

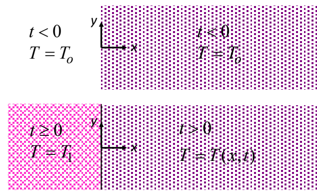


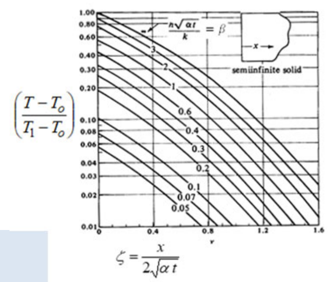
$$\zeta = \frac{x}{2\sqrt{\alpha\, t}}$$

Geankoplis 4th ed., Figure 5.3-3, page 364

Example: When will my pipes freeze?

1D Heat Transfer: Unsteady State Heat Conduction in a Semi-Infinite Slab





$$\frac{T - T_0}{T_1 - T_0} = \text{erfc}\,\zeta - e^{\beta(2\zeta+\beta)}\,\text{erfc}(\zeta + \beta)$$

Answer:

$t \approx 480\ hours \approx 20\ days$

$$Y \equiv \frac{(T_1 - T)}{(T_1 - T_0)}$$

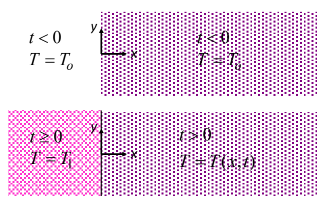
$$1 - Y = \frac{(T - T_0)}{(T_1 - T_0)}$$

29

Example: When will my pipes freeze?

1D Heat Transfer: Unsteady State Heat Conduction in a Semi-Infinite Slab

Or, use Excel. (How exactly?)



$$\frac{T - T_0}{T_1 - T_0} = \text{erfc}\,\zeta - e^{\beta(2\zeta+\beta)}\,\text{erfc}(\zeta + \beta)$$

$$\zeta \equiv \frac{x}{2\sqrt{\alpha t}}$$

$$\beta \equiv \frac{h\sqrt{\alpha\, t}}{k}$$

You try.

T0=		
T1=		
T=		
h=		
alpha=		
k=		
x=		

Answer:

$t = 21.2\ days$

$\beta = 12.1$

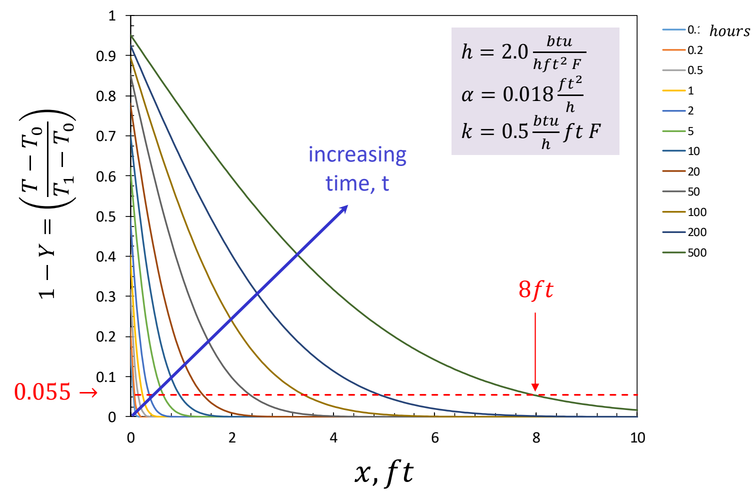
pages.mtu.edu/~fmorriso/cm310/2019PracticeProblemsInHeatTransfer%28Geankoplis%29.pdf

30

Example: When will my pipes freeze?

1D Heat Transfer: Unsteady State Heat Conduction in a Semi-Infinite Slab

With modern tools, we can plot the evolution of the model directly (evaluated in Excel)



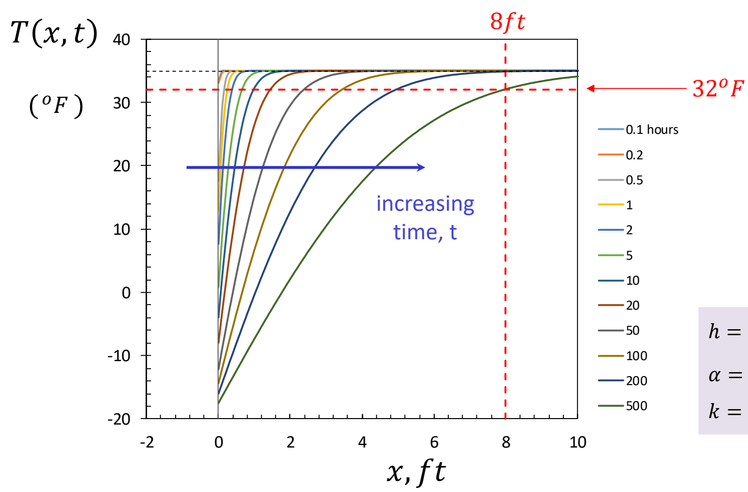
aa_solutions/Unsteady semi infinite solid plots.xlsx

31

Example: When will my pipes freeze?

1D Heat Transfer: Unsteady State Heat Conduction in a Semi-Infinite Slab

With modern tools, we can plot the evolution of the model directly (evaluated in Excel)



$h = 2.0 \frac{btu}{hft^2 F}$

$\alpha = 0.018 \frac{ft^2}{h}$

$k = 0.5 \frac{btu}{h} ft\ F$

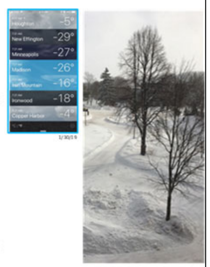
aa_solutions/Unsteady semi infinite solid plots.xlsx

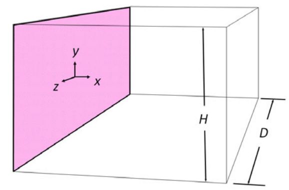
32

Solution Summary:

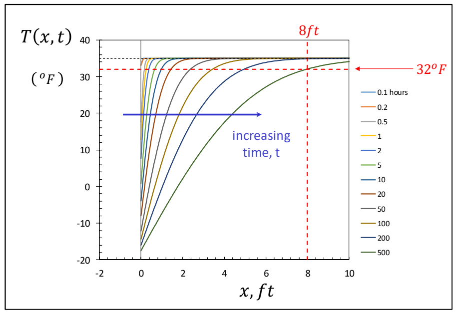
Example: When will my pipes freeze?

The temperature has been 35°F for a while now, sufficient to chill the ground to this temperature for many tens of feet below the surface. Suddenly the temperature drops to -20°F. How long will it take for freezing temperatures (32°F) to reach my pipes, which are 8 *ft* under ground?





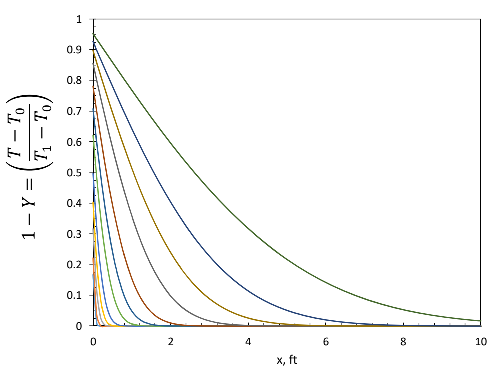
Answer:
$t = 509\ hours \approx 21\ days$



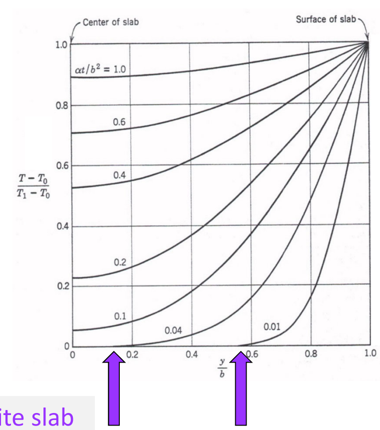
Note:

We can use the <u>semi-infinite</u> slab solution for <u>finite</u> slabs, within limits

Semi-Infinite Slab



Finite Slab



BSL1, p356, 1960

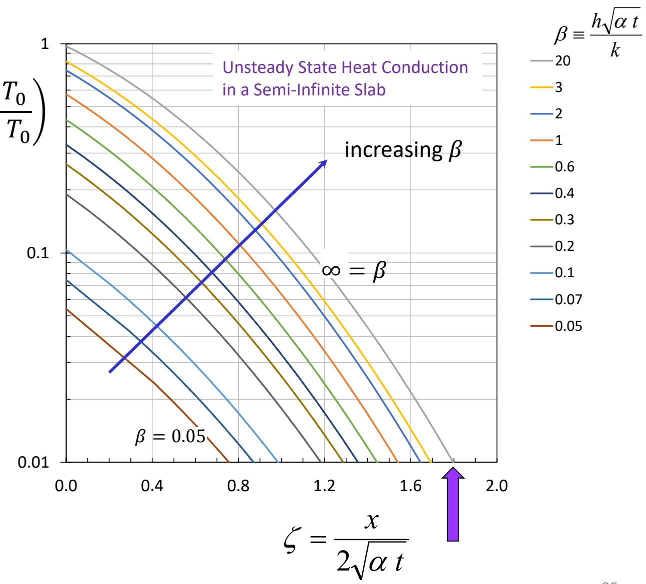
For some cases, the finite slab looks semi-infinite

- Short time
- Thicker slab

Note:

We can use the <u>semi-infinite</u> slab solution for <u>finite</u> slabs, within limits

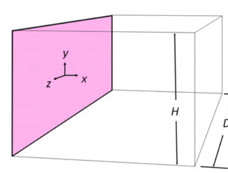
$$1 - Y = \left(\frac{T - T_0}{T_1 - T_0}\right)$$



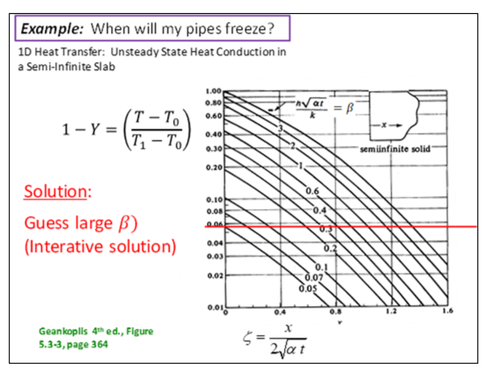
For values of $\zeta(t, x)$ that predict $T \approx T_0$ or $(1 - Y) = 0.01$ at a distance equal to the half-slab thickness b, the semi-infinite slab solution is equivalent to the finite slab; this occurs $\forall\ \beta$ when $\zeta_{half-slab} = \frac{b}{2\sqrt{\alpha t}} > 1.8$

35

TAKEAWAY:



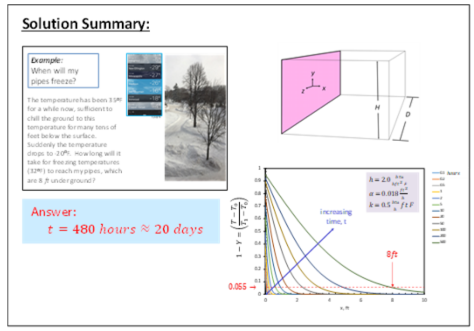
We're not making the case that this ONE solution is so important to learn. . .



Rather, knowing that solutions are available, *and* being able to set up and walk yourself through the published graph (chart, plot, equation) to answer a question of interest, *is* **an important engineering thinking skill**.

36

We used the **unsteady state microscopic energy balance** to solve one practical problem.



Next, we explore the **unsteady macroscopic energy balance.**

This adds another tool to our tool belt.