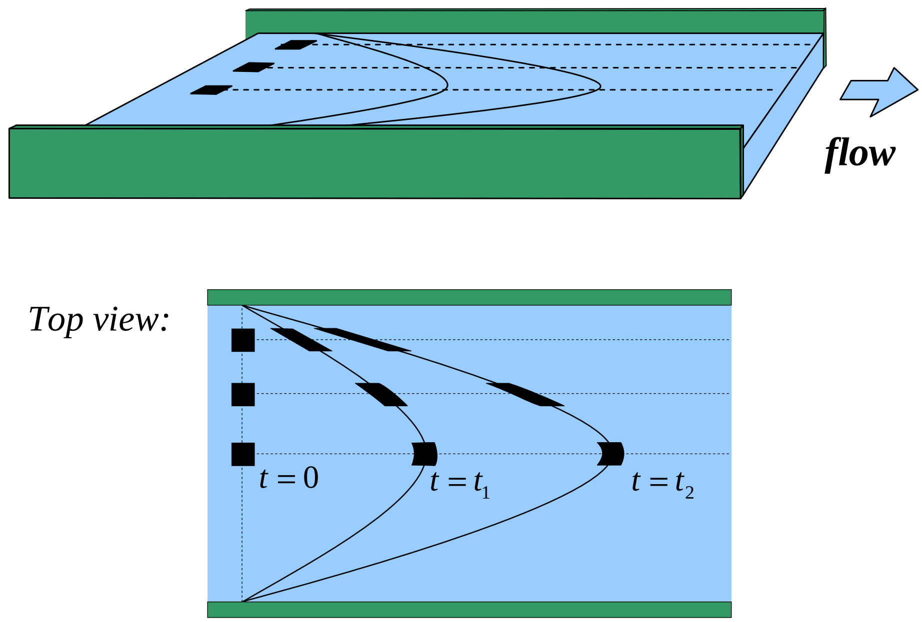


Figure 3.18: We begin again, considering a very simple flow, steady flow in a straight channel. Even in this simple flow, however, fluid particles undergo considerable deformation.

may proceed ahead to section 3.2.3 and return subsequently to this derivation section.

3.2.2.1 Momentum Balance on a Control Volume

The volume on which we do our balances is called the control volume. The control volume (CV) is an imaginary container through which fluid particles move (Figure 3.20). For the derivation of the momentum balance on a control volume, we consider an arbitrarily shaped control volume fixed in position and shape in an arbitrary flow (Figure 3.21).

At chosen time t, the control volume contains certain fluid particles. These fluid particles are a body in the sense of Newton's laws. We imagine that the fluid in the control volume at time t is colored red (Figure 3.22, left). The red fluid is subject to forces on it, and the relationship between the net forces on the red fluid and the momentum of the red

Momentum Conservation Equations

Individual Bodies

Newton's second law of motion

$$\sum_{\substack{\text{on}\\\text{body}}} \underline{f} = m\underline{a} = \frac{d\left(m\underline{v}\right)_{\text{body}}}{dt}$$

Control Volumes

Reynolds transport theorem

$$\sum_{\substack{\text{on}\\\text{CV}}} \underline{f} = \frac{d\left(m\underline{v}\right)_{\text{CV}}}{dt} + \iint_{\text{CS}} \left(\hat{n}\cdot\underline{v}\right)\rho\,\underline{v}\,dS$$

Figure 3.19: Momentum is conserved. For individual bodies, Newton's second law is a convenient equation for making calculations of motion and forces. For fluids, it is often more convenient to use an equivalent expression, the Reynolds transport theorem, the momentum balance written on a control volume.

fluid is given by Newton's second law.

$$\sum_{\substack{\text{on}\\\text{body}}} \underline{f} = m\underline{a} \quad = \quad \frac{d(m\underline{v})_{\text{body}}}{dt} \tag{3.52}$$

$$\sum_{\substack{\text{on}\\\text{body}}} \underline{f} = \left(\begin{array}{c}\text{net force}\\ \text{on red fluid at } t\\ = \text{net force}\\ \text{on CV at } t\end{array}\right) \quad = \quad \left(\begin{array}{c}\text{rate of change}\\ \text{of momentum of}\\ \text{red fluid at } t\end{array}\right) \tag{3.53}$$

We need to work on this equation to see how the momentum of the fluid in the control volume changes with time.

We can use the definition of derivative (equation 3.39) to rewrite the derivative that appears on the right-hand side of equation 3.52 as a limit of a rate of change of momentum

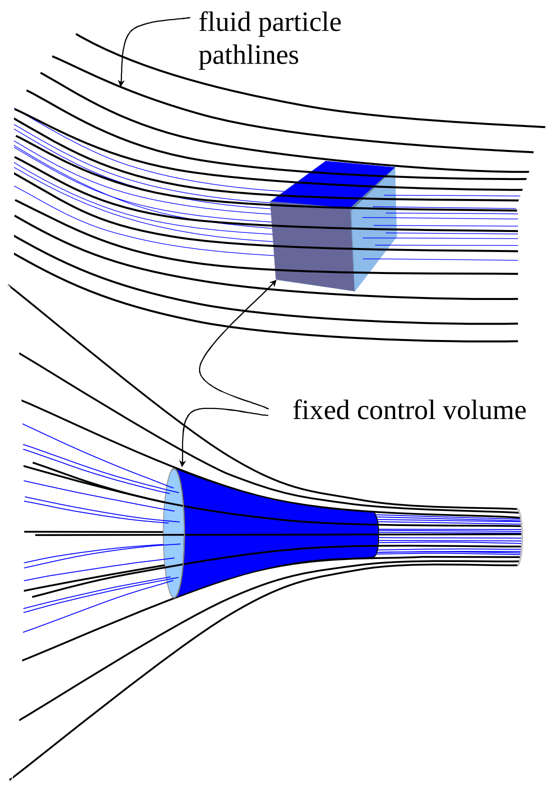


Figure 3.20: A control volume is an imagined region in space through which fluid moves. In our discussion we take the control volume to be fixed in shape and position. The shape of the control volume is arbitrary, and we usually choose a shape that mimics the flow pattern because this choice simplifies mass, momentum, and energy balance calculations. In this figure, the paths of the particles that pass through the control volume have been emphasized.

over the interval between time t and a slightly later time $t + \Delta t$.

$$\sum_{\substack{\text{on}\\\text{body}}} \underline{f}\Bigg|_t = \sum_{\substack{\text{on}\\\text{CV}}} \underline{f}\Bigg|_t = \left.\frac{d(m\underline{v})}{dt}\right|_t \tag{3.54}$$

$$= \lim_{\Delta t \longrightarrow 0} \left[\frac{(m\underline{v})|_{t+\Delta t} - (m\underline{v})|_t}{\Delta t} \right] \tag{3.55}$$

To fill-in the terms on the right-hand side of equation 3.55, we need to think about the momentum of the red fluid at t and at $t + \Delta t$. Our goal is to relate these quantities to the forces on the control volume.

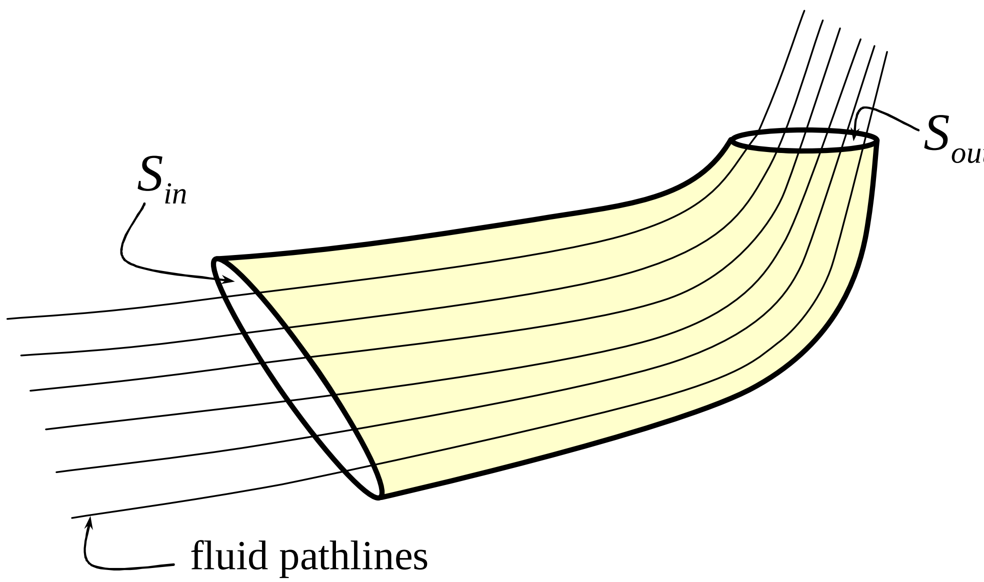


Figure 3.21: For the derivation of the momentum balance as applied to a control volume we do not assume any special shape for the control volume, but we do take the control volume to be fixed in shape and position.

Going back to our picture of the control volume (Figure 3.22) we can visualize the process of the red fluid passing through the control volume between times t and $t + \Delta t$. At time t, all the red fluid is in the control volume. At time $t + \Delta t$, some of the red fluid has left the control volume, and some of the upstream fluid has entered the control volume. For simplicity, we call the upstream fluid the blue fluid. First, we divide the red fluid into the red fluid that stays in the control volume between t and $t + \Delta t$ and the red fluid that leaves during that interval. Dropping the limit symbol, equation 3.55 becomes:

$$\Delta t \sum_{\substack{\text{on}\\ \text{CV}}} \underline{f}\Bigg|_t = \left(\begin{array}{c}\text{momentum of}\\ \text{red fluid}\end{array}\right)\Bigg|_{t+\Delta t} - \left(\begin{array}{c}\text{momentum of}\\ \text{red fluid}\end{array}\right)\Bigg|_{t} \tag{3.56}$$

$$= \left[\left(\begin{array}{c}\text{momentum of}\\ \text{red fluid}\\ \text{that stays}\end{array}\right) + \left(\begin{array}{c}\text{momentum of}\\ \text{red fluid}\\ \text{that exits}\end{array}\right)\right]\Bigg|_{t+\Delta t}$$
$$- \left[\left(\begin{array}{c}\text{momentum of}\\ \text{red fluid}\\ \text{that stays}\end{array}\right) + \left(\begin{array}{c}\text{momentum of}\\ \text{red fluid}\\ \text{that exits}\end{array}\right)\right]\Bigg|_{t} \tag{3.57}$$

Although we have temporarily omitted the limit symbol, at the end of this derivation we again take the limit as Δt goes to zero. We distinguish here between red fluid that ultimately

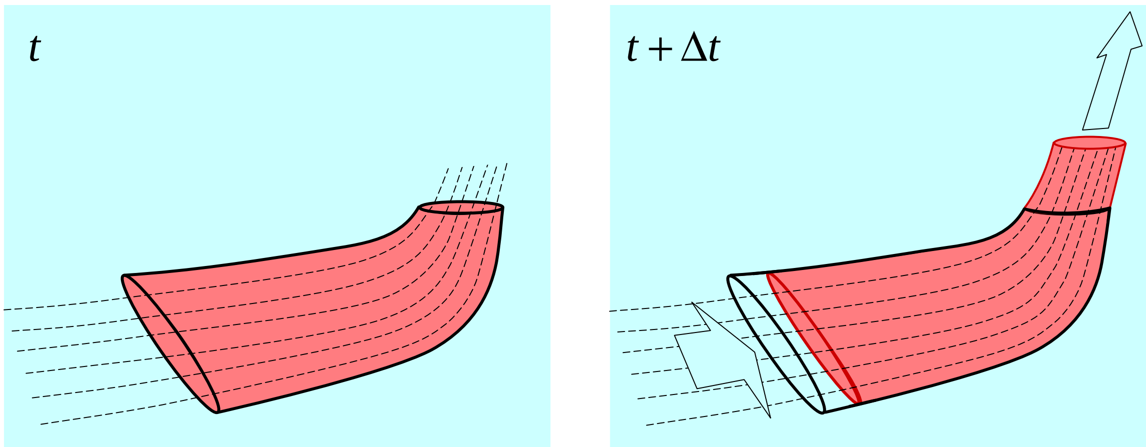


Figure 3.22: At time t the fluid in the control volume is imagined to be colored red and all fluid outside of the control volume is colored blue. At a slightly later time $t + \Delta t$, some of the red fluid has exited the control volume and some new (blue) fluid has entered through the inlet surface(s).

stays and red fluid that ultimately exits because this separation will be convenient in a later step in the derivation.

Newton's second law relates the net forces on a body (the red fluid) to the rate of change of momentum of the body. We are trying now to relate forces in a fluid to the rate of change of momentum of the fluid in the control volume. The fluid in the control volume is different fluid at different times, and that is the complicating factor. Beginning with the red-fluid momentum balance as written in equation 3.57, we can make some definitions and rearrangements that allow us to isolate the rate of change of momentum of the fluid in the control volume at a time of interest.

We define a variable $\underline{\mathbf{P}}$ to represent the momentum of the fluid in the control volume at any time.

$$\begin{pmatrix} \text{momentum} \\ \text{of fluid} \\ \text{in the CV} \end{pmatrix} \equiv \underline{\mathbf{P}} \tag{3.58}$$

Since the fluid in the control volume at time t is different fluid from the fluid in the control volume at time $t + \Delta t$, the momentum of the fluid in the control volume is different at these two times. We write the momentum of the fluid in the control volume at these two times in terms of red and blue fluid as follows.

First, at time t, the red fluid fills the control volume, so $\underline{\mathbf{P}}\big|_t$ is the momentum of all of the red fluid at t.

$$\left.\begin{pmatrix}\text{momentum}\\\text{of fluid}\\\text{in CV}\end{pmatrix}\right|_t = \underline{\mathbf{P}}|_t = \left.\begin{pmatrix}\text{momentum}\\\text{of red fluid}\\\text{that stays}\end{pmatrix}\right|_t + \left.\begin{pmatrix}\text{momentum}\\\text{of red fluid}\\\text{that exits}\end{pmatrix}\right|_t \tag{3.59}$$

Second, we write the momentum in the control volume at time $t+\Delta t$. At this time, the fluid in the control volume is the red fluid that stayed and the new blue fluid that entered.

$$\left.\begin{pmatrix}\text{momentum}\\\text{of fluid}\\\text{in CV}\end{pmatrix}\right|_{t+\Delta t} = \underline{\mathbf{P}}|_{t+\Delta t} = \left.\begin{pmatrix}\text{momentum}\\\text{of red fluid}\\\text{that stays}\end{pmatrix}\right|_{t+\Delta t} + \left.\begin{pmatrix}\text{momentum}\\\text{of blue fluid}\\\text{that enters}\end{pmatrix}\right|_{t+\Delta t} \tag{3.60}$$

We now combine the last two equations above with equation 3.57, the momentum balance on the red fluid; this yields a new relationship between forces and the fluid in the control volume.

First we solve equation 3.59 for the momentum at time t of red fluid that stays:

$$\left.\begin{pmatrix}\text{momentum}\\\text{of red fluid}\\\text{that stays}\end{pmatrix}\right|_t = \underline{\mathbf{P}}|_t - \left.\begin{pmatrix}\text{momentum}\\\text{of red fluid}\\\text{that exits}\end{pmatrix}\right|_t \tag{3.61}$$

Second, we solve equation 3.60 for the momentum at time $t+\Delta t$ of red fluid that stays:

$$\left.\begin{pmatrix}\text{momentum}\\\text{of red fluid}\\\text{that stays}\end{pmatrix}\right|_{t+\Delta t} = \underline{\mathbf{P}}|_{t+\Delta t} - \left.\begin{pmatrix}\text{momentum}\\\text{of blue fluid}\\\text{that enters}\end{pmatrix}\right|_{t+\Delta t} \tag{3.62}$$

Combining these two expressions with equation 3.57 results in:

$$\begin{aligned}\Delta t \left.\sum_{\substack{\text{on}\\\text{CV}}} \underline{f}\right|_t &= \left.\begin{pmatrix}\text{momentum}\\\text{of red fluid}\\\text{that stays}\end{pmatrix}\right|_{t+\Delta t} + \left.\begin{pmatrix}\text{momentum of}\\\text{red fluid}\\\text{that exits}\end{pmatrix}\right|_{t+\Delta t}\\ &\qquad - \left.\begin{pmatrix}\text{momentum}\\\text{of red fluid}\\\text{that stays}\end{pmatrix}\right|_t - \left.\begin{pmatrix}\text{momentum}\\\text{of red fluid}\\\text{that exits}\end{pmatrix}\right|_t &(3.63)\\ &= \underline{\mathbf{P}}|_{t+\Delta t} - \left.\begin{pmatrix}\text{momentum}\\\text{of blue fluid}\\\text{that enters}\end{pmatrix}\right|_{t+\Delta t} + \left.\begin{pmatrix}\text{momentum}\\\text{of red fluid}\\\text{that exits}\end{pmatrix}\right|_{t+\Delta t}\\ &\qquad - \underline{\mathbf{P}}|_t + \left.\begin{pmatrix}\text{momentum}\\\text{of red fluid}\\\text{that exits}\end{pmatrix}\right|_t - \left.\begin{pmatrix}\text{momentum}\\\text{of red fluid}\\\text{that exits}\end{pmatrix}\right|_t &(3.64)\end{aligned}$$

The final two terms cancel, yielding,

$$\boxed{\Delta t \sum_{\substack{\text{on} \\ \text{CV}}} \underline{f}\Bigg|_t = \underline{\mathbf{P}}\big|_{t+\Delta t} - \underline{\mathbf{P}}\big|_t - \left(\begin{array}{c}\text{momentum}\\ \text{of blue fluid}\\ \text{that enters}\end{array}\right)\Bigg|_{t+\Delta t} + \left(\begin{array}{c}\text{momentum}\\ \text{of red fluid}\\ \text{that exits}\end{array}\right)\Bigg|_{t+\Delta t}} \qquad (3.65)$$

We have made considerable progress in our quest to relate red-fluid momentum changes to momentum changes of the fluid in the control volume. To proceed further, we write mathematical expressions for the two quantities expressed in words on the right-hand side of equation 3.65. These two quantities are entering and exiting fluid momenta at $t + \Delta t$, that is, momenta of fluid that crosses the control-volume boundaries. Both of these expressions can be written following the same approach; the calculation results in a double integral over the control-volume bounding surfaces.

The final mathematical expression for the terms in equation 3.65 are derived in the next section. The final results, derived as equation 3.132, are given below. The two integrals are called the convective terms.

$$\sum_{\substack{\text{on} \\ \text{CV}}} \underline{f}\Bigg|_t = \frac{\underline{\mathbf{P}}\big|_{t+\Delta t} - \underline{\mathbf{P}}\big|_t}{\Delta t} + \left(\iint_{S_{in}} (\hat{n} \cdot \underline{v})\ \rho \underline{v}\ dS\right)\Bigg|_{t+\Delta t} + \left(\iint_{S_{out}} (\hat{n} \cdot \underline{v})\ \rho \underline{v}\ dS\right)\Bigg|_{t+\Delta t} \qquad (3.66)$$

3.2.2.2 The Convective Term

To convert the word expressions in equation 3.65 to mathematical terms, we need to consider how to use the continuum model to keep track of mass or momentum flow in through a surface. We begin by considering the simplest case of direct mass and momentum flow through a flat surface. We derive some key mathematical tools in the next two examples.

EXAMPLE 3.6 *Liquid passes through a chosen area A as shown in Figure 3.23. The velocity is perpendicular to the surface A at every point and does not vary across the cross-section. What are the volumetric flow rate (volume liquid/time), mass flow rate (mass/time), and momentum flow rate (momentum/time) through A?*

SOLUTION Figure 3.23 shows that for the case under consideration, the velocity of the fluid is perpendicular to the surface A and is constant (does not vary with position). Consider the fluid that passes through A during a short time interval Δt (Figure 3.24). The volume of fluid that passes through A during the

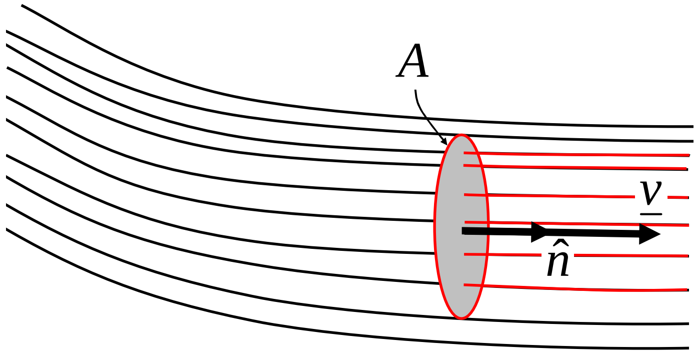


Figure 3.23: For this example we consider the flow through a surface A. The velocity of the fluid is perpendicular to the surface A.

interval Δt forms a solid whose volume is given by

$$\begin{pmatrix} \text{Volume of fluid} \\ \text{passing through } A \\ \text{in time } \Delta t \end{pmatrix} = \begin{pmatrix} \text{height} \\ \text{of solid} \end{pmatrix} \begin{pmatrix} \text{cross-section} \\ \text{of solid} \end{pmatrix} \tag{3.67}$$

$$= \Delta x \; A \tag{3.68}$$

where Δx is the change in location of fluid that started at A and has moved in the x-direction for time Δt. The magnitude of the fluid velocity, v, can be written as

$$\begin{array}{c}\text{Magnitude of} \\ \text{fluid velocity}\end{array} \qquad |\underline{v}| = v = \frac{\Delta x}{\Delta t} \tag{3.69}$$

With these two expressions we can calculate all the quantities of interest. The volumetric flow rate is the volume of fluid divided by the time interval.

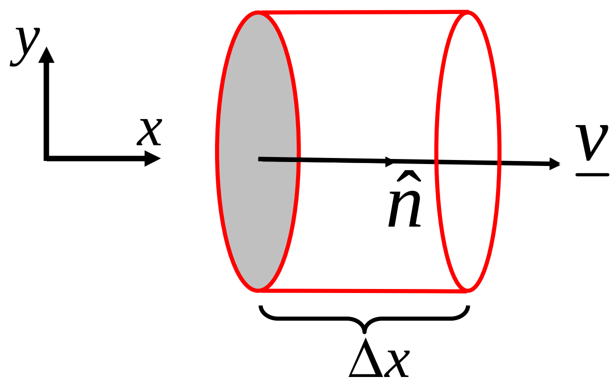
$$Q = \frac{\text{fluid volume}}{\text{time interval}} = \frac{\Delta x \; A}{\Delta t} = v \; A \tag{3.70}$$

$$\begin{array}{c}\text{Volumetric flow} \\ \text{of liquid through } A \\ \text{(velocity perpendicular to } A; \\ \underline{v} \text{ does not vary across } A)\end{array} \qquad \boxed{Q = v \; A} \tag{3.71}$$

The mass flow rate can be calculated from the volumetric flow rate and the density.

$$m = \left(\frac{\text{mass}}{\text{volume}}\right)\left(\frac{\text{volume}}{\text{time}}\right) \tag{3.72}$$

$$= (\rho)(v \; A) \tag{3.73}$$



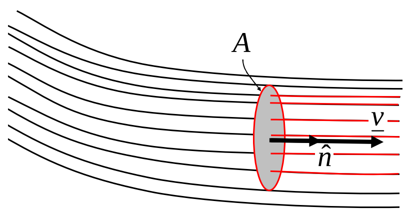


Figure 3.24: During the time interval Δt, a volume of fluid of height Δx and of cross-sectional area A passes through the area A.

$$\begin{array}{c}\text{Mass flow}\\ \text{of liquid through } A\\ \text{(velocity perpendicular to } A;\\ \underline{v} \text{ does not vary across } A)\end{array} \quad \boxed{m = (\rho)(v\ A)} \tag{3.74}$$

Finally, the momentum flow rate (a vector quantity) can be calculated from the definition of momentum and the previous results.

$$\left(\begin{array}{c}\text{Momentum flow}\\ \text{of liquid through } A\end{array}\right) = \left(\frac{\text{momentum}}{\text{volume}}\right)\left(\frac{\text{volume}}{\text{time}}\right) \tag{3.75}$$

$$= \frac{(\text{mass})(\text{velocity})}{\text{volume}}\left(\frac{\text{volume}}{\text{time}}\right) \tag{3.76}$$

$$= \left(\frac{\text{mass}}{\text{volume}}\right)(\underline{v})\left(\frac{\text{volume}}{\text{time}}\right) \tag{3.77}$$

$$= \rho\ \underline{v}\ (vA) \tag{3.78}$$

Note that for this example the velocity of the fluid was perpendicular to the surface A and $\underline{v}$ does not vary across A.

$$\boxed{\begin{array}{c}\text{Momentum flow}\\ \text{of liquid through } A\\ \text{(velocity perpendicular to } A;\\ \underline{v} \text{ does not vary across } A)\end{array} = \rho\ \underline{v}\ (vA)} \tag{3.79}$$

The previous example shows how powerful the continuum approach is. With very simple logic (essentially unit matching), we are able to express volume, mass, and momentum flows for a chosen system in terms of two field variables, density and velocity. For more complex systems, we build on these relationships and employ some vector tools, as we show in the next example.

EXAMPLE 3.7 *Liquid passes through a chosen area A as shown in Figure 3.25. The velocity of the fluid makes an angle θ with the unit normal to A, which is called $\hat{n}$. The velocity does not vary across the surface A. What are the volumetric flow rate (volume liquid/time), mass flow rate (mass/time), and momentum flow rate (momentum/time) through A?*

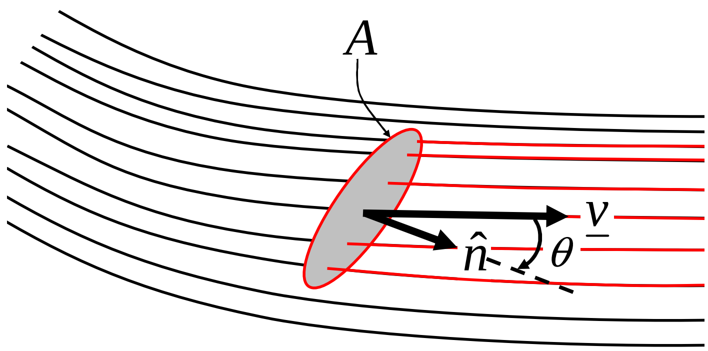


Figure 3.25: For this example we consider the flow through a surface A. The velocity of the fluid is not perpendicular to the surface A; instead, the velocity makes an angle θ with the surface unit normal $\hat{n}$.

SOLUTION The logic of the solution is the same for this case as in the previous example; there is, however, a difference in the volume of fluid that passes through A in time interval Δt.

Consider the fluid that passes through A during the short time interval Δt (Figure 3.26). The x-direction is the direction of flow. In time interval Δt fluid that started on the surface A moved along x a distance Δx. The volume of fluid that passed through A in this time interval is the volume of the solid shown. The height of the solid is $\Delta x \cos\theta$. The volume of fluid that passes through A during

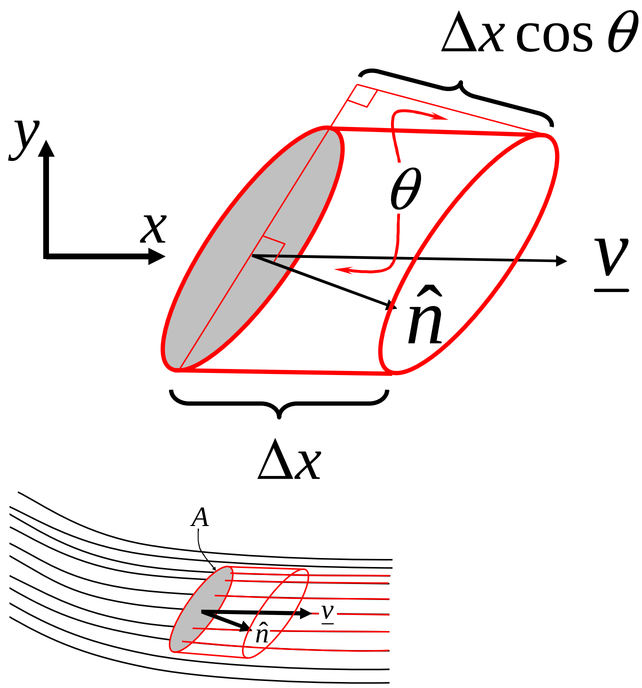


Figure 3.26: During the time interval Δt, a volume of fluid of height $\Delta x \cos\theta$ and of cross-sectional area A passes through the area A.

the interval Δt is thus given by

$$\begin{aligned} \left(\begin{array}{c}\text{Volume of fluid}\\ \text{passing through } A\\ \text{in time } \Delta t\end{array}\right) &= \left(\begin{array}{c}\text{height}\\ \text{of solid}\end{array}\right)\left(\begin{array}{c}\text{cross-section}\\ \text{of solid}\end{array}\right) && (3.80)\\ &= (\Delta x \cos\theta)\ A && (3.81)\end{aligned}$$

The magnitude of the fluid velocity, v, can be written as before as

$$\begin{array}{c}\text{Magnitude of}\\ \text{fluid velocity}\end{array} \qquad |\underline{v}| = v = \frac{\Delta x}{\Delta t} \tag{3.82}$$

With these two expressions we can calculate all the quantities of interest.

$$\begin{aligned} \begin{array}{c}\text{Volumetric flow}\\ \text{of liquid through } A\end{array} \qquad Q &= \frac{\text{fluid volume}}{\text{time interval}} && (3.83)\\ &= \frac{\Delta x \cos\theta\ A}{\Delta t} && (3.84)\\ &= v \cos\theta\ A && (3.85)\\ &= (\hat{n}\cdot\underline{v})A && (3.86)\end{aligned}$$

$$\begin{array}{c}\text{Volumetric flow}\\ \text{of liquid through } A\\ \text{(general orientation case;}\\ \underline{v} \text{ does not vary across } A)\end{array} \qquad \boxed{Q = v\cos\theta\ A = (\hat{n}\cdot\underline{v})A} \tag{3.87}$$

We have used the definition of the dot product to write the final result (equation 3.85) in vector notation ($\hat{n}\cdot\underline{v} = |\hat{n}||\underline{v}|\cos\theta = v\cos\theta$; see equation 1.165). As before, the mass flow rate can be calculated from the volumetric flow rate and the density.

$$\begin{array}{c}\text{Mass flow}\\ \text{of liquid through } A\end{array} \qquad m = \left(\frac{\text{mass}}{\text{volume}}\right)\left(\frac{\text{volume}}{\text{time}}\right) \tag{3.88}$$

$$= (\rho)\ (v\cos\theta\ A) = \rho\ (\hat{n}\cdot\underline{v})\ A \tag{3.89}$$

$$\begin{array}{c}\text{Mass flow}\\ \text{of liquid through } A\\ \text{(general orientation case;}\\ \underline{v} \text{ does not vary across } A)\end{array} \qquad \boxed{m = \rho\ (\hat{n}\cdot\underline{v})\ A} \tag{3.90}$$

Finally, the momentum flow rate can be calculated as before from the definition of momentum and the previous results.

$$\left(\begin{array}{c}\text{Momentum flow}\\ \text{of liquid through } A\end{array}\right) = \left(\frac{\text{momentum}}{\text{volume}}\right)\left(\frac{\text{volume}}{\text{time}}\right) \tag{3.91}$$

$$= \frac{\text{(mass)(velocity)}}{\text{volume}}\left(\frac{\text{volume}}{\text{time}}\right) \tag{3.92}$$

$$= \left(\frac{\text{mass}}{\text{volume}}\right)(\underline{v})\left(\frac{\text{volume}}{\text{time}}\right) \tag{3.93}$$

$$= \rho\ \underline{v}\ (v\cos\theta\ A) = \rho\ \underline{v}\ (\hat{n}\cdot\underline{v})A \tag{3.94}$$

This is the general result when $\underline{v}$ is not necessarily perpendicular to A.

$$\boxed{\left(\begin{array}{c}\text{Momentum flow}\\ \text{of liquid through } A\\ \text{(general orientation case;}\\ \underline{v} \text{ does not vary across } A)\end{array}\right) = \rho\underline{v}\ (\hat{n}\cdot\underline{v})A} \tag{3.95}$$

We recover the case of velocity perpendicular to A (equation 3.79) when $\theta = 0$ ($\cos 0 = 1$, $\hat{n}\cdot\underline{v} = v$).

The relationship we obtained in equation 3.87 for volumetric flow rate through an area as a function of the locally constant velocity $\underline{v}$ ($Q = (\hat{n}\cdot\underline{v})A$) is similar to an equation

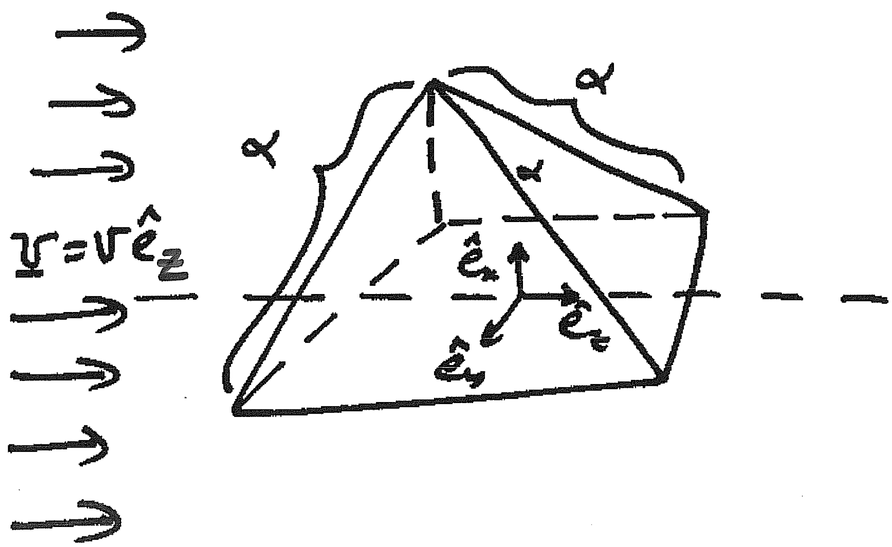


Figure 3.27: The control volume is a square pyramid, which has five sides, four of which are equilateral triangles.

introduced in Chapter 1 that relates overall volumetric flow rate through a pipe to the *average* velocity in the pipe $\langle v \rangle$ (equation 1.2). If we write equation 3.87 on a microscopic piece of cross-sectional area in a pipe flow with varying $\underline{v}$ and integrate over the pipe cross section (recall equation 1.157) we obtain equation 1.2; this calculation is shown in Chapter 6 (equation 6.254). In the example below, we practice a bit with the relations we have just developed.

EXAMPLE 3.8 *Consider a control volume in the shape of the square pyramid as shown in Figures 3.27 and 3.28. The square pyramid is a pentahedron with a square for a base and four triangles for sides; the one in Figure 3.27 has four equilateral triangles for sides (a Johnson solid). The pyramid is a control volume placed in a uniform flow (velocity $\underline{v}$ in the flow is constant at every position in space). The flow direction is parallel at all points to a vector in the plane of the pyramid's base that bisects two opposite sides of the base. Calculate the mass flow rate of fluid of density ρ through each of the five sides of the pentahedron. Write your answer in terms of the speed of the fluid v and the pyramid edge-length α.*

SOLUTION The use of a pentahedron as a control volume is unusual, but the calculations involved in solving this problem are not unusual at all when making calculations of the convective contribution to the momentum balance.

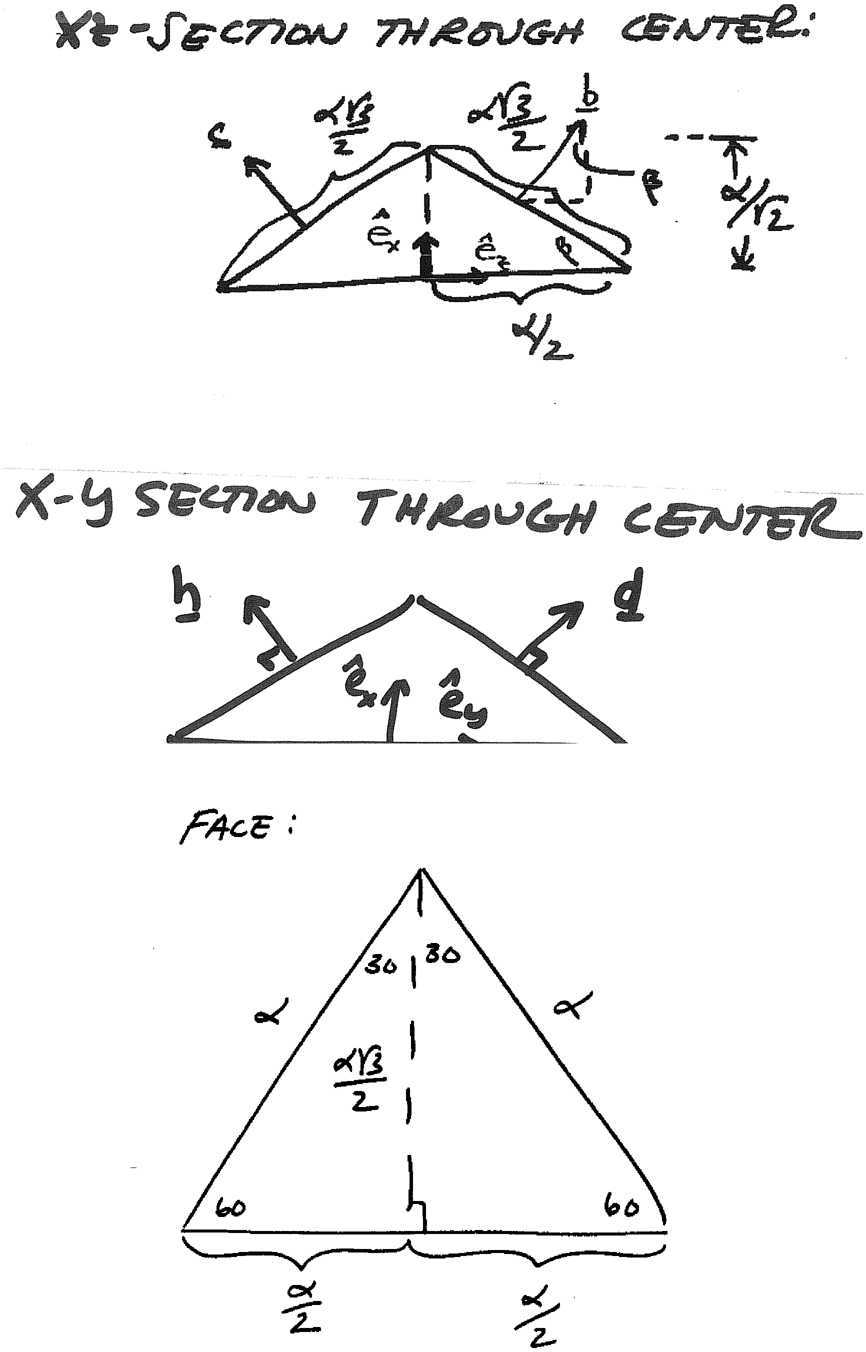


Figure 3.28: The unit normals needed for the calculations in the example can be determined through the geometry of sections cut through the center of the control volume.

This problem provides us with an opportunity to practice with angles, geometry, the dot product, and the relations in this section.

The mass flow through a surface is given by equation 3.90.

$$\text{Mass flow of liquid through surface } A \qquad m = \rho\ (\hat{n} \cdot \underline{v})\ A \tag{3.96}$$

For each of the five surfaces of the control volume we need the unit normal $\hat{n}$ and the area A. The density ρ is constant, and the velocity vector $\underline{v} = v\hat{e}_z$ is the same at all locations for uniform flow.

We choose as our coordinate system a Cartesian coordinate system with the flow direction as the z-direction.

$$\underline{v} = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}_{xyz} = v\hat{e}_z \tag{3.97}$$

The outwardly pointing unit normal vectors for each surface of the control volume are shown in Figure 3.27. For the bottom of the pyramid, the outwardly pointing unit vector $\underline{a}$ points downward, $\underline{a} = -\hat{e}_x$. The dot product of $\underline{a}$ and $\underline{v} = v\hat{e}_z$ is therefore zero, and the mass flow rate through the bottom is zero:

$$\begin{aligned} m &= \rho\ (\hat{n} \cdot \underline{v})\ A && (3.98)\\ m|_a &= \rho(\underline{a} \cdot \underline{v})\alpha^2 && (3.99)\\ &= \rho\alpha^2 \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}_{xyz} \cdot \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}_{xyz} && (3.100)\\ &= 0 && (3.101) \end{aligned}$$

For surface b, the geometry in the inset of Figure 3.27 shows us that the outwardly pointing unit normal vector $\underline{b}$ is

$$\text{from geometry:} \qquad \hat{n}|_b \equiv \underline{b} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ \sqrt{\frac{2}{3}} \end{pmatrix}_{xyz} \tag{3.102}$$

and the area of the equilateral triangle that makes up the face is $A =$

$(1/2)(\alpha)(\alpha\sqrt{3}/2)$. The mass flow rate through surface b is therefore

$$m = \rho\ (\hat{n}\cdot\underline{v})\ A \tag{3.103}$$

$$m|_b = \rho(\underline{b}\cdot\underline{v})\frac{\alpha^2\sqrt{3}}{4} \tag{3.104}$$

$$= \frac{\rho\alpha^2\sqrt{3}}{4}\begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & \sqrt{\frac{2}{3}} \end{pmatrix}_{xyz}\cdot\begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}_{xyz} \tag{3.105}$$

$$= \frac{\rho v\alpha^2}{2\sqrt{2}} \tag{3.106}$$

For surface c, also shown in the insert, the outwardly pointing unit normal vector $\underline{c}$ is similar to $\underline{b}$, but the z-component points in the opposite direction.

$$\text{from geometry:} \qquad \hat{n}|_c \equiv \underline{c} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ -\sqrt{\frac{2}{3}} \end{pmatrix}_{xyz} \tag{3.107}$$

The mass flow rate through surface c is therefore

$$m|_c = \rho(\underline{c}\cdot\underline{v})\frac{\alpha^2\sqrt{3}}{4} \tag{3.108}$$

$$= \frac{\rho\alpha^2\sqrt{3}}{4}\begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & -\sqrt{\frac{2}{3}} \end{pmatrix}_{xyz}\cdot\begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}_{xyz} \tag{3.109}$$

$$= -\frac{\rho v\alpha^2}{2\sqrt{2}} \tag{3.110}$$

The mass flow rates out through surfaces b and c are the same but one is positive, indicating that the flow is outwards (surface $\underline{b}$) and one is negative, indicating that the flow is inwards (surface c).

For surfaces d and h, the two side faces of the pyramid, the unit normal vectors are in the xy-plane, and thus when the outwardly pointed unit normal $\hat{n}$ is dotted with $\underline{v} = v\hat{e}_z$ in each case, we get zero; there is no mass flow out of the control volume through either of these surfaces.

$$\hat{n}|_d = \underline{d} = \begin{pmatrix} d_x \\ d_y \\ 0 \end{pmatrix}_{xyz} \tag{3.111}$$

$$\hat{d}\cdot\underline{v} = (d_x\hat{e}_x + d_y\hat{e}_y)\cdot v\hat{e}_z = 0 \tag{3.112}$$

$$\hat{n}|_h = \underline{h} = \begin{pmatrix} h_x \\ h_y \\ 0 \end{pmatrix}_{xyz} \tag{3.113}$$

$$\hat{h}\cdot\underline{v} = (h_x\hat{e}_x + h_y\hat{e}_y)\cdot\underline{v}\hat{e}_z = 0 \tag{3.114}$$

Finally, notice that the sum of all the mass flow rates is zero; this is in accord with the mass balance that at steady state the net outflow of mass from the control volume is zero.

$$\begin{aligned}
\begin{pmatrix} \text{net outflow} \\ \text{of mass from} \\ \text{control volume (CV)} \end{pmatrix} &= m|_a + m|_b + m|_c + m|_d + m|_h && (3.115) \\
&= 0 + \frac{\rho v \alpha^2}{2\sqrt{2}} - \frac{\rho v \alpha^2}{2\sqrt{2}} + 0 + 0 && (3.116) \\
&= 0 && (3.117)
\end{aligned}$$

We return now to our work with equation 3.65. We seek to covert the two word-expressions in that equation to mathematical terms. Both of the word-expressions under consideration account for momentum flows of fluid through the surfaces that bound the control volume. In example 3.8 we practiced writing momentum flows through a surface (equation 3.95), and we now turn to applying this technique to the control volume.

Beginning with the blue fluid that enters the control volume, consider the surface area S_{in} through which blue fluid enters (Figure 3.29). We have chosen a surface with an

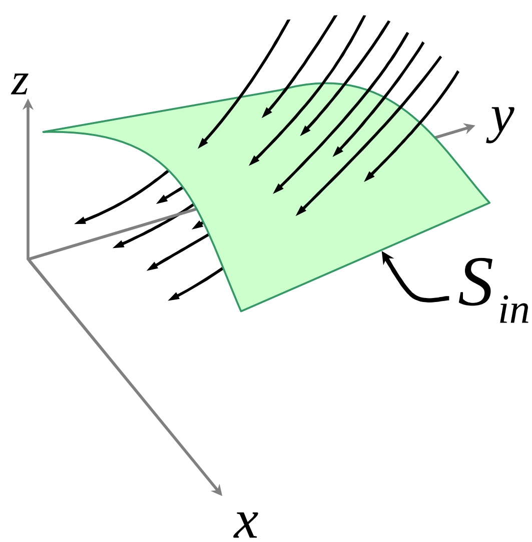


Figure 3.29: The momentum carried by fluid moving across a curved surface is calculated with a surface integral.

arbitrary shape and orientation for this derivation. In a general flow, fluid velocity varies with position, and therefore some care must be taken when calculating the momentum entering the control volume through S_{in}. We must divide up the surface S_{in} in some way and sum the contributions from various regions. In addition, the surface S_{in} is not generally flat, and therefore the task of dividing S_{in} is itself a challenge. This very problem has been addressed in the development of integral calculus (for review see the web appendix [124]), and we can directly apply these methods to the calculation of the flow of momentum through S_{in}.

Our approach is to project S_{in} onto a plane we arbitrarily call the xy-plane (Figure 3.30). The area of the projection is $\mathcal{R}$. Since $\mathcal{R}$ is in the xy-plane, the unit normal to $\mathcal{R}$

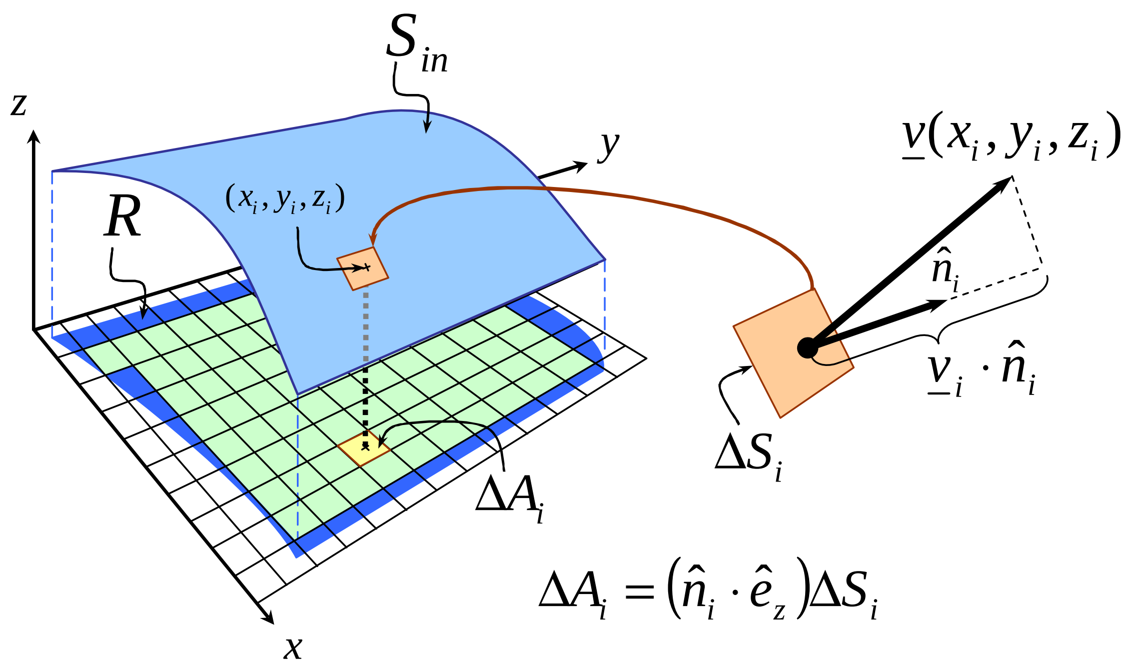


Figure 3.30: For a surface that is not flat, we first project the surface onto a plane called the xy-plane. We then divide up the projection and proceed to write and sum up the momentum flow rate through each small piece. The surface differential ΔS_i can be related to ΔA_i, its projection onto the xy-plane, by $\Delta S_i = \Delta A_i/(\hat{n}_i \cdot \hat{e}_z)$.

is $\hat{e}_z$. We divide the projection $\mathcal{R}$ into areas $\Delta A = \Delta x \Delta y$ and seek to write the momentum flow rate in different regions of S_{in} associated with their projections ΔA_i. By focusing on $\mathcal{R}$ and equal-sized divisions of $\mathcal{R}$ (rather than dividing the curvy surface S_{in} directly), we can arrive at the appropriate integral expression.

Figure 3.30 shows the area S_{in} and its projection $\mathcal{R}$ in the xy-plane. The area $\mathcal{R}$ has been divided into rectangles of area ΔA_i, and we only consider the ΔA_i that are wholly contained within the boundaries of $\mathcal{R}$. For each ΔA_i in the xy-plane, we choose a point within ΔA_i, and we call this point $(x_i, y_i, 0)$. The point (x_i, y_i, z_i) is located on the surface S_{in} directly above $(x_i, y_i, 0)$. If we draw a plane tangent to S_{in} through the point (x_i, y_i, z_i), we can construct an area ΔS_i that is a portion of the tangent plane whose projection onto the xy-plane is ΔA_i (Figure 3.30). We shall soon take a limit as ΔA_i becomes infinitesimally small, and therefore it is not important which point $(x_i, y_i, 0)$ is chosen so long as it is in ΔA_i.

Each tangent-plane area ΔS_i approximates a portion of the surface S_{in}, and we can write an estimate of the total momentum flow through S_{in} as a sum of the momentum flows through all the tangent planes ΔS_i. The momentum entering the control volume (CV) between t and $t + \Delta t$ through one such ΔS_i can be calculated as:

$$\begin{pmatrix} \text{momentum} \\ \text{entering CV} \\ \text{through } i^{th} \\ \text{tangent plane } \Delta S_i \end{pmatrix} = \left(\frac{\text{momentum}}{\text{volume}}\right)\left(\frac{\text{volume flow inwards}}{\text{time}}\right)\Delta t \quad (3.118)$$

Volumetric flow inwards may be written with the aid of equation 3.87.

$$\begin{aligned} \begin{pmatrix} \text{momentum} \\ \text{entering CV} \\ \text{through } i^{th} \\ \text{tangent plane } \Delta S_i \end{pmatrix} &= \left(\frac{\text{mass} \cdot \text{velocity}}{\text{volume}}\right)\begin{pmatrix} \text{inflow} & \\ \text{velocity} & \cdot \text{area} \\ \text{magnitude} & \end{pmatrix}\Delta t \quad (3.119) \\ &= \left(\rho_i \, \underline{v}|_i\right)\left(-(\hat{n}_i \cdot \underline{v}|_i)\Delta S_i\right)\Delta t \quad (3.120) \end{aligned}$$

where ρ_i and $\underline{v}|_i$ are the density and velocity at (x_i, y_i, z_i), and $\hat{n}_i$ is the outwardly pointing unit normal vector at (x_i, y_i, z_i) (compare with equation 3.95). Note that we have a choice here for unit normal vector $\hat{n}_i$ since any surface has two unit normal vectors, one pointing into the control volume and one pointing out of the control volume. The convention in fluid mechanics is to choose the outwardly pointing unit normal. The negative sign in equation 3.120 is a consequence of this choice, and the expression $\hat{n}_i \cdot \underline{v}|_i$ corresponds to the outwardly moving component of the velocity. Since we are interested in the inwardly moving flow in equation 3.120, we must include a negative sign.

Equation 3.120 gives the contribution of momentum passing through each ΔS_i. To approximate the total momentum flow through S_{in} we now sum over all tangent-planes ΔS_i. Note that we are only including the ΔS_i that are associated with those projections ΔA_i that are fully contained within $\mathcal{R}$. Subsequently we take the limit as ΔA becomes small to make the calculation exact.

$$\begin{pmatrix} \text{momentum} \\ \text{of blue fluid} \\ \text{that enters CV} \end{pmatrix} \approx \sum_{i=1}^{N} \begin{pmatrix} \text{momentum} \\ \text{entering CV} \\ \text{through } i^{th} \\ \text{tangent plane } \Delta S_i \end{pmatrix} \tag{3.121}$$

$$= -\sum_{i=1}^{N} \left(\rho_i \, \underline{v}|_i\right) \left((\hat{n}_i \cdot \underline{v}|_i)\Delta S_i\right) \Delta t \tag{3.122}$$

$$= -\Delta t \sum_{i=1}^{N} \left((\hat{n}_i \cdot \underline{v}|_i)\rho_i \, \underline{v}|_i \, \Delta S_i\right) \tag{3.123}$$

$$\begin{pmatrix} \text{momentum} \\ \text{of blue fluid} \\ \text{that enters CV} \end{pmatrix} = -\Delta t \lim_{\Delta A \longrightarrow 0} \left[\sum_{i=1}^{N} \left((\hat{n}_i \cdot \underline{v}|_i)\rho_i \, \underline{v}|_i \, \Delta S_i\right) \right] \tag{3.124}$$

where N is the number of projections ΔA_i that are wholly within $\mathcal{R}$. We can relate the tangent-plane area ΔS_i and the projected area ΔA_i through geometry (see the web appendix [124]). The result is

$$\Delta A_i = (\hat{n}_i \cdot \hat{e}_z)\Delta S_i \tag{3.125}$$

where $\hat{e}_z$ is the unit normal of the ΔA_i, and $\hat{n}_i$ is the unit normal of ΔS_i. Substituting this relationship, equation 3.124 becomes

$$\begin{pmatrix} \text{momentum} \\ \text{of blue fluid} \\ \text{that enters CV} \end{pmatrix} = -\Delta t \lim_{\Delta A \longrightarrow 0} \left[\sum_{i=1}^{N} \frac{(\hat{n}_i \cdot \underline{v}|_i)\rho_i \, \underline{v}|_i}{\hat{n}_i \cdot \hat{e}_z} \Delta A_i \right] \tag{3.126}$$

The limit of the sum on the right-hand side of equation 3.126 is related to the definition of a double integral[124]. The double integral is defined as

$$\begin{matrix} \text{Double integral} \\ \text{of a function} \\ \text{(general version)} \end{matrix} \qquad \boxed{I = \iint_{\mathcal{R}} f(x,y) \; dA \equiv \lim_{\Delta A \longrightarrow 0} \left[\sum_{i=1}^{N} f(x_i, y_i)\Delta A_i \right]} \tag{3.127}$$

where $\mathcal{R}$ is the region in the xy-plane over which f is being integrated (summed). Comparing equations 3.126 and 3.127 we write

$$\begin{pmatrix} \text{momentum} \\ \text{of blue fluid} \\ \text{that enters CV} \end{pmatrix} = -\Delta t \iint_{\mathcal{R}} \frac{(\hat{n} \cdot \underline{v})\rho \underline{v}}{\hat{n} \cdot \hat{e}_z} \; dA \tag{3.128}$$

If we define $dS \equiv dA/(\hat{n} \cdot \hat{e}_z)$, then equation 3.128 becomes[124]

$$\begin{pmatrix} \text{momentum} \\ \text{of blue fluid} \\ \text{that enters CV} \end{pmatrix} = -\Delta t \iint_{S_{in}} (\hat{n} \cdot \underline{v}) \; \rho \underline{v} \; dS \tag{3.129}$$

This is the expression we needed to finish equation 3.65. Our calculations show that the momentum of blue fluid that enters the control volume is equal to the surface integral of the crossing momentum per unit volume $(\hat{n} \cdot \underline{v})\ \rho\underline{v}$ over the inlet surface S_{in}.

The momentum of the red fluid that exits the control volume may be written in a similar way, resulting in an analogous integral over the outflow surface S_{out}.

$$\left(\begin{array}{c} \text{momentum} \\ \text{of red fluid} \\ \text{that exits CV} \end{array} \right) = \Delta t \iint_{S_{out}} (\hat{n} \cdot \underline{v})\ \rho\underline{v}\ dS \tag{3.130}$$

Notice that there is no negative sign in equation 3.130 (recall the discussion related to equation 3.120), since in this case we are accounting for fluid that is exiting, and the outwardly pointing normal dotted with the velocity vector gives the component of velocity corresponding to outflow. We now substitute the results in equation 3.130 and equation 3.129 into equation 3.65 to replace the word expressions.

$$\Delta t \sum_{\substack{\text{on}\\ \text{CV}}} \underline{f}\Big|_t = \underline{\mathbf{P}}\big|_{t+\Delta t} - \underline{\mathbf{P}}\big|_t - \left(\begin{array}{c} \text{momentum} \\ \text{of blue fluid} \\ \text{that enters} \end{array} \right)\Bigg|_{t+\Delta t} + \left(\begin{array}{c} \text{momentum} \\ \text{of red fluid} \\ \text{that exits} \end{array} \right)\Bigg|_{t+\Delta t} \tag{3.131}$$

$$\sum_{\substack{\text{on}\\ \text{CV}}} \underline{f}\Big|_t = \frac{\underline{\mathbf{P}}\big|_{t+\Delta t} - \underline{\mathbf{P}}\big|_t}{\Delta t} + \left(\iint_{S_{in}} (\hat{n} \cdot \underline{v})\ \rho\underline{v}\ dS \right)\Bigg|_{t+\Delta t} + \left(\iint_{S_{out}} (\hat{n} \cdot \underline{v})\ \rho\underline{v}\ dS \right)\Bigg|_{t+\Delta t} \tag{3.132}$$

The two integrals in equation 3.132 may be combined, since the first is over all inlet surfaces and the second is over all outlet surfaces. All surfaces of the control volume are either inlet surfaces or outlet surfaces, or surfaces through which no fluid passes. Surfaces through which no fluids pass would have $\hat{n} \cdot \underline{v} = 0$ since $\underline{v} = 0$ there. We can therefore safely write these two integrals together as the integral over the entire enclosing surface of the control volume, CS.

$$\left(\iint_{S_{in}} (\hat{n} \cdot \underline{v})\ \rho\underline{v}\ dS \right)\Bigg|_{t+\Delta t} + \left(\iint_{S_{out}} (\hat{n} \cdot \underline{v})\ \rho\underline{v}\ dS \right)\Bigg|_{t+\Delta t} = \left(\iint_{CS} (\hat{n} \cdot \underline{v})\ \rho\underline{v}\ dS \right)\Bigg|_{t+\Delta t} \tag{3.133}$$

Making this change in equation 3.132 and taking the limit as Δt goes to zero, we arrive at the final relationship we seek, the relationship between the forces on the control volume and the rate of change of momentum of the fluid in the control volume.

$$\sum_{\substack{\text{on}\\ \text{CV}}} \underline{f}\Big|_t = \lim_{\Delta t \longrightarrow 0} \left(\frac{\underline{\mathbf{P}}\big|_{t+\Delta t} - \underline{\mathbf{P}}\big|_t}{\Delta t} \right) + \lim_{\Delta t \longrightarrow 0} \left(\iint_{CS} (\hat{n} \cdot \underline{v})\ \rho\underline{v}\ dS \right)\Bigg|_{t+\Delta t} \tag{3.134}$$

Reynolds transport theorem (momentum balance on CV)

$$\boxed{\sum_{\substack{\text{on}\\ \text{CV}}} \underline{f} = \frac{d\underline{\mathbf{P}}}{dt} + \iint_{CS} (\hat{n} \cdot \underline{v})\ \rho \underline{v}\ dS} \tag{3.135}$$

$$\left(\begin{array}{c}\text{sum of}\\ \text{forces}\\ \text{on a CV}\end{array}\right) = \left(\begin{array}{c}\text{rate of}\\ \text{increase of}\\ \text{momentum of}\\ \text{fluid in CV}\end{array}\right) + \left(\begin{array}{c}\text{net outflow of}\\ \text{momentum}\\ \text{through bounding}\\ \text{surfaces of CV}\end{array}\right) \tag{3.136}$$

In going from equation 3.134 to equation 3.135 we have once again employed the fundamental definition of a derivative (equation 3.39).[7] The integral term is called the convective term.

Equation 3.135, called the Reynolds transport theorem, gives the equivalent of Newton's second law ($\sum \underline{f} = m\underline{a}$) for a control volume. The Reynolds transport theorem states that the sum of forces on a control volume is equal to the rate of increase of momentum of the fluid in the control volume plus the net outward flux of momentum through the surfaces bounding the control volume (Figure 3.19). In the next section we turn to learning how to apply this equation to control volumes that interest us in fluid mechanics.

3.2.3 Problem-Solving with Control Volumes

With the development of the Reynolds transport theorem we have the main tool that we need to be able to solve a wide variety of flow problems.

Reynolds transport theorem (momentum balance on CV)

$$\boxed{\sum_{\substack{\text{on}\\ \text{CV}}} \underline{f} = \frac{d\underline{\mathbf{P}}}{dt} + \iint_{CS} (\hat{n} \cdot \underline{v})\ \rho \underline{v}\ dS} \tag{3.137}$$

$$\left(\begin{array}{c}\text{sum of}\\ \text{forces}\\ \text{on a CV}\end{array}\right) = \left(\begin{array}{c}\text{rate of}\\ \text{increase of}\\ \text{momentum of}\\ \text{fluid in CV}\end{array}\right) + \left(\begin{array}{c}\text{net outflow of}\\ \text{momentum}\\ \text{through bounding}\\ \text{surfaces of CV}\end{array}\right) \tag{3.138}$$

The Reynolds transport theorem gives the equivalent of Newton's second law ($\sum \underline{f} = m\underline{a}$) for a control volume. This expression states that the sum of forces on a control volume is equal to the rate of change of momentum of the fluid in the control volume plus the net outward flux of momentum through the surfaces bounding the control volume (Figure 3.19). When properly applied to a flow situation and solved, the momentum balance gives the velocity field and information on how forces interact in a fluid. The Reynolds transport theorem is a

[7] The momentum of the fluid in the control volume $\underline{\mathbf{P}}$ is only a function of time; for more discussion of this point see Deen[45] and the supplemental web materials [124].