Polymer Rheology

Rhe-

$\underbrace{\text{Rhe-}}_{}$ *rei* – *Greek* for flow

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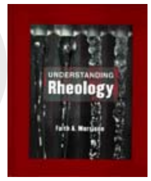
Polymer Rheology
Michigan Tech



What is rheology anyway?

Rheology = the study of deformation and flow.

"What is Rheology Anyway?" Faith A. Morrison, *The Industrial Physicist*, 10(2) 29-31, April/May 2004.

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Chapter 1: Introduction

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1. What is rheology, anyway?
2. Newtonian versus non-Newtonian
3. Key features of non-Newtonian behavior: ***Nonlinearity*** and ***Memory***

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What is rheology anyway?

To the layperson, rheology is:

- Mayonnaise does not flow even under stress for a long time; honey always flows
- Silly Putty bounces (is elastic) but also flows (is viscous)
- Dilute flour-water solutions are easy to work with but doughs can be quite temperamental
- Corn starch and water can display strange behavior – poke it slowly and it deforms easily around your finger; punch it rapidly and your fist bounces off of the surface

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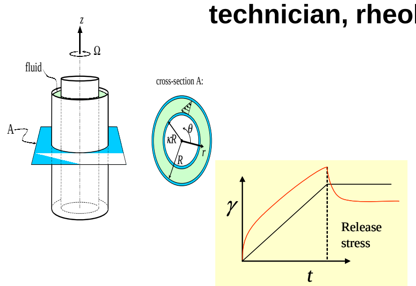
What is rheology anyway?

To the scientist, engineer, or technician, rheology is



- Yield stresses
- Viscoelastic effects
- Memory effects
- Shear thickening and shear thinning

For both the layperson and the technical person, rheology is a set of problems or observations related to how the stress in a material or force applied to a material is related to deformation (change of shape) of the material.

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What is rheology anyway?

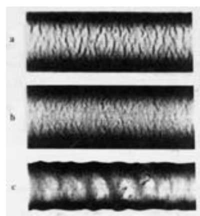
Rheology affects:

•Processing (design, costs, production rates)

www.corrugatorman.com/pic/akron%20extruder.JPG

www.math.utwente.nl/mpcm/aamp/examples.html

•End use (food texture, product pour, motor-oil function)



Pomar et al. JNNFM 54 143 1994

•Product quality (surface distortions, anisotropy, strength, structure development)

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Goal of the scientist, engineer, or technician:

•**Understand** the kinds of flow and deformation **effects** exhibited by complex systems

•**Apply qualitative** rheological **knowledge** to diagnostic, design, or optimization problems

•In diagnostic, design, or optimization problems, **use or devise quantitative** analytical **tools** that correctly capture rheological effects

How do we reach these goals?

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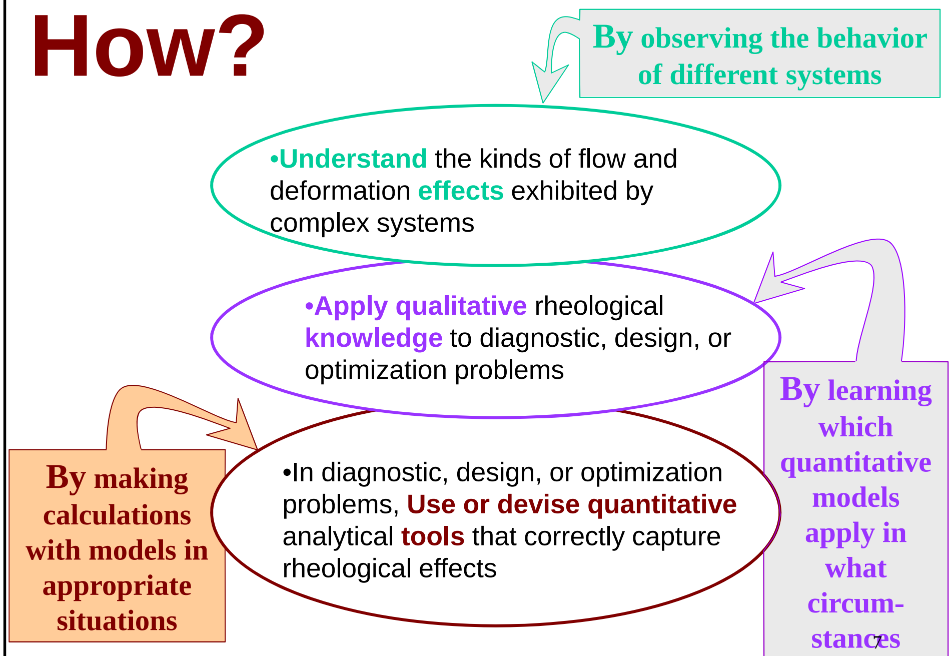
How?



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Learning Rheology (bibliography)

Descriptive Rheology

Barnes, H., J. Hutton, and K. Walters, *An Introduction to Rheology* (Elsevier, 1989)

Quantitative Rheology

Morrison, Faith, *Understanding Rheology* (Oxford, 2001)
Bird, R., R. Armstrong, and O. Hassager, *Dynamics of Polymeric Liquids, Volume 1* (Wiley, 1987)

Industrial Rheology

Dealy, John and Kurt Wissbrun, *Melt Rheology and Its Role in Plastics Processing* (Van Nostrand Reinhold, 1990)

Polymer Behavior

Larson, Ron, *The Structure and Rheology of Complex Fluids* (Oxford, 1999)
Ferry, John, *Viscoelastic Properties of Polymers* (Wiley, 1980)

Suspension Behavior

Mewis, Jan and Norm Wagner, *Colloidal Suspension* (Cambridge, 2012)
Macosko, Chris, *Rheology: Principles, Measurements, and Applications* (VCH Publishers, 1994)

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The Physics Behind Rheology:

1. Conservation laws
   - mass
   - momentum
   - energy

Cauchy Momentum Equation

$$\rho\left(\frac{\partial \underline{v}}{\partial t}+\underline{v}\cdot\nabla\underline{v}\right)=-\nabla p-\nabla\cdot\underline{\underline{\tau}}+\rho\underline{g}$$

2. Mathematics
   - differential equations
   - vectors
   - tensors
3. Constitutive law = law that relates **stress** to **deformation** for a particular fluid

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Polymer Rheology Non-Newtonian Fluid Mechanics

Newtonian fluids: (fluid mechanics)

$$\underbrace{\tau_{21}=-\mu\frac{dv_1}{dx_2}}_{\text{Newton's Law of Viscosity}}$$

material parameter

deformation

•This is an empirical law (measured or observed)

•May be derived theoretically for some systems

Non-Newtonian fluids: (rheology)

Need a new law or new <u>laws</u>

•These laws will also either be empirical or will be derived theoretically

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Polymer Rheology — Non-Newtonian Fluid Mechanics

Newtonian fluids: (shear flow only)

$$\tau_{21} = -\frac{dv_1}{dx_2}$$

Constitutive Equation

Non-Newtonian fluids: (all flows)

stress tensor

$$\underline{\underline{\tau}} = f\left(\dot{\underline{\underline{\gamma}}}\right)$$

Rate-of-deformation tensor

non-linear function (in time and position)

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Introduction to Non-Newtonian Behavior

Rheological Behavior of Fluids, National Committee on Fluid Mechanics Films, 1964

Velocity gradient tensor $\dot{\underline{\underline{\gamma}}}$

Type of fluid	Momentum balance	Stress –Deformation relationship (constitutive equation)
Inviscid (zero viscosity, μ=0)	Euler equation (Navier-Stokes with zero viscosity)	Stress is isotropic
Newtonian (finite. constant viscosity, μ)	Navier-Stokes (Cauchy momentum equation with Newtonian constitutive equation)	Stress is a function of the **instantaneous** velocity gradient
Non-Newtonian (finite, variable viscosity η plus **memory** effects)	Cauchy momentum equation with memory constitutive equation	Stress is a function of the **history** of the velocity gradient

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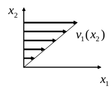
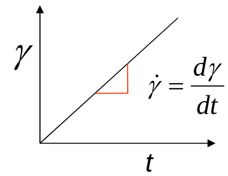
Rheological Behavior of Fluids - Newtonian

1. Strain response to imposed shear stress

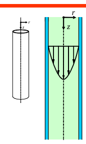
•shear rate is constant





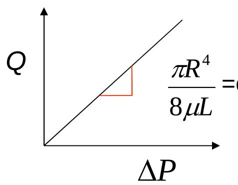
$$\dot{\gamma} = \frac{d\gamma}{dt} = \text{constant}$$

2. Pressure-driven flow in a tube (Poiseuille flow)



•viscosity is constant

$$Q = \frac{\pi \Delta P R^4}{8\mu L}$$



$$\frac{\pi R^4}{8\mu L} = \text{constant}$$

3. Stress tensor in shear flow

•only two components are nonzero

$$\underline{\underline{\tau}} = \begin{pmatrix} 0 & \tau_{12} & 0 \\ \tau_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{123}$$

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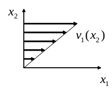
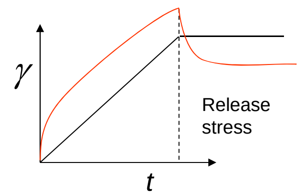
Rheological Behavior of Fluids – non-Newtonian

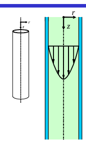
1. Strain response to imposed shear stress

•shear rate is variable

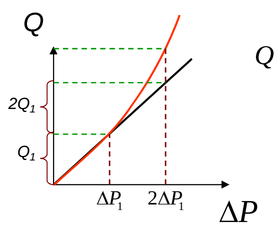




2. Pressure-driven flow in a tube (Poiseuille flow)



•viscosity is variable

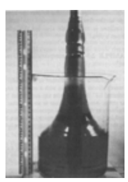


$$Q = f(\Delta P)$$

3. Stress tensor in shear flow

•all 9 components are nonzero

Normal stresses



$$\underline{\underline{\tau}} = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}_{123}$$

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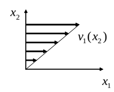
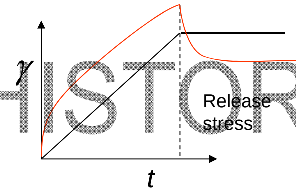
Rheological Behavior of Fluids – non-Newtonian

1. Strain response to imposed shear stress

•shear rate is variable



HISTORY



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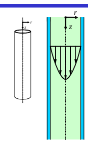
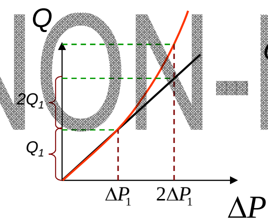
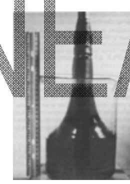


•viscosity is variable

NON-LINEARITY



$Q = f(\Delta P)$



•all 9 components are nonzero

Normal stresses

$$\underline{\underline{\tau}} = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}_{123}$$

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Examples from the film of . . . .

Dependence on the history of the deformation gradient

•Polymer fluid pours, but springs back
•Elastic ball bounces, but flows if given enough time
•Steel ball dropped in polymer solution "bounces"
•Polymer solution in concentric cylinders – has fading memory
•Quantitative measurements in concentric cylinders show memory and need a finite time to come to steady state

Non-linearity of the function $\underline{\underline{\tau}} = f\left(\dot{\underline{\underline{\gamma}}}\right)$

•Polymer solution draining from a tube is first slower, then faster than a Newtonian fluid
•Double the static head on a draining tube, and the flow rate does not necessarily double (as it does for Newtonian fluids); sometimes more than doubles, sometimes less
•Normal stresses in shear flow
•Die swell

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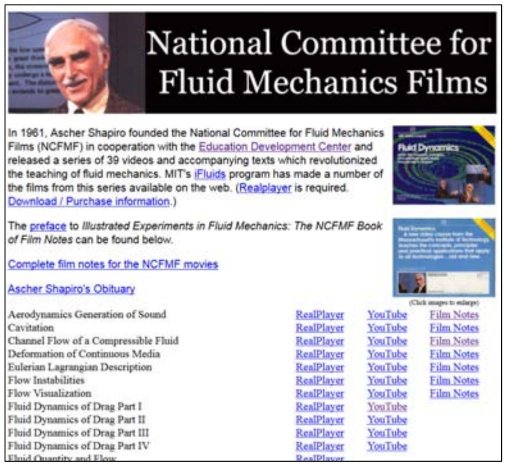
National Committee for Fluid Mechanics Films

In 1961, Ascher Shapiro founded the National Committee for Fluid Mechanics Films (NCFMF) in cooperation with the Education Development Center and released a series of 39 videos and accompanying texts which revolutionized the teaching of fluid mechanics. MIT's iFluids program has made a number of the films from this series available on the web. (Realplayer is required. Download / Purchase information.)

The preface to *Illustrated Experiments in Fluid Mechanics: The NCFMF Book of Film Notes* can be found below.

Complete film notes for the NCFMF movies

Ascher Shapiro's Obituary

(Click images to enlarge)

Aerodynamics Generation of Sound	RealPlayer	YouTube	Film Notes
Cavitation	RealPlayer	YouTube	Film Notes
Channel Flow of a Compressible Fluid	RealPlayer	YouTube	Film Notes
Deformation of Continuous Media	RealPlayer	YouTube	Film Notes
Eulerian Lagrangian Description	RealPlayer	YouTube	Film Notes
Flow Instabilities	RealPlayer	YouTube	Film Notes
Flow Visualization	RealPlayer	YouTube	Film Notes
Fluid Dynamics of Drag Part I	RealPlayer	YouTube	
Fluid Dynamics of Drag Part II	RealPlayer	YouTube	
Fluid Dynamics of Drag Part III	RealPlayer	YouTube	
Fluid Dynamics of Drag Part IV	RealPlayer	YouTube	
Fluid Quantity and Flow	RealPlayer		

Show NCFM Film on *Rheological Behavior of Fluids*

- Search for "NCFMF"
- web.mit.edu/hml/ncfmf.html
- Also on YouTube

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Chapter 2: Mathematics Review

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1. Vector review
2. Einstein notation
3. Tensors

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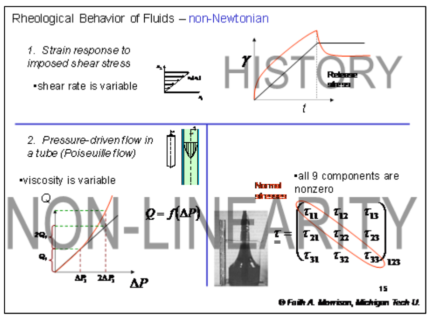
Motivation: We will be solving the momentum balance:

$$\rho\left(\frac{\partial \underline{v}}{\partial t}+\underline{v}\cdot\nabla\underline{v}\right)=-\nabla p-\nabla\cdot\underline{\underline{\tau}}+\rho\underline{g}$$

Newtonian fluids:
- Linear
- Instantaneous
- $\underline{\underline{\tau}}(t)=-\mu\dot{\underline{\underline{\gamma}}}(t)$

Non-Newtonian fluids:



- Non-linear
- Non-instantaneous
- $\underline{\underline{\tau}}(t)=?$

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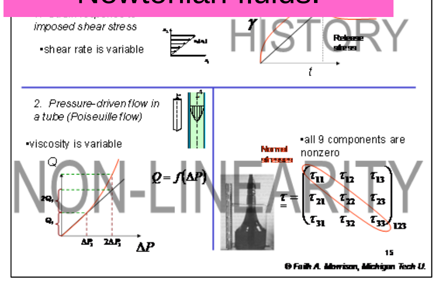
Motivation: We will be solving the momentum balance:

$$\rho\left(\frac{\partial \underline{v}}{\partial t}+\underline{v}\cdot\nabla\underline{v}\right)=-\nabla p-\nabla\cdot\underline{\underline{\tau}}+\rho\underline{g}$$

Newtonian fluids:
- Linear
- Instantaneous
- $\underline{\underline{\tau}}(t)=-\mu\dot{\underline{\underline{\gamma}}}(t)$

We're going to be trying to identify the constitutive equation $\underline{\underline{\tau}}(t)$ for non-Newtonian fluids.



- Non-linear
- Non-instantaneous
- $\underline{\underline{\tau}}(t)=?$

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Motivation: We will be solving the momentum balance:

$$\rho\left(\frac{\partial \underline{v}}{\partial t}+\underline{v}\cdot\nabla\underline{v}\right)=-\nabla p-\nabla\cdot\underline{\underline{\tau}}+\rho\underline{g}$$

Newtonian fluids:

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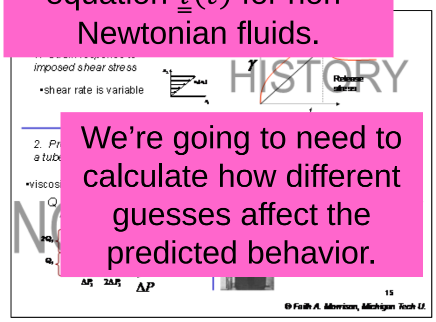

We're going to need to calculate how different guesses affect the predicted behavior.

- Non-linear
- Non-instantaneous
- $\underline{\underline{\tau}}(t)=?$

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Motivation: We will be solving the momentum balance:

$$\rho\left(\frac{\partial \underline{v}}{\partial t}+\underline{v}\cdot\nabla\underline{v}\right)=-\nabla p-\nabla\cdot\underline{\underline{\tau}}+\rho\underline{g}$$

Newtonian fluids:

- Linear
- Instantaneous
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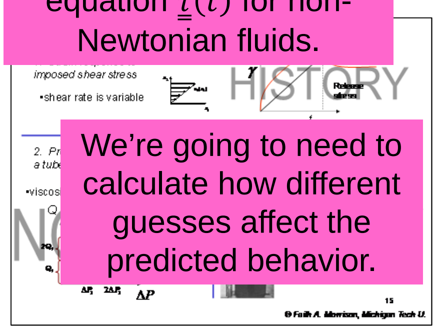

We're going to need to calculate how different guesses affect the predicted behavior.

- Non-linear
- Non-instantaneous
- $\underline{\underline{\tau}}(t)=?$

We need to understand and be able to manipulate this mathematical notation.

Chapter 2: Mathematics Review

1. Scalar – *a mathematical entity that has magnitude only*

e.g.: temperature T
speed *v*
time *t*
density r

– *scalars may be constant or may be variable*

Laws of Algebra for Scalars:		
	yes commutative	ab = ba
	yes associative	a(bc) = (ab)c
	yes distributive	a(b+c) = ab+ac

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2. Vector – *a mathematical entity that has magnitude and direction*

e.g.: force on a surface $\underline{f}$
velocity $\underline{v}$

– *vectors may be constant or may be variable*

Definitions

magnitude of a vector – a scalar associated with a vector

$$|\underline{v}| = v \quad |\underline{f}| = f$$

unit vector – a vector of unit length

$$\frac{\underline{v}}{|\underline{v}|} = \hat{v}$$

a unit vector in the direction of $\underline{v}$

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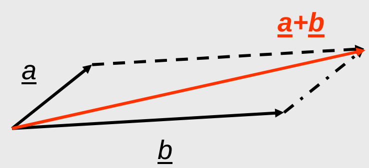
Mathematics Review

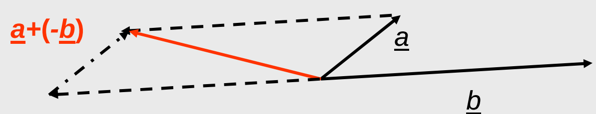
Polymer Rheology

Laws of Algebra for Vectors:

1. Addition



2. Subtraction



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Laws of Algebra for Vectors (continued):

3. Multiplication by scalar $\alpha \underline{v}$

yes commutative $\alpha \underline{v} = \underline{v} \alpha$

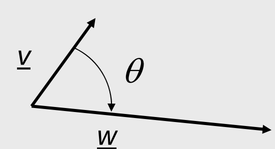
yes associative $\alpha(\beta \underline{v}) = (\alpha\beta)\underline{v} = \alpha\beta\underline{v}$

yes distributive $\alpha(\underline{v} + \underline{w}) = \alpha\underline{v} + \alpha\underline{w}$

4. Multiplication of vector by vector

4a. scalar (dot) (inner) product

$$\underline{v} \cdot \underline{w} = vw\cos\theta$$



Note: we can find magnitude with dot product

$$\underline{v} \cdot \underline{v} = vv\cos 0 = v^2$$

$$v = |\underline{v}| = \sqrt{\underline{v} \cdot \underline{v}}$$

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Laws of Algebra for Vectors (continued):

4a. scalar (dot) (inner) product (con't)

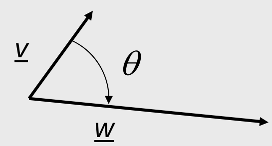
yes commutative $\underline{v} \cdot \underline{w} = \underline{w} \cdot \underline{v}$

NO associative $\cancel{\underline{v} \cdot \underline{w} \cdot \underline{z}}$ no such operation

yes distributive $\underline{z} \cdot \left(\underline{v} + \underline{w}\right) = \underline{z} \cdot \underline{v} + \underline{z} \cdot \underline{w}$

4b. vector (cross) (outer) product

$$\underline{v} \times \underline{w} = vw \sin\theta \, \hat{e}$$



$\hat{e}$ is a unit vector perpendicular to both $\underline{v}$ and $\underline{w}$ following the right-hand rule

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Laws of Algebra for Vectors (continued):

4b. vector (cross) (outer) product (con't)

NO commutative $\underline{v} \times \underline{w} \neq \underline{w} \times \underline{v}$

NO associative $\underline{v} \times \underline{w} \times \underline{z} \neq \left(\underline{v} \times \underline{w}\right) \times \underline{z} \neq \underline{v} \times \left(\underline{w} \times \underline{z}\right)$

yes distributive $\underline{z} \times \left(\underline{v} + \underline{w}\right) = \left(\underline{z} \times \underline{v}\right) + \left(\underline{z} \times \underline{w}\right)$

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Mathematics Review Polymer Rheology

Coordinate Systems

•Allow us to make actual calculations with vectors

Rule: any three vectors that are *non-zero* and *linearly independent* (non-coplanar) may form a coordinate basis

Three vectors are linearly dependent if a, b, and g can be found such that:

$$\alpha \underline{a} + \beta \underline{b} + \gamma \underline{c} = \underline{0}$$
$$for \quad \alpha, \beta, \gamma \neq 0$$

If α, β, and γ are found to be zero, the vectors are linearly independent.

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Mathematics Review Polymer Rheology

How can we do actual calculations with vectors?

Rule: *any* vector may be expressed as the linear combination of three, non-zero, non-coplanar basis vectors

coefficient of a in the $\hat{e}_y$ direction

any vector

$$\underline{a} = a_x \hat{e}_x + a_y \hat{e}_y + a_z \hat{e}_z = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}_{xyz}$$
$$= a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3$$
$$= \sum_{j=1}^{3} a_j \hat{e}_j$$

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Mathematics Review

Polymer Rheology

Trial calculation: dot product of two vectors

$$
\begin{aligned}
\underline{a}\cdot\underline{b} &= \left(a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3\right)\cdot\left(b_1\hat{e}_1 + b_2\hat{e}_2 + b_3\hat{e}_3\right) \\
&= a_1\hat{e}_1\cdot\left(b_1\hat{e}_1 + b_2\hat{e}_2 + b_3\hat{e}_3\right) + \\
&\qquad a_2\hat{e}_2\cdot\left(b_1\hat{e}_1 + b_2\hat{e}_2 + b_3\hat{e}_3\right) + \\
&\qquad\quad a_3\hat{e}_3\cdot\left(b_1\hat{e}_1 + b_2\hat{e}_2 + b_3\hat{e}_3\right) \\
&= a_1\hat{e}_1\cdot b_1\hat{e}_1 + a_1\hat{e}_1\cdot b_2\hat{e}_2 + a_1\hat{e}_1\cdot b_3\hat{e}_3 + \\
&\qquad a_2\hat{e}_2\cdot b_1\hat{e}_1 + a_2\hat{e}_2\cdot b_2\hat{e}_2 + a_2\hat{e}_2\cdot b_3\hat{e}_3 + \\
&\qquad\quad a_3\hat{e}_3\cdot b_1\hat{e}_1 + a_3\hat{e}_3\cdot b_2\hat{e}_2 + a_3\hat{e}_3\cdot b_3\hat{e}_3
\end{aligned}
$$

If we choose the basis to be orthonormal - mutually perpendicular and of unit length - then we can simplify.

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Mathematics Review

Polymer Rheology

If we choose the basis to be orthonormal - mutually perpendicular and of unit length, then we can simplify.

$$
\begin{aligned}
\hat{e}_1\cdot\hat{e}_1 &= 1 \\
\hat{e}_1\cdot\hat{e}_2 &= 0 \\
\hat{e}_1\cdot\hat{e}_3 &= 0 \\
&\cdots
\end{aligned}
$$

$$
\begin{aligned}
\underline{a}\cdot\underline{b} &= a_1\hat{e}_1\cdot b_1\hat{e}_1 + a_1\hat{e}_1\cdot b_2\hat{e}_2 + a_1\hat{e}_1\cdot b_3\hat{e}_3 + \\
&\qquad a_2\hat{e}_2\cdot b_1\hat{e}_1 + a_2\hat{e}_2\cdot b_2\hat{e}_2 + a_2\hat{e}_2\cdot b_3\hat{e}_3 + \\
&\qquad\quad a_3\hat{e}_3\cdot b_1\hat{e}_1 + a_3\hat{e}_3\cdot b_2\hat{e}_2 + a_3\hat{e}_3\cdot b_3\hat{e}_3 \\
&= a_1b_1 + a_2b_2 + a_3b_3
\end{aligned}
$$

We can generalize this operation with a technique called Einstein notation.

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Einstein Notation

a system of notation for vectors and tensors that allows for the calculation of results in Cartesian coordinate systems.

$$\underline{a} = a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3$$
$$= \sum_{j=1}^{3} a_j\hat{e}_j$$
$$= a_j\hat{e}_j = a_m\hat{e}_m$$

•the initial choice of subscript letter is *arbitrary*

•the presence of a pair of like subscripts implies a missing summation sign

Einstein Notation (con't)

The result of the dot products of basis vectors can be summarized by the Kronecker delta function

$$\hat{e}_1 \cdot \hat{e}_1 = 1$$
$$\hat{e}_1 \cdot \hat{e}_2 = 0$$
$$\hat{e}_1 \cdot \hat{e}_3 = 0$$
$$\cdots$$

$$\underbrace{\hat{e}_i \cdot \hat{e}_p = \delta_{ip} = \begin{cases} 1 & i = p \\ 0 & i \neq p \end{cases}}_{\text{Kronecker delta}}$$

Einstein Notation (con't)

To carry out a dot product of two arbitrary vectors . . .

Detailed Notation

$$\underline{a}\cdot\underline{b} = (a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3)\cdot(b_1\hat{e}_1 + b_2\hat{e}_2 + b_3\hat{e}_3)$$
$$= a_1\hat{e}_1\cdot b_1\hat{e}_1 + a_1\hat{e}_1\cdot b_2\hat{e}_2 + a_1\hat{e}_1\cdot b_3\hat{e}_3 +$$
$$a_2\hat{e}_2\cdot b_1\hat{e}_1 + a_2\hat{e}_2\cdot b_2\hat{e}_2 + a_2\hat{e}_2\cdot b_3\hat{e}_3 +$$
$$a_3\hat{e}_3\cdot b_1\hat{e}_1 + a_3\hat{e}_3\cdot b_2\hat{e}_2 + a_3\hat{e}_3\cdot b_3\hat{e}_3$$
$$= a_1b_1 + a_2b_2 + a_3b_3$$

Einstein Notation

$$\underline{a}\cdot\underline{b} = a_j\hat{e}_j\cdot b_m\hat{e}_m$$
$$= a_j\delta_{jm}b_m$$
$$= a_jb_j$$

3. Tensor – *the indeterminate vector product of two (or more) vectors*

e.g.: stress $\underline{\underline{\tau}}$
velocity gradient $\dot{\underline{\underline{\gamma}}}$

– *tensors may be constant or may be variable*

Definitions

dyad or dyadic product – a tensor written explicitly as the indeterminate vector product of two vectors

$\underline{a}\,\underline{d}$ dyad

$\underline{\underline{A}}$ general representation of a tensor

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Laws of Algebra for Indeterminate Product of Vectors:

NO commutative $\underline{a}\,\underline{v} \neq \underline{v}\,\underline{a}$

yes associative $\underline{b}\,(\underline{a}\,\underline{v}) = (\underline{b}\,\underline{a})\,\underline{v} = \underline{b}\,\underline{a}\,\underline{v}$

yes distributive $\underline{a}\,(\underline{v} + \underline{w}) = \underline{a}\,\underline{v} + \underline{a}\,\underline{w}$

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How can we represent tensors with respect to a chosen coordinate system?

Just follow the rules of tensor algebra

$$
\begin{aligned}
\underline{a}\,\underline{m} &= (a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3)(m_1\hat{e}_1 + m_2\hat{e}_2 + m_3\hat{e}_3) \\
&= a_1\hat{e}_1 m_1\hat{e}_1 + a_1\hat{e}_1 m_2\hat{e}_2 + a_1\hat{e}_1 m_3\hat{e}_3 + \\
&\quad a_2\hat{e}_2 m_1\hat{e}_1 + a_2\hat{e}_2 m_2\hat{e}_2 + a_2\hat{e}_2 m_3\hat{e}_3 + \\
&\quad a_3\hat{e}_3 m_1\hat{e}_1 + a_3\hat{e}_3 m_2\hat{e}_2 + a_3\hat{e}_3 m_3\hat{e}_3 \\
&= \sum_{k=1}^{3}\sum_{w=1}^{3} a_k\hat{e}_k m_w\hat{e}_w \\
&= \sum_{k=1}^{3}\sum_{w=1}^{3} a_k m_w\, \hat{e}_k\hat{e}_w
\end{aligned}
$$

Any tensor may be written as the sum of 9 dyadic products of basis vectors

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What about $\underline{\underline{A}}$? Same.

$$\underline{\underline{A}} = \sum_{i=1}^{3} \sum_{j=1}^{3} A_{ij}\, \hat{e}_i \hat{e}_j$$

Einstein notation for tensors: *drop the summation sign; every double index implies a summation sign has been dropped.*

$$\underline{\underline{A}} = A_{ij}\, \hat{e}_i \hat{e}_j = A_{pk}\, \hat{e}_p \hat{e}_k$$

Reminder: the initial choice of subscript letters is arbitrary

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How can we use Einstein Notation to calculate dot products between vectors and tensors?

It's the same as between vectors.

$$\underline{a} \cdot \underline{b} =$$
$$\underline{a} \cdot \underline{u}\, \underline{v} =$$
$$\underline{b} \cdot \underline{\underline{A}} =$$

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Summary of Einstein Notation

1. Express vectors, tensors, (later, vector operators) in a Cartesian coordinate system as the sums of coefficients multiplying basis vectors - each separate summation has a different index
2. Drop the summation signs
3. Dot products between basis vectors result in the Kronecker delta function because the Cartesian system is orthonormal.

Note:

•In Einstein notation, the presence of repeated indices implies a missing summation sign

•The choice of initial index (i, m, p, etc.) is *arbitrary* - it merely indicates which indices change together

3. Tensor – *(continued)*

Definitions

Scalar product of two tensors

$$\underline{\underline{A}} : \underline{\underline{M}} = A_{ip}\hat{e}_i\hat{e}_p : M_{km}\hat{e}_k\hat{e}_m$$

$$= A_{ip}M_{km}\hat{e}_i\hat{e}_p : \hat{e}_k\hat{e}_m$$

carry out the dot products indicated

$$= A_{ip}M_{km} \left(\hat{e}_p \cdot \hat{e}_k\right)\left(\hat{e}_i \cdot \hat{e}_m\right)$$

$$= A_{ip}M_{km} \quad \delta_{pk}\delta_{im}$$

"p" becomes "k"
"i" becomes "m"

$$= A_{mk}M_{km}$$

But, what is a tensor really?

A tensor is a handy representation of a *Linear Vector Function*

scalar function: $y = f(x) = x^2 + 2x + 3$

a mapping of values of *x* onto values of *y*

vector function: $\underline{w} = f(\underline{v})$

a mapping of vectors of $\underline{v}$ into vectors $\underline{w}$

How do we express a vector function?

What is a linear function?

Linear, in this usage, has a precise, mathematical definition.

Linear functions (scalar and vector) have the following two properties:

$$f(\lambda x) = \lambda f(x)$$
$$f(x+w) = f(x) + f(w)$$

It turns out . . .

Multiplying vectors and tensors is a convenient way of representing the actions of a linear vector function (as we will now show).

Tensors are *Linear Vector Functions*

Let $f(\underline{a}) = \underline{b}$ be a linear vector function.

We can write $\underline{a}$ in Cartesian coordinates.

$$\underline{a} = a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3$$

$$f(\underline{a}) = f(a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3) = \underline{b}$$

Using the linear properties of *f*, we can distribute the function action:

$$f(\underline{a}) = a_1 \underbrace{f(\hat{e}_1)} + a_2 \underbrace{f(\hat{e}_2)} + a_3 \underbrace{f(\hat{e}_3)} = \underline{b}$$

These results are just vectors, we will name them $\underline{v}$, $\underline{w}$, and $\underline{m}$.

Tensors are *Linear Vector Functions* (continued)

$$f(\underline{a}) = a_1 \underbrace{f(\hat{e}_1)}_{\underline{v}} + a_2 \underbrace{f(\hat{e}_2)}_{\underline{w}} + a_3 \underbrace{f(\hat{e}_3)}_{\underline{m}} = \underline{b}$$

$$f(\underline{a}) = a_1\underline{v} + a_2\underline{w} + a_3\underline{m} = \underline{b}$$

Now we note that the coefficients a_i may be written as,

$$a_1 = \underline{a}\cdot\hat{e}_1 \quad a_2 = \underline{a}\cdot\hat{e}_2 \quad a_3 = \underline{a}\cdot\hat{e}_3$$

Substituting,

$$f(\underline{a}) = \underline{a}\cdot\hat{e}_1\,\underline{v} + \underline{a}\cdot\hat{e}_2\,\underline{w} + \underline{a}\cdot\hat{e}_3\,\underline{m} = \underline{b}$$

The indeterminate vector product has appeared!

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Using the distributive law, we can factor out the dot product with $\underline{a}$:

$$f(\underline{a}) = \underline{a} \cdot \underbrace{\left(\hat{e}_1\,\underline{v} + \hat{e}_2\,\underline{w} + \hat{e}_3\,\underline{m}\right)} = \underline{b}$$

This is just a tensor (the sum of dyadic products of vectors)

$$\left(\hat{e}_1\,\underline{v} + \hat{e}_2\,\underline{w} + \hat{e}_3\,\underline{m}\right) \equiv \underline{\underline{M}}$$

$$\boxed{f(\underline{a}) = \underline{a} \cdot \underline{\underline{M}} = \underline{b}}$$

CONCLUSION: Tensor operations are convenient to use to express linear vector functions.

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3. Tensor – (continued)

More Definitions

Identity Tensor

$$\underline{\underline{I}} = \hat{e}_i\hat{e}_i = \hat{e}_1\hat{e}_1 + \hat{e}_2\hat{e}_2 + \hat{e}_3\hat{e}_3$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123}$$

$$\begin{aligned} \underline{\underline{A}} \cdot \underline{\underline{I}} &= A_{ip}\hat{e}_i\hat{e}_p \cdot \hat{e}_k\hat{e}_k \\ &= A_{ip}\hat{e}_i\delta_{pk}\hat{e}_k \\ &= A_{ik}\hat{e}_i\hat{e}_k \\ &= \underline{\underline{A}} \end{aligned}$$

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3. Tensor – (continued) More Definitions

Zero Tensor

$$\underline{\underline{0}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{123}$$

Magnitude of a Tensor

$$\left|\underline{\underline{A}}\right| \equiv +\sqrt{\frac{\underline{\underline{A}}:\underline{\underline{A}}}{2}}$$

Note that the book has a typo on this equation: the "2" is under the square root.

$$\begin{aligned} \underline{\underline{A}}:\underline{\underline{A}} &= A_{ip}\hat{e}_i\hat{e}_p : A_{km}\hat{e}_k\hat{e}_m \\ &= A_{ip}A_{km}\left(\hat{e}_p\cdot\hat{e}_k\right)\left(\hat{e}_i\cdot\hat{e}_m\right) \\ &= A_{mk}A_{km} \end{aligned}$$

products across the diagonal

3. Tensor – (continued) More Definitions

Tensor Transpose

$$\underline{\underline{M}}^T = \left(M_{ik}\hat{e}_i\hat{e}_k\right)^T = M_{ik}\hat{e}_k\hat{e}_i$$

Exchange the coefficients across the diagonal

CAUTION:

$$\begin{aligned} \left(\underline{\underline{A}}\cdot\underline{\underline{C}}\right)^T &= \left(A_{ik}\hat{e}_i\hat{e}_k\cdot C_{pj}\hat{e}_p\hat{e}_j\right)^T = \left(A_{ik}C_{pj}\,\hat{e}_i\hat{e}_j\delta_{kp}\right)^T \\ &= \left(A_{ip}C_{pj}\,\hat{e}_i\hat{e}_j\right)^T \\ &= A_{ip}C_{pj}\,\hat{e}_j\hat{e}_i \end{aligned}$$

It is **not** equal to: $\left(\underline{\underline{A}}\cdot\underline{\underline{C}}\right)^T = \left(A_{ip}C_{pj}\,\hat{e}_i\hat{e}_j\right)^T \neq \cancel{A_{pi}C_{jp}\,\hat{e}_i\hat{e}_j}$

I recommend you always interchange the indices on the **basis vectors** rather than on the coefficients.

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3. *Tensor – (continued)*

More Definitions

Symmetric Tensor

$$\underline{\underline{M}} = \underline{\underline{M}}^T$$
$$M_{ik} = M_{ki}$$

e.g.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}_{123}$$

Antisymmetric Tensor

$$\underline{\underline{M}} = -\underline{\underline{M}}^T$$
$$M_{ik} = -M_{ki}$$

e.g.

$$\begin{pmatrix} 0 & -2 & -3 \\ 2 & 0 & -5 \\ 3 & 5 & 0 \end{pmatrix}_{123}$$

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3. *Tensor – (continued)*

More Definitions

Tensor order

Scalars, vectors, and tensors may all be considered to be tensors (entities that exist independent of coordinate system). They are tensors of different orders, however.

order = degree of complexity

scalars	0^{th}-order tensors	3^0
vectors	1^{st}-order tensors	3^1
tensors	2^{nd}-order tensors	3^2
higher-order tensors	3^{rd}-order tensors	3^3

Number of coefficients needed to express the tensor in 3D space

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3. Tensor – (continued)

More Definitions

Tensor Invariants

Scalars that are associated with tensors; these are numbers that are independent of coordinate system.

vectors: $|\underline{v}| = v$

The magnitude of a vector is a scalar associated with the vector

It is independent of coordinate system, i.e. it is an invariant.

tensors: $\underline{\underline{A}}$

There are three invariants associated with a second-order tensor.

Tensor Invariants

$$I_{\underline{\underline{A}}} \equiv trace\underline{\underline{A}} = tr\underline{\underline{A}}$$

For the tensor written in Cartesian coordinates:

$$trace\underline{\underline{A}} = A_{pp} = A_{11} + A_{22} + A_{33}$$

$$II_{\underline{\underline{A}}} \equiv trace\left(\underline{\underline{A}} \cdot \underline{\underline{A}}\right) = \underline{\underline{A}} : \underline{\underline{A}} = A_{pk}A_{kp}$$

$$III_{\underline{\underline{A}}} \equiv trace\left(\underline{\underline{A}} \cdot \underline{\underline{A}} \cdot \underline{\underline{A}}\right) = A_{pj}A_{jh}A_{hp}$$

Note: the definitions of invariants written in terms of coefficients are only valid when the tensor is written in Cartesian coordinates.

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4. Differential Operations with Vectors, Tensors

Scalars, vectors, and tensors are differentiated to determine rates of change (with respect to time, position)

- To carryout the differentiation with respect to a *single variable*, differentiate each coefficient individually.
- There is no change in order (vectors remain vectors, scalars remain scalars, etc.

$$\frac{\partial \alpha}{\partial t} \qquad \frac{\partial \underline{w}}{\partial t} = \begin{pmatrix} \frac{\partial w_1}{\partial t} \\ \frac{\partial w_2}{\partial t} \\ \frac{\partial w_3}{\partial t} \end{pmatrix}_{123} \qquad \frac{\partial \underline{\underline{B}}}{\partial t} = \begin{pmatrix} \frac{\partial B_{11}}{\partial t} & \frac{\partial B_{21}}{\partial t} & \frac{\partial B_{31}}{\partial t} \\ \frac{\partial B_{21}}{\partial t} & \frac{\partial B_{22}}{\partial t} & \frac{\partial B_{23}}{\partial t} \\ \frac{\partial B_{31}}{\partial t} & \frac{\partial B_{32}}{\partial t} & \frac{\partial B_{33}}{\partial t} \end{pmatrix}_{123}$$

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4. *Differential Operations with Vectors, Tensors (continued)*

- To carryout the differentiation with respect to *3D spatial variation*, use the del (nabla) operator.
- This is a vector operator
- Del may be applied in three different ways
- Del may operate on scalars, vectors, or tensors

Del Operator

This is written in Cartesian coordinates

$$\nabla \equiv \hat{e}_1 \frac{\partial}{\partial x_1} + \hat{e}_2 \frac{\partial}{\partial x_2} + \hat{e}_3 \frac{\partial}{\partial x_3} = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix}_{123}$$

$$= \sum_{p=1}^{3} \hat{e}_p \frac{\partial}{\partial x_p} = \underbrace{\hat{e}_p \frac{\partial}{\partial x_p}}_{\text{Einstein notation for del}}$$

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4. *Differential Operations with Vectors, Tensors (continued)*

A. Scalars - gradient

Gibbs notation

$$\nabla\beta \equiv \hat{e}_1\frac{\partial}{\partial x_1}\beta + \hat{e}_2\frac{\partial}{\partial x_2}\beta + \hat{e}_3\frac{\partial}{\partial x_3}\beta = \begin{pmatrix} \frac{\partial\beta}{\partial x_1} \\ \frac{\partial\beta}{\partial x_2} \\ \frac{\partial\beta}{\partial x_3} \end{pmatrix}_{123}$$

This is written in Cartesian coordinates

Gradient of a scalar field $= \hat{e}_p\frac{\partial\beta}{\partial x_p}$

The gradient of a scalar field is a *vector*

The gradient operation captures the total spatial variation of a scalar, vector, or tensor field.

•gradient operation increases the order of the entity operated upon

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4. *Differential Operations with Vectors, Tensors (continued)*

B. Vectors - gradient

$$\nabla\underline{w} \equiv \hat{e}_1\frac{\partial}{\partial x_1}\underline{w} + \hat{e}_2\frac{\partial}{\partial x_2}\underline{w} + \hat{e}_3\frac{\partial}{\partial x_3}\underline{w}$$

$$= \hat{e}_1\frac{\partial}{\partial x_1}\left(w_1\hat{e}_1 + w_2\hat{e}_2 + w_3\hat{e}_3\right)$$

$$+ \hat{e}_2\frac{\partial}{\partial x_2}\left(w_1\hat{e}_1 + w_2\hat{e}_2 + w_3\hat{e}_3\right)$$

$$+ \hat{e}_3\frac{\partial}{\partial x_3}\left(w_1\hat{e}_1 + w_2\hat{e}_2 + w_3\hat{e}_3\right)$$

$$= \hat{e}_1\hat{e}_1\frac{\partial w_1}{\partial x_1} + \hat{e}_1\hat{e}_2\frac{\partial w_2}{\partial x_1} + \hat{e}_1\hat{e}_3\frac{\partial w_3}{\partial x_1} + \hat{e}_2\hat{e}_1\frac{\partial w_1}{\partial x_2} +$$

$$\hat{e}_2\hat{e}_2\frac{\partial w_2}{\partial x_2} + \hat{e}_2\hat{e}_3\frac{\partial w_3}{\partial x_2} + \hat{e}_3\hat{e}_1\frac{\partial w_1}{\partial x_3} + \hat{e}_3\hat{e}_2\frac{\partial w_2}{\partial x_3} + \hat{e}_3\hat{e}_3\frac{\partial w_3}{\partial x_3}$$

The basis vectors can move out of the derivatives because they are constant (do not change with position)

This is all written in Cartesian coordinates (basis vectors are constant)

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4. *Differential Operations with Vectors, Tensors (continued)*

B. Vectors - gradient *(continued)*

Gradient of a vector field

constants may appear on either side of the differential operator

$$\nabla \underline{w} \equiv \sum_{j=1}^{3}\sum_{k=1}^{3} \hat{e}_j \hat{e}_k \frac{\partial w_k}{\partial x_j} = \hat{e}_j \hat{e}_k \frac{\partial w_k}{\partial x_j} = \underbrace{\frac{\partial w_k}{\partial x_j}}_{\text{Einstein notation for gradient of a vector}} \hat{e}_j \hat{e}_k$$

The gradient of a vector field is a *tensor*

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4. *Differential Operations with Vectors, Tensors (continued)*

C. Vectors - divergence

Divergence of a vector field

$$\nabla \cdot \underline{w} \equiv \left(\hat{e}_1 \frac{\partial}{\partial x_1} + \hat{e}_2 \frac{\partial}{\partial x_2} + \hat{e}_3 \frac{\partial}{\partial x_3} \right) \cdot w_1 \hat{e}_1 + w_2 \hat{e}_2 + w_3 \hat{e}_3$$

$$= \frac{\partial w_1}{\partial x_1} + \frac{\partial w_2}{\partial x_2} + \frac{\partial w_3}{\partial x_3}$$

$$= \sum_{i=1}^{3} \frac{\partial w_i}{\partial x_i} = \underbrace{\frac{\partial w_i}{\partial x_i}}_{\text{Einstein notation for divergence of a vector}}$$

The Divergence of a vector field is a *scalar*

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4. *Differential Operations with Vectors, Tensors (continued)*

C. Vectors - divergence *(continued)*

This is all written in Cartesian coordinates (basis vectors are constant)

constants may appear on either side of the differential operator

Using Einstein notation

$$\nabla \cdot \underline{w} \equiv \hat{e}_m \frac{\partial}{\partial x_m} \cdot w_j \hat{e}_j = \frac{\partial w_j}{\partial x_m} \hat{e}_m \cdot \hat{e}_j = \frac{\partial w_j}{\partial x_m} \delta_{mj}$$

$$= \frac{\partial w_j}{\partial x_j}$$

•divergence operation decreases the order of the entity operated upon

4. *Differential Operations with Vectors, Tensors (continued)*

D. Vectors - Laplacian

Using Einstein notation:

$$\nabla \cdot \nabla \underline{w} \equiv \hat{e}_m \frac{\partial}{\partial x_m} \cdot \hat{e}_p \frac{\partial}{\partial x_p} w_j \hat{e}_j = \frac{\partial}{\partial x_m} \frac{\partial}{\partial x_p} w_j \left(\hat{e}_m \cdot \hat{e}_p \right) \hat{e}_j$$

$$= \frac{\partial}{\partial x_m} \frac{\partial}{\partial x_p} w_j \left(\delta_{mp} \right) \hat{e}_j$$

$$= \frac{\partial}{\partial x_p} \frac{\partial}{\partial x_p} w_j \, \hat{e}_j$$

The Laplacian of a vector field is a *vector*

$$= \begin{pmatrix} \frac{\partial^2 w_1}{\partial x_1} + \frac{\partial^2 w_1}{\partial x_2} + \frac{\partial^2 w_1}{\partial x_3} \\ \frac{\partial^2 w_2}{\partial x_1} + \frac{\partial^2 w_2}{\partial x_2} + \frac{\partial^2 w_2}{\partial x_3} \\ \frac{\partial^2 w_3}{\partial x_1} + \frac{\partial^2 w_3}{\partial x_2} + \frac{\partial^2 w_3}{\partial x_3} \end{pmatrix}_{123}$$

•Laplacian operation does not change the order of the entity operated upon

4. *Differential Operations with Vectors, Tensors (continued)*

E. Scalar - divergence $\cancel{\nabla \cdot \alpha}$ *(impossible; cannot decrease order of a scalar)*

F. Scalar - Laplacian $\nabla \cdot \nabla \alpha$

G. Tensor - gradient $\nabla \underline{\underline{A}}$

H. Tensor - divergence $\nabla \cdot \underline{\underline{A}}$

I. Tensor - Laplacian $\nabla \cdot \nabla \underline{\underline{A}}$

5. Curvilinear Coordinates

Cylindrical	$\bar{r},\theta,z$	$\hat{e}_{\bar{r}},\hat{e}_\theta,\hat{e}_z$
Spherical	r,θ,ϕ	$\hat{e}_r,\hat{e}_\theta,\hat{e}_\phi$

See figures 2.11 and 2.12

These coordinate systems are ortho-normal, *but they are not constant* (they vary with position).

This causes some non-intuitive effects when derivatives are taken.

5. Curvilinear Coordinates (continued)

$$\underline{v} = v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_z \hat{e}_z$$

$$\nabla \cdot \underline{v} = \nabla \cdot \left(v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_z \hat{e}_z \right)$$

$$= \underbrace{\left(\frac{\partial}{\partial x}\hat{e}_x + \frac{\partial}{\partial y}\hat{e}_y + \frac{\partial}{\partial z}\hat{e}_z \right)} \cdot \left(v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_z \hat{e}_z \right)$$

First, we need to write this in cylindrical coordinates.

solve for Cartesian basis vectors and substitute above

$$\begin{cases} \hat{e}_r = \cos\theta\, \hat{e}_x + \sin\theta\, \hat{e}_y \\ \hat{e}_\theta = -\sin\theta\, \hat{e}_x + \cos\theta\, \hat{e}_y \\ \hat{e}_z = \hat{e}_z \end{cases}$$

$$\left. \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{array} \right\}$$

substitute above using chain rule (see next slide for details)

$$\hat{e}_x = \cos\theta\, \hat{e}_r - \sin\theta\, \hat{e}_\theta$$

$$\hat{e}_y = \sin\theta\, \hat{e}_r + \cos\theta\, \hat{e}_\theta$$

$$x = r\cos\theta \qquad r = \sqrt{x^2 + y^2}$$

$$y = r\sin\theta \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$z = z$$

$$\boxed{\nabla \psi = \left(\frac{\partial \psi}{\partial x}\hat{e}_x + \frac{\partial \psi}{\partial y}\hat{e}_y + \frac{\partial \psi}{\partial z}\hat{e}_z \right)}$$

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial \psi}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial \psi}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial \psi}{\partial r}\cos\theta + \frac{\partial \psi}{\partial \theta}\left(\frac{-\sin\theta}{r}\right)$$

$$\frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial \psi}{\partial \theta}\frac{\partial \theta}{\partial y} + \frac{\partial \psi}{\partial z}\frac{\partial z}{\partial y} = \frac{\partial \psi}{\partial r}\sin\theta + \frac{\partial \psi}{\partial \theta}\left(\frac{\cos\theta}{r}\right)$$

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5. Curvilinear Coordinates (continued)

Result:
$$\nabla = \left(\frac{\partial}{\partial x}\hat{e}_x + \frac{\partial}{\partial y}\hat{e}_y + \frac{\partial}{\partial z}\hat{e}_z\right)$$
$$= \hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{e}_z\frac{\partial}{\partial z}$$

Now, proceed:

$$\nabla\cdot\underline{v} = \left(\hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{e}_z\frac{\partial}{\partial z}\right)\cdot\left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z\right)$$

(We cannot use Einstein notation because these are not Cartesian coordinates)

$$= \hat{e}_r\frac{\partial}{\partial r}\cdot\left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z\right) +$$
$$\hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\cdot\left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z\right) +$$
$$\hat{e}_z\frac{\partial}{\partial z}\cdot\left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z\right)$$

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5. Curvilinear Coordinates (continued)

$$\nabla\cdot\underline{v} = \hat{e}_r\frac{\partial}{\partial r}\cdot\left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z\right) +$$
$$\hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\cdot\left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z\right) +$$
$$\hat{e}_z\frac{\partial}{\partial z}\cdot\left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z\right)$$

$$\hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\cdot v_r\hat{e}_r = \hat{e}_\theta\cdot\frac{1}{r}\frac{\partial v_r\hat{e}_r}{\partial\theta}$$
$$= \hat{e}_\theta\cdot\frac{1}{r}\left(v_r\frac{\partial\hat{e}_r}{\partial\theta} + \hat{e}_r\frac{\partial v_r}{\partial\theta}\right)$$

$$\frac{\partial\hat{e}_r}{\partial\theta} = \frac{\partial}{\partial\theta}\left(\cos\theta\,\hat{e}_x + \sin\theta\,\hat{e}_y\right)$$
$$= -\sin\theta\,\hat{e}_x + \cos\theta\,\hat{e}_y$$
$$= \hat{e}_\theta$$

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Mathematics Review

Polymer Rheology

5. Curvilinear Coordinates (continued)

$$\hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial\theta} \cdot v_r \hat{e}_r = \hat{e}_\theta \cdot \frac{1}{r}\frac{\partial v_r \hat{e}_r}{\partial\theta}$$

$$= \hat{e}_\theta \cdot \frac{1}{r}\left(v_r \frac{\partial \hat{e}_r}{\partial\theta} + \hat{e}_r \frac{\partial v_r}{\partial\theta} \right)$$

$$= \hat{e}_\theta \cdot \frac{1}{r}\left(v_r \hat{e}_\theta + \hat{e}_r \frac{\partial v_r}{\partial\theta} \right)$$

$$= \frac{1}{r} v_r$$

This term is not intuitive, and appears because the basis vectors in the curvilinear coordinate systems vary with position.

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Mathematics Review

Polymer Rheology

5. Curvilinear Coordinates (continued)

Final result for divergence of a vector in cylindrical coordinates:

$$\nabla \cdot \underline{v} = \hat{e}_r \frac{\partial}{\partial r} \cdot \left(v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_z \hat{e}_z \right) + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial\theta} \cdot \left(v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_z \hat{e}_z \right) + \hat{e}_z \frac{\partial}{\partial z} \cdot \left(v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_z \hat{e}_z \right)$$

$$\boxed{\nabla \cdot \underline{v} = \frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z}}$$

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5. Curvilinear Coordinates (continued)

Curvilinear Coordinates (summary)

- The basis vectors are ortho-normal
- The basis vectors are non-constant (vary with position)
- These systems are convenient when the flow system mimics the coordinate surfaces in curvilinear coordinate systems.
- *We cannot use Einstein notation* – must use Tables in Appendix C2 (pp464-468).

6. Vector and Tensor Theorems and definitions

In Chapter 3 we review Newtonian fluid mechanics using the vector/tensor vocabulary we have learned thus far. We just need a few more theorems to prepare us for those studies. These are presented without proof.

Gauss Divergence Theorem

$$\iiint_V \nabla \cdot \underline{b}\, dV = \iint_S \hat{n} \cdot \underline{b}\, dS$$

outwardly directed unit normal

This theorem establishes the utility of the divergence operation. The integral of the divergence of a vector field over a volume is equal to the net outward flow of that property through the bounding surface.

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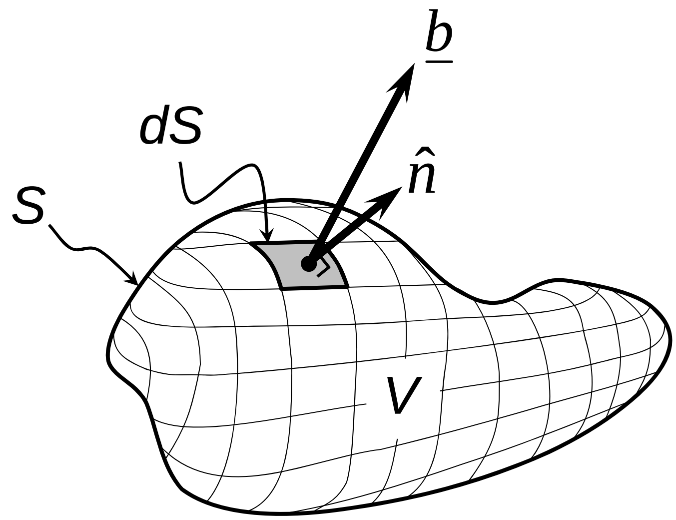

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6. Vector and Tensor Theorems *(continued)*

Leibnitz Rule for differentiating integrals

constant limits

$$I = \int_{\alpha}^{\beta} f(x,t)\,dx$$

$$\frac{dI}{dt} = \frac{d}{dt}\int_{\alpha}^{\beta} f(x,t)\,dx$$

$$= \int_{\alpha}^{\beta} \frac{\partial f(x,t)}{\partial t}\,dx$$

one dimension, constant limits

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6. Vector and Tensor Theorems *(continued)*

Leibnitz Rule for differentiating integrals

$$J=\int_{\alpha(t)}^{\beta(t)} f(x,t)\,dx$$

variable limits

$$\frac{dJ}{dt}=\frac{d}{dt}\int_{\alpha(t)}^{\beta(t)} f(x,t)\,dx$$

$$=\int_{\alpha(t)}^{\beta(t)} \frac{\partial f(x,t)}{\partial t}\,dx+\boxed{\frac{d\beta}{dt}f(\beta,t)-\frac{d\alpha}{dt}f(\alpha,t)}$$

one dimension, *variable* limits

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6. Vector and Tensor Theorems *(continued)*

Leibnitz Rule for differentiating integrals

$$J=\iiint_{V(t)} f(x,y,z,t)\,dV$$

$$\frac{dJ}{dt}=\frac{d}{dt}\iiint_{V(t)} f(x,y,z,t)\,dV$$

$$=\iiint_{V(t)} \frac{\partial f(x,y,z,t)}{\partial t}\,dV+\iint_{S(t)} f\left(\underline{v}_{surface}\cdot\hat{\underline{n}}\right)dS$$

velocity of the surface element *dS*

three dimensions, *variable* limits

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6. Vector and Tensor Theorems *(continued)*

Substantial Derivative

Consider a function $f(x, y, z, t)$

true for any path:

$$df \equiv \left(\frac{\partial f}{\partial x}\right)_{yzt} dx + \left(\frac{\partial f}{\partial y}\right)_{xzt} dy + \left(\frac{\partial f}{\partial z}\right)_{xyt} dz + \left(\frac{\partial f}{\partial t}\right)_{xyz} dt$$

choose special path:

$$\frac{df}{dt} \equiv \left(\frac{\partial f}{\partial x}\right)_{yzt} \frac{dx}{dt} + \left(\frac{\partial f}{\partial y}\right)_{xzt} \frac{dy}{dt} + \left(\frac{\partial f}{\partial z}\right)_{xyt} \frac{dz}{dt} + \left(\frac{\partial f}{\partial t}\right)_{xyz}$$

time rate of change of *f* along a chosen path

x-component of velocity along that path

When the chosen path is the path of a fluid particle, then these are the components of the particle velocities.

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6. Vector and Tensor Theorems *(continued)*

Substantial Derivative

When the chosen path is the path of a fluid particle, then the space derivatives are the components of the particle velocities.

$$\frac{df}{dt} \equiv \left(\frac{\partial f}{\partial x}\right)_{yzt} \frac{dx}{dt} + \left(\frac{\partial f}{\partial y}\right)_{xzt} \frac{dy}{dt} + \left(\frac{\partial f}{\partial z}\right)_{xyt} \frac{dz}{dt} + \left(\frac{\partial f}{\partial t}\right)_{xyz}$$

$$\left(\frac{df}{dt}\right)_{\substack{along\\ a\ particle\\ path}} \equiv \underbrace{\left(\frac{\partial f}{\partial x}\right)_{yzt} v_x + \left(\frac{\partial f}{\partial y}\right)_{xzt} v_y + \left(\frac{\partial f}{\partial z}\right)_{xyt} v_z}_{\underline{v} \cdot \nabla f} + \left(\frac{\partial f}{\partial t}\right)_{xyz}$$

Substantial Derivative

$$\boxed{\left(\frac{df}{dt}\right)_{\substack{along\\ a\ particle\\ path}} \equiv \frac{Df}{Dt} = \frac{\partial f}{\partial t} + \underline{v} \cdot \nabla f}$$

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Done with math background.

Chapter 2: Mathematics Review

CM4650
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1. Vector review
2. Einstein notation
3. Tensors

Let's use it with Newtonian fluids

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Chapter 3: Newtonian Fluids

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Navier-Stokes Equation

$$\boxed{\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\right) = -\nabla p + \mu\nabla^2\underline{v} + \rho\underline{g}}$$

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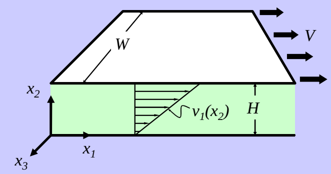
Chapter 3: Newtonian Fluid Mechanics

TWO GOALS

•Derive governing equations (mass and momentum balances

•Solve governing equations for velocity and stress fields

QUICK START



First, before we get deep into derivation, let's do a Navier-Stokes problem to get you started in the mechanics of this type of problem solving.

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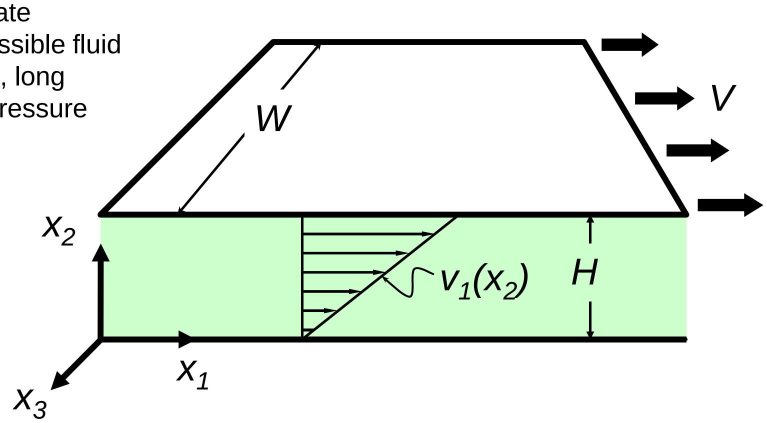
EXAMPLE: Drag flow between infinite parallel plates

$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}_{123}$$

•Newtonian
•steady state
•incompressible fluid
•very wide, long
•uniform pressure



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Chapter 3: Newtonian Fluid Mechanics

TWO GOALS

- Derive governing equations (mass and momentum balances
- Solve governing equations for velocity and stress fields

Mass Balance

Consider an arbitrary control volume V enclosed by a surface S

$$\begin{pmatrix} \textit{rate of increase} \\ \textit{of mass in } CV \end{pmatrix} = \begin{pmatrix} \textit{net flux of} \\ \textit{mass into } CV \end{pmatrix}$$

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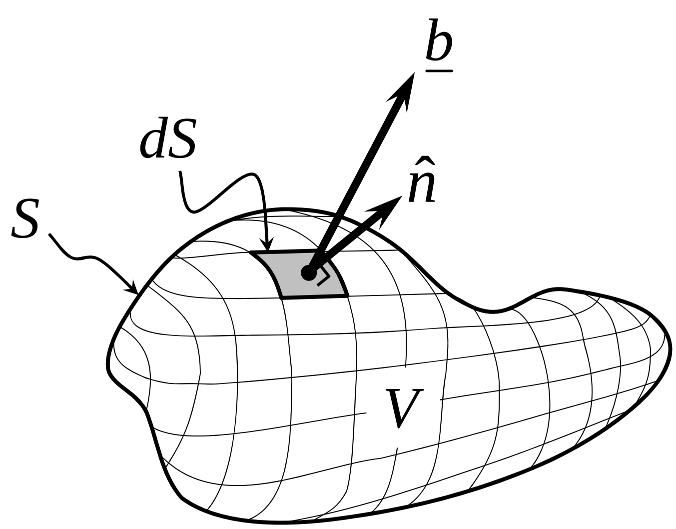

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Mass Balance *(continued)*

Consider an arbitrary volume *V* enclosed by a surface *S*

$$\begin{pmatrix} \text{rate of increase} \\ \text{of mass in } V \end{pmatrix} = \frac{d}{dt}\left(\iiint_V \rho \, dV \right)$$

outwardly pointing unit normal

$$\begin{pmatrix} \text{net flux of} \\ \text{mass into } V \\ \text{through surface } S \end{pmatrix} = -\iint_S \rho \, \hat{n} \cdot \underline{v} \, dS$$

Mass Balance *(continued)*

$$\frac{d}{dt}\left(\iiint_V \rho \, dV \right) = -\iint_S \rho \, \hat{n} \cdot \underline{v} \, dS$$

Leibnitz rule

$$\iiint_V \frac{\partial \rho}{\partial t} dV = -\iint_S \hat{n} \cdot (\rho \underline{v}) dS$$

Gauss Divergence Theorem

$$= -\iiint_V \nabla \cdot (\rho \underline{v}) \, dV$$

$$\iiint_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) \right) dV = 0$$

Chapter 3: Newtonian Fluid Mechanics Polymer Rheology

Mass Balance *(continued)*

$$\iiint_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) \right) dV = 0$$

Since V is *arbitrary*,

Continuity equation:
microscopic mass balance

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0}$$

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Chapter 3: Newtonian Fluid Mechanics Polymer Rheology

Mass Balance *(continued)*

Continuity equation (general fluids)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho (\nabla \cdot \underline{v}) + \underline{v} \cdot \nabla \rho = 0$$

$$\frac{D\rho}{Dt} + \rho (\nabla \cdot \underline{v}) = 0$$

For ρ=constant (incompressible fluids):

$$\nabla \cdot \underline{v} = 0$$

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Momentum Balance

Consider an arbitrary control volume *V* enclosed by a surface *S*

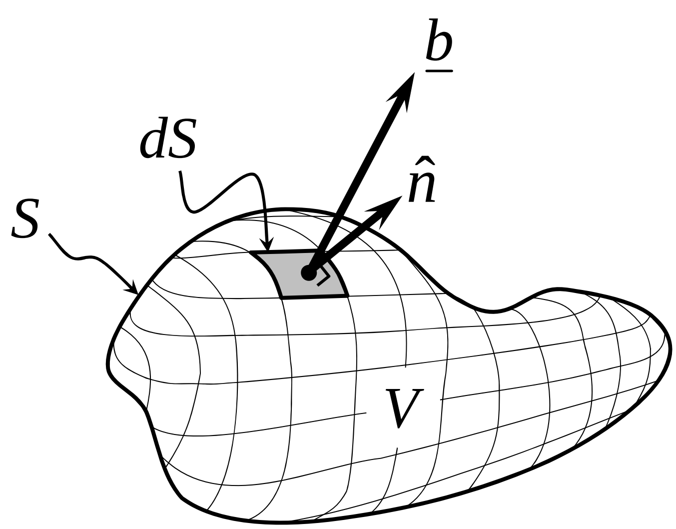
Momentum is conserved.

$$\begin{pmatrix} \textit{rate of increase} \\ \textit{of momentum in } CV \end{pmatrix} = \begin{pmatrix} \textit{net flux of} \\ \textit{momentum into } CV \end{pmatrix} + \begin{pmatrix} \textit{sum of} \\ \textit{forces on } CV \end{pmatrix}$$

resembles the rate term in the mass balance

resembles the flux term in the mass balance

Forces: body (gravity) molecular forces



Momentum Balance *(continued)* Polymer Rheology

$$\begin{pmatrix} \text{rate of increase} \\ \text{of momentum in } V \end{pmatrix} = \frac{d}{dt}\left(\iiint_V \rho \underline{v}\, dV \right)$$
$$= \iiint_V \frac{\partial}{\partial t}\left(\rho \underline{v}\right) dV$$

Leibnitz rule

$$\begin{pmatrix} \text{net flux of} \\ \text{momentum into } V \end{pmatrix} = -\iint_S \hat{n}\cdot\left(\rho \underline{v}\,\underline{v}\right) dS$$
$$= -\iiint_V \nabla\cdot\left(\rho \underline{v}\,\underline{v}\right) dV$$

Gauss Divergence Theorem

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Momentum Balance *(continued)* Polymer Rheology

Forces on *V*

Body Forces (non-contact)

$$\begin{pmatrix} \text{force on } V \\ \text{due to } \underline{g} \end{pmatrix} = \iiint_V \rho \underline{g}\; dV$$

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Chapter 3: Newtonian Fluid Mechanics

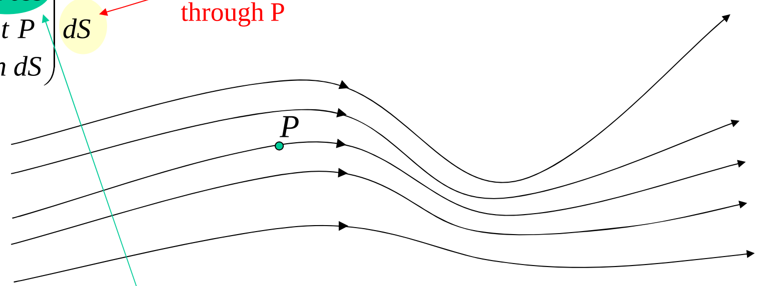
Polymer Rheology

Molecular Forces (contact) – this is the tough one

$$\underline{f} = \begin{pmatrix} stress \\ at\ P \\ on\ dS \end{pmatrix} dS$$

choose a surface through P

the force on that surface



We need an expression for the state of **stress** at an arbitrary point P in a flow.

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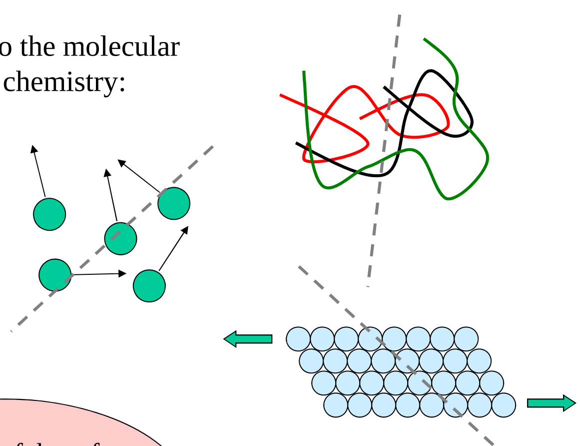
Molecular Forces *(continued)*

Think back to the molecular picture from chemistry:



The specifics of these forces, connections, and interactions must be captured by the molecular forces term that we seek.

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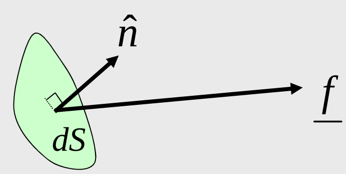
Molecular Forces *(continued)*

•We will concentrate on **expressing the molecular forces** mathematically;

•We leave to later the task of relating the resulting mathematical expression to experimental observations.

First, choose a surface:

•arbitrary shape
•small



$$\begin{pmatrix} stress \\ at\ P \\ on\ dS \end{pmatrix} dS = \underline{f}$$

What is f?

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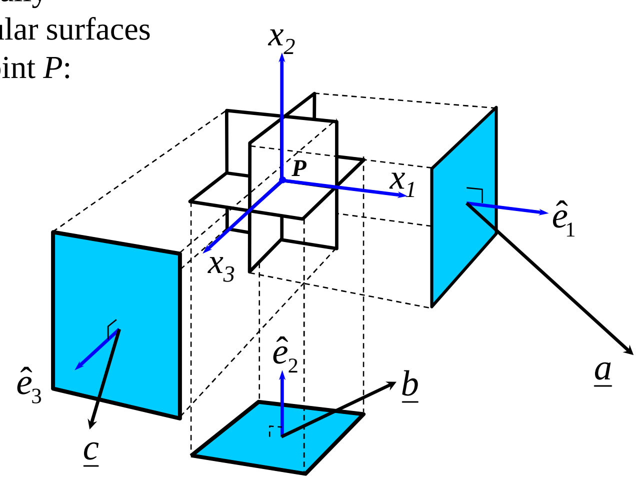
Consider the forces on three mutually perpendicular surfaces through point P:



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Molecular Forces *(continued)*

$\underline{a}$ is stress on a $\underbrace{\text{“1” surface}}_{\text{a surface with unit normal } \hat{e}_1}$ at P

$\underline{b}$ is stress on a “2” surface at P

$\underline{c}$ is stress on a “3” surface at P

We can write these vectors in a Cartesian coordinate system:

$$\underline{a} = a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3$$
$$= \Pi_{11}\hat{e}_1 + \Pi_{12}\hat{e}_2 + \Pi_{13}\hat{e}_3$$

stress on a “1” surface in the 1-direction

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Molecular Forces *(continued)*

$$\underline{a} = a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3$$
$$= \Pi_{11}\hat{e}_1 + \Pi_{12}\hat{e}_2 + \Pi_{13}\hat{e}_3$$
$$\underline{b} = b_1\hat{e}_1 + b_2\hat{e}_2 + b_3\hat{e}_3$$
$$= \Pi_{21}\hat{e}_1 + \Pi_{22}\hat{e}_2 + \Pi_{23}\hat{e}_3$$
$$\underline{c} = c_1\hat{e}_1 + c_2\hat{e}_2 + c_3\hat{e}_3$$
$$= \Pi_{31}\hat{e}_1 + \Pi_{32}\hat{e}_2 + \Pi_{33}\hat{e}_3$$

$\underline{a}$ is stress on a “1” surface at P

$\underline{b}$ is stress on a “2” surface at P

$\underline{c}$ is stress on a “3” surface at P

So far, this is nomenclature; next we relate these expressions to force on an arbitrary surface.

Π_{pk}

Stress on a “*p*” surface in the *k*-direction

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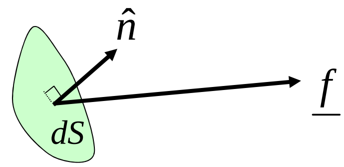
Molecular Forces *(continued)*

How can we write $\underline{f}$ (the force on an arbitrary surface dS) in terms of the Π_{pk}?



$$\underline{f} = f_1\hat{e}_1 + f_2\hat{e}_2 + f_3\hat{e}_3$$

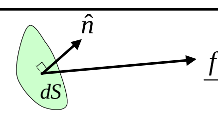
f_1 is force on dS in 1-direction

f_2 is force on dS in 2-direction

f_3 is force on dS in 3-direction

There are <u>three</u> Π_{pk} that relate to forces in the 1-direction: $\Pi_{11}, \Pi_{21}, \Pi_{31}$

Molecular Forces *(continued)*



How can we write $\underline{f}$ (the force on an arbitrary surface dS) in terms of the quantities Π_{pk}?

$$\underline{f} = f_1\hat{e}_1 + f_2\hat{e}_2 + f_3\hat{e}_3$$

f_1 , the force on dS in 1-direction, can be broken into <u>three</u> parts associated with the three stress components: $\Pi_{11}, \Pi_{21}, \Pi_{31}$

first part:

$$(\Pi_{11})\overbrace{\begin{pmatrix}\text{projection of}\\ dA \text{ onto the}\\ 1-\text{surface}\end{pmatrix}}^{\hat{n}\cdot\hat{e}_1\, dS} = \Pi_{11}\hat{n}\cdot\hat{e}_1\, dS$$

$$\left(\frac{force}{area}\right) \cdot (area)$$

Molecular Forces *(continued)*

f_1 , the force on dS in 1-direction, is composed of THREE parts:

first part: $\left\{ (\Pi_{11}) \begin{pmatrix} projection\ of \\ dA\ onto\ the \\ 1-surface \end{pmatrix} \right. = \Pi_{11}\hat{n}\cdot\hat{e}_1\, dS$

second part: $\left\{ (\Pi_{21}) \begin{pmatrix} projection\ of \\ dA\ onto\ the \\ 2-surface \end{pmatrix} \right. = \Pi_{21}\hat{n}\cdot\hat{e}_2\, dS$

third part: $\left\{ (\Pi_{31}) \begin{pmatrix} projection\ of \\ dA\ onto\ the \\ 3-surface \end{pmatrix} \right. = \Pi_{31}\hat{n}\cdot\hat{e}_3\, dS$

stress on a 2 -surface in the 1-direction

the sum of these three = f_1

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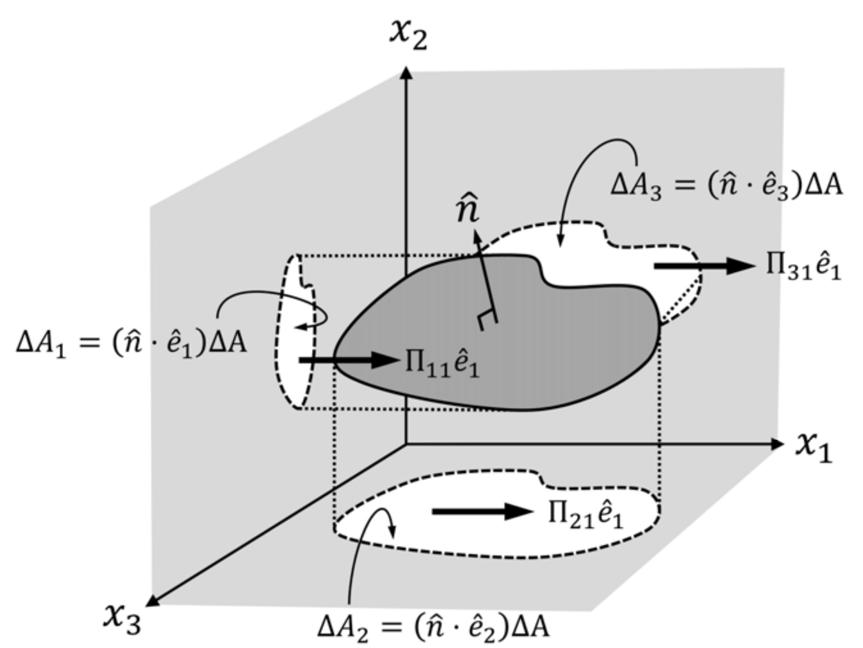

Molecular Forces *(continued)*

f_1, the force in the 1-direction on an arbitrary surface dS is composed of THREE parts.

$$f_1 = \underset{\uparrow \text{ stress}}{\Pi_{11}}\hat{n}\cdot\hat{e}_1\, dS + \Pi_{21}\underbrace{\hat{n}\cdot\hat{e}_2\, dS}_{\text{appropriate area}} + \Pi_{31}\hat{n}\cdot\hat{e}_3\, dS$$

Using the distributive law:

$$\boxed{f_1 = \hat{n}\cdot\left(\Pi_{11}\hat{e}_1 + \Pi_{21}\hat{e}_2 + \Pi_{31}\hat{e}_3\right) dS}$$

Force in the 1-direction on an arbitrary surface dS

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Molecular Forces *(continued)*

The same logic applies in the 2-direction and the 3-direction

$$\boxed{\begin{aligned} f_1 &= \hat{n}\cdot\left(\Pi_{11}\hat{e}_1 + \Pi_{21}\hat{e}_2 + \Pi_{31}\hat{e}_3\right) dS \\ f_2 &= \hat{n}\cdot\left(\Pi_{12}\hat{e}_1 + \Pi_{22}\hat{e}_2 + \Pi_{32}\hat{e}_3\right) dS \\ f_3 &= \hat{n}\cdot\left(\Pi_{13}\hat{e}_1 + \Pi_{23}\hat{e}_2 + \Pi_{33}\hat{e}_3\right) dS \end{aligned}}$$

Assembling the force vector:

$$\begin{aligned} \underline{f} &= f_1\hat{e}_1 + f_2\hat{e}_2 + f_3\hat{e}_3 \\ &= dS\ \hat{n}\cdot\left(\Pi_{11}\hat{e}_1 + \Pi_{21}\hat{e}_2 + \Pi_{31}\hat{e}_3\right)\hat{e}_1 \\ &\quad + dS\ \hat{n}\cdot\left(\Pi_{12}\hat{e}_1 + \Pi_{22}\hat{e}_2 + \Pi_{32}\hat{e}_3\right)\hat{e}_2 \\ &\quad\quad + dS\ \hat{n}\cdot\left(\Pi_{13}\hat{e}_1 + \Pi_{23}\hat{e}_2 + \Pi_{33}\hat{e}_3\right)\hat{e}_3 \end{aligned}$$

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Molecular Forces *(continued)*

Assembling the force vector:

$$\underline{f} = f_1\hat{e}_1 + f_2\hat{e}_2 + f_3\hat{e}_3$$

$$= dS\ \hat{n}\cdot\left(\Pi_{11}\hat{e}_1 + \Pi_{21}\hat{e}_2 + \Pi_{31}\hat{e}_3\right)\hat{e}_1 + dS\ \hat{n}\cdot\left(\Pi_{12}\hat{e}_1 + \Pi_{22}\hat{e}_2 + \Pi_{32}\hat{e}_3\right)\hat{e}_2 + dS\ \hat{n}\cdot\left(\Pi_{13}\hat{e}_1 + \Pi_{23}\hat{e}_2 + \Pi_{33}\hat{e}_3\right)\hat{e}_3$$

$$= dS\ \hat{n}\cdot\underbrace{\left[\Pi_{11}\hat{e}_1\hat{e}_1 + \Pi_{21}\hat{e}_2\hat{e}_1 + \Pi_{31}\hat{e}_3\hat{e}_1 + \Pi_{12}\hat{e}_1\hat{e}_2 + \Pi_{22}\hat{e}_2\hat{e}_2 + \Pi_{32}\hat{e}_3\hat{e}_2 + \Pi_{13}\hat{e}_1\hat{e}_3 + \Pi_{23}\hat{e}_2\hat{e}_3 + \Pi_{33}\hat{e}_3\hat{e}_3\right]}$$

linear combination of dyadic products = **tensor**

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Molecular Forces *(continued)*

Assembling the force vector:

$$\underline{f} = dS\ \hat{n}\cdot\left[\Pi_{11}\hat{e}_1\hat{e}_1 + \Pi_{21}\hat{e}_2\hat{e}_1 + \Pi_{31}\hat{e}_3\hat{e}_1 + \Pi_{12}\hat{e}_1\hat{e}_2 + \Pi_{22}\hat{e}_2\hat{e}_2 + \Pi_{32}\hat{e}_3\hat{e}_2 + \Pi_{13}\hat{e}_1\hat{e}_3 + \Pi_{23}\hat{e}_2\hat{e}_3 + \Pi_{33}\hat{e}_3\hat{e}_3\right]$$

$$= dS\ \hat{n}\cdot\sum_{p=1}^{3}\sum_{m=1}^{3}\Pi_{pm}\hat{e}_p\hat{e}_m$$

$$= dS\ \hat{n}\cdot\Pi_{pm}\hat{e}_p\hat{e}_m$$

$$\boxed{\underline{f} = dS\ \hat{n}\cdot\underline{\underline{\Pi}}}$$

Total stress tensor (molecular stresses)

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Momentum Balance *(continued)* Polymer Rheology

$$\begin{pmatrix} \text{rate of increase} \\ \text{of momentum in } V \end{pmatrix} = \begin{pmatrix} \text{net flux of} \\ \text{momentum into } V \end{pmatrix} + \begin{pmatrix} \text{sum of} \\ \text{forces on } V \end{pmatrix}$$

$$\iiint_V \frac{\partial}{\partial t}\left(\rho \underline{v}\right) dV = -\iiint_V \nabla \cdot \left(\rho \underline{v}\underline{v}\right) dV + \iiint_V \rho \underline{g}\; dV + \begin{matrix}\text{molecular}\\ \text{forces}\end{matrix}$$

$$\begin{matrix}\text{molecular}\\ \text{forces}\end{matrix} = \iint_S \begin{pmatrix}\text{molecular}\\ \text{forces on}\\ dS\end{pmatrix}$$

$$= \iint_S \hat{n}\cdot\left(-\underline{\underline{\Pi}}\right) dS$$

$$= \iiint_V \nabla\cdot\left(-\underline{\underline{\Pi}}\right) dV$$

We use a stress sign convention that requires a negative sign here.

Gauss Divergence Theorem

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Momentum Balance *(continued)* Polymer Rheology

$$\begin{pmatrix} \text{rate of increase} \\ \text{of momentum in } V \end{pmatrix} = \begin{pmatrix} \text{net flux of} \\ \text{momentum into } V \end{pmatrix} + \begin{pmatrix} \text{sum of} \\ \text{forces on } V \end{pmatrix}$$

$$\iiint_V \frac{\partial}{\partial t}\left(\rho \underline{v}\right) dV = -\iiint_V \nabla \cdot \left(\rho \underline{v}\underline{v}\right) dV + \iiint_V \rho \underline{g}\; dV + \begin{matrix}\text{molecular}\\ \text{forces}\end{matrix}$$

$$\begin{matrix}\text{molecular}\\ \text{forces}\end{matrix} = \iint_S \begin{pmatrix}\text{molecular}\\ \text{forces on}\\ dS\end{pmatrix}$$

$$= \iint_S \hat{n}\cdot\left(-\underline{\underline{\Pi}}\right) dS$$

$$= \iiint_V \nabla\cdot\left(-\underline{\underline{\Pi}}\right) dV$$

<u>UR/Bird choice</u>: positive compression (pressure is positive)

Gauss Divergence Theorem

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Momentum Balance *(continued)*

Polymer Rheology

$$\underline{F}_{\substack{on\\surface}} = \iint_S \hat{n}\cdot\left(-\underline{\underline{\Pi}}\right)dS = \iint_S \hat{n}\cdot\left(\underline{\underline{\tilde{\Pi}}}\right)dS$$

$$\Pi_{yx} \qquad \tilde{\Pi}_{yx}$$

<u>UR/Bird choice</u>: fluid at lesser *y* exerts force on fluid at greater *y*

<u>(IFM/Mechanics choice</u>: (opposite)

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Momentum Balance *(continued)*

Polymer Rheology

Final Assembly:

$$\begin{pmatrix}\text{rate of increase}\\ \text{of momentum in } V\end{pmatrix} = \begin{pmatrix}\text{net flux of}\\ \text{momentum into } V\end{pmatrix} + \begin{pmatrix}\text{sum of}\\ \text{forces on } V\end{pmatrix}$$

$$\iiint_V \frac{\partial}{\partial t}\left(\rho\underline{v}\right)dV = -\iiint_V \nabla\cdot\left(\rho\underline{v}\,\underline{v}\right)dV + \iiint_V \rho\underline{g}\;dV - \iiint_V \nabla\cdot\underline{\underline{\Pi}}\;dV$$

$$\iiint_V \left[\frac{\partial\rho\underline{v}}{\partial t} + \nabla\cdot\left(\rho\underline{v}\,\underline{v}\right) - \rho\underline{g} + \nabla\cdot\underline{\underline{\Pi}}\right]dV = 0$$

Because *V* is *arbitrary*, we may conclude:

$$\boxed{\frac{\partial\rho\underline{v}}{\partial t} + \nabla\cdot\left(\rho\underline{v}\,\underline{v}\right) - \rho\underline{g} + \nabla\cdot\underline{\underline{\Pi}} = 0}$$

Microscopic momentum balance

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Momentum Balance *(continued)* Polymer Rheology

Microscopic momentum balance

$$\boxed{\frac{\partial \rho \underline{v}}{\partial t} + \nabla \cdot \left(\rho \underline{v}\,\underline{v} \right) - \rho \underline{g} + \nabla \cdot \underline{\underline{\Pi}} = 0}$$

After some rearrangement:

$$\rho \left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} \right) = -\nabla \cdot \underline{\underline{\Pi}} + \rho \underline{g}$$

$$\rho \frac{D\underline{v}}{Dt} = -\nabla \cdot \underline{\underline{\Pi}} + \rho \underline{g}$$

Equation of Motion

Now, what to do with $\underline{\underline{\Pi}}$?

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Momentum Balance *(continued)* Polymer Rheology

Now, what to do with $\underline{\underline{\Pi}}$? Pressure is part of it.

Pressure

definition: An isotropic force/area of molecular origin. Pressure is the same on any surface drawn through a point and acts normally to the chosen surface.

$$pressure = p\,\underline{\underline{I}} = p\,\hat{e}_1\hat{e}_1 + p\,\hat{e}_2\hat{e}_2 + p\,\hat{e}_3\hat{e}_3 = \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}_{123}$$

Test: what is the force on a surface with unit normal $\hat{n}$?

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Momentum Balance *(continued)* Polymer Rheology

back to our question,

Now, what to do with $\underline{\underline{\Pi}}$? Pressure is part of it.

There are other, nonisotropic stresses

Extra Molecular Stresses

definition: The extra stresses are the molecular stresses that are not isotropic

$$\underbrace{\underline{\underline{\tau}}}_{\text{Extra stress tensor,}} \equiv \underline{\underline{\Pi}} - p\,\underline{\underline{I}}$$

i.e. everything complicated in molecular deformation

$\boxed{\text{Now, what to do with } \underline{\underline{\tau}}\ ?}$ This becomes the central question of rheological study

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Momentum Balance *(continued)* Polymer Rheology

Stress sign convention affects any expressions with $\underline{\underline{\Pi}}, \underline{\underline{\tilde{\Pi}}}$ or $\underline{\underline{\tau}}, \underline{\underline{\tilde{\tau}}}$

$$\underline{\underline{\Pi}} \equiv \underline{\underline{\tau}} + p\,\underline{\underline{I}}$$

UR/Bird choice: fluid at lesser *y* exerts force on fluid at greater *y*

$$\underline{\underline{\tilde{\Pi}}} \equiv \underline{\underline{\tilde{\tau}}} - p\,\underline{\underline{I}}$$

(IFM/Mechanics choice: (opposite)

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Momentum Balance *(continued)* Polymer Rheology

Constitutive equations for Stress

- are tensor equations
- relate the velocity field to the stresses generated by molecular forces
- are based on observations (empirical) or are based on molecular models (theoretical)
- are typically found by trial-and-error
- are justified by how well they work for a system of interest
- are observed to be symmetric

$$\underline{\underline{\tau}} = f(\nabla \underline{v}, \ \mathit{material\ properties})$$

Observation: the stress tensor is symmetric

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Momentum Balance *(continued)* Polymer Rheology

Microscopic momentum balance

$$\boxed{\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\right) = -\nabla\cdot\underline{\underline{\Pi}} + \rho\underline{g}}$$

Equation of Motion

In terms of the extra stress tensor:

$$\boxed{\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\right) = -\nabla p - \nabla\cdot\underline{\underline{\tau}} + \rho\underline{g}}$$

Equation of Motion

Cauchy Momentum Equation

Components in three coordinate systems (our sign convention):
http://www.chem.mtu.edu/~fmorriso/cm310/Navier2007.pdf

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Momentum Balance *(continued)* Polymer Rheology

Newtonian Constitutive equation

$$\underline{\underline{\tau}} = -\mu\left(\nabla \underline{v} + \left(\nabla \underline{v}\right)^T\right)$$

- for incompressible fluids (see text for compressible fluids)
- is empirical
- may be justified for some systems with molecular modeling calculations

Note: $\underline{\underline{\tilde{\tau}}} = +\mu\left(\nabla \underline{v} + \left(\nabla \underline{v}\right)^T\right)$

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Momentum Balance *(continued)* Polymer Rheology

How is the Newtonian Constitutive equation related to Newton's Law of Viscosity?

$$\underline{\underline{\tau}} = -\mu\left(\nabla \underline{v} + \left(\nabla \underline{v}\right)^T\right)$$

- incompressible fluids

$$\tau_{21} = -\mu \frac{\partial v_1}{\partial x_2}$$

- incompressible fluids
- rectilinear flow (straight lines)
- no variation in x_3-direction

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Momentum Balance *(continued)* Polymer Rheology

Back to the momentum balance . . .

$$\rho\left(\frac{\partial \underline{v}}{\partial t}+\underline{v}\cdot\nabla\underline{v}\right)=-\nabla p-\nabla\cdot\underline{\underline{\tau}}+\rho\underline{g}$$

Equation of Motion

$$\underline{\underline{\tau}}=-\mu\left(\nabla\underline{v}+\left(\nabla\underline{v}\right)^T\right)$$

We can incorporate the Newtonian constitutive equation into the momentum balance to obtain a momentum-balance equation that is specific to incompressible, Newtonian fluids

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Momentum Balance *(continued)* Polymer Rheology

Navier-Stokes Equation

$$\boxed{\rho\left(\frac{\partial \underline{v}}{\partial t}+\underline{v}\cdot\nabla\underline{v}\right)=-\nabla p+\mu\nabla^2\underline{v}+\rho\underline{g}}$$

- incompressible fluids
- Newtonian fluids

Note: The Navier-Stokes is unaffected by the stress sign convention.

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Next?

Navier-Stokes Equation

$$\boxed{\rho\left(\frac{\partial \underline{v}}{\partial t}+\underline{v}\cdot\nabla\underline{v}\right)=-\nabla p+\mu\nabla^2\underline{v}+\rho\underline{g}}$$

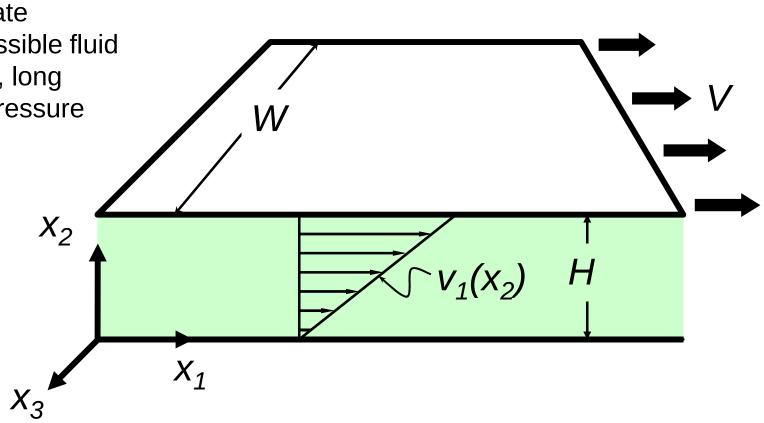
Newtonian Problem Solving

***from* QUICK START**

EXAMPLE: Drag flow between infinite parallel plates

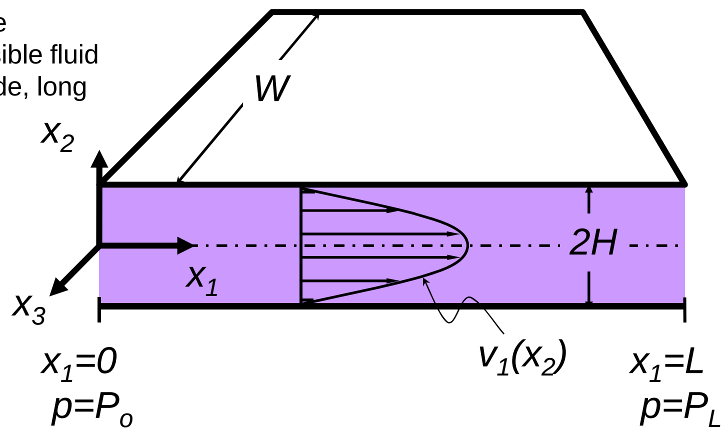
- Newtonian
- steady state
- incompressible fluid
- very wide, long
- uniform pressure

$$\underline{v}=\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}_{123}$$



EXAMPLE: Poiseuille flow between infinite parallel plates

- Newtonian
- steady state
- Incompressible fluid
- infinitely wide, long



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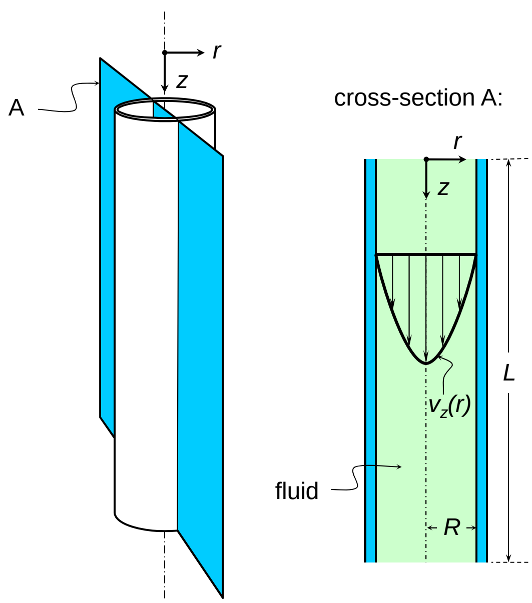
EXAMPLE: Poiseuille flow in a tube

- Newtonian
- Steady state
- incompressible fluid
- long tube



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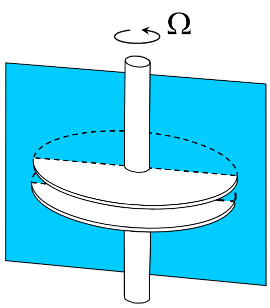
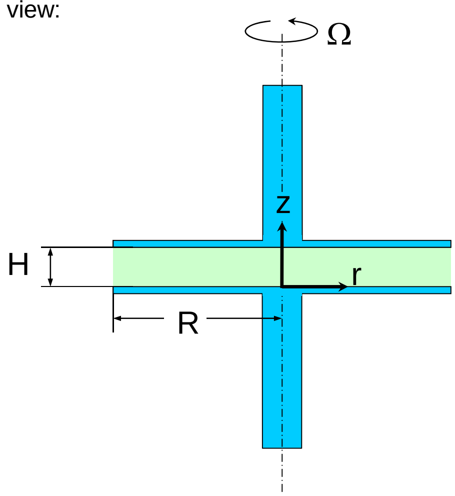
EXAMPLE: Torsional flow between parallel plates

- Newtonian
- Steady state
- incompressible fluid
- $v_\theta = zf(r)$



cross-sectional view:



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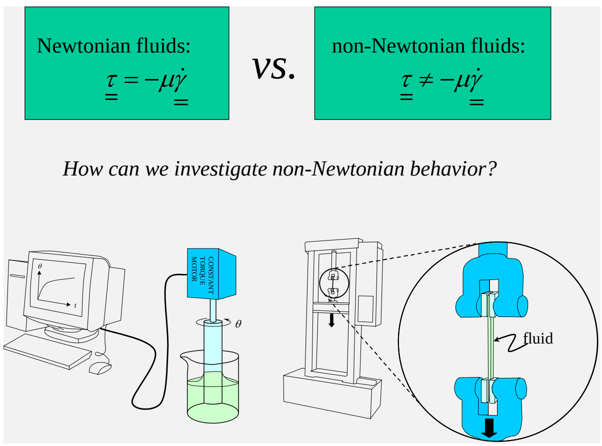
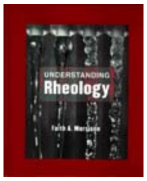
Chapter 4: Standard Flows

CM4650
Polymer Rheology
Michigan Tech





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Chapter 4: Standard Flows for Rheology

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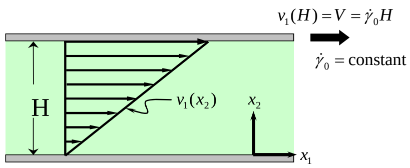
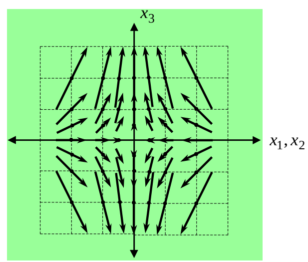
Polymer Rheology
Michigan Tech



shear



elongation

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On to . . . Polymer Rheology . . .

We now know how to model Newtonian fluid motion, $\underline{v}(\underline{x},t),\ p(\underline{x},t)$:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \underline{v}\right) = 0$$

Continuity equation

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v}\right) = -\nabla p - \nabla \cdot \underline{\underline{\tau}} + \rho \underline{g}$$

Cauchy momentum equation

$$\underline{\underline{\tau}} = -\mu\left(\nabla \underline{v} + \left(\nabla \underline{v}\right)^T\right)$$

Newtonian constitutive equation

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Rheological Behavior of Fluids – Non-Newtonian

How do we model the motion of Non-Newtonian fluid fluids?

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0$$

Continuity equation

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v}\right) = -\nabla p - \nabla \cdot \underline{\underline{\tau}} + \rho \underline{g}$$

Cauchy Momentum Equation

$$\underline{\underline{\tau}} = f(\underline{x}, t)$$

Non-Newtonian constitutive equation

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Rheological Behavior of Fluids – Non-Newtonian

How do we model the motion of Non-Newtonian fluid fluids?

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0$$

Continuity equation

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v}\right) = -\nabla p - \nabla \cdot \underline{\underline{\tau}} + \rho \underline{g}$$

Cauchy Momentum Equation

$$\underline{\underline{\tau}} = f(\underline{x}, t)$$

Non-Newtonian constitutive equation

This is the missing piece

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Chapter 4: Standard Flows for Rheology

Chapter 4: Standard flows
Chapter 5: Material Functions
Chapter 6: Experimental Data

To get to constitutive equations, we must first **quantify** how non-Newtonian fluids behave

Chapter 7: GNF
Chapter 8: GLVE
Chapter 9: Advanced

Constitutive equations

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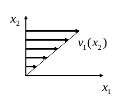
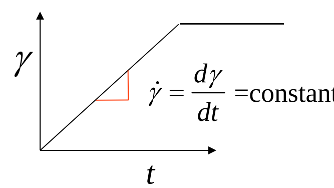
What do we observe?

Rheological Behavior of Fluids – Newtonian

1. Strain response to imposed shear stress

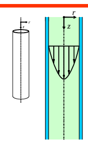
•shear rate is constant





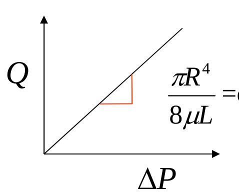
$$\dot{\gamma} = \frac{d\gamma}{dt} = \text{constant}$$

2. Pressure-driven flow in a tube (Poiseuille flow)



•viscosity is constant

$$Q = \frac{\pi \Delta P R^4}{8\mu L}$$



$$\frac{\pi R^4}{8\mu L} = \text{constant}$$

3. Stress tensor in shear flow

•only two components are nonzero

$$\underline{\underline{\tau}} = \begin{pmatrix} 0 & \tau_{12} & 0 \\ \tau_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{123}$$

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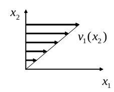
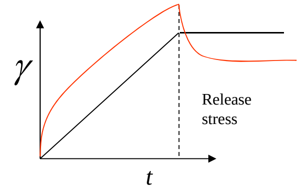
What do we observe?

Rheological Behavior of Fluids – Non-Newtonian

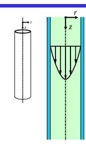
1. Strain response to imposed shear stress

•shear rate is variable

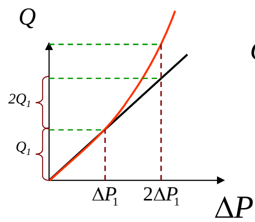




2. Pressure-driven flow in a tube (Poiseuille flow)



•viscosity is variable



$$Q = f(\Delta P)$$

3. Stress tensor in shear flow

Normal stresses

•all 9 components are nonzero

$$\underline{\underline{\tau}} = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}_{123}$$

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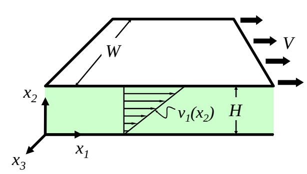
Non-Newtonian Constitutive Equations

- We have observations that some materials are not like Newtonian fluids.
- How can we be systematic about developing new, unknown models for these materials?

Need measurements

For Newtonian fluids, measurements were easy:

- shear flow
- one stress, τ_{21}
- one material constant, μ (viscosity)



$$\underline{\underline{\tau}} = -\mu\left(\nabla \underline{v} + (\nabla \underline{v})^T\right)$$

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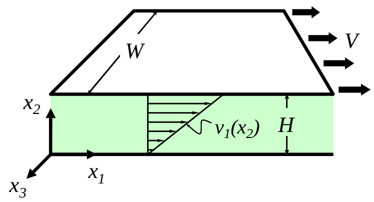
Non-Newtonian Constitutive Equations

Need measurements

For non-Newtonian fluids, measurements are **not easy**:

- shear flow (not the only choice)
- Four stresses in shear, $\tau_{21}, \tau_{11}, \tau_{22}, \tau_{33}$
- Unknown number of material constants in $\underline{\underline{\tau}}(\underline{v})$
- Unknown number of material *functions* in $\underline{\underline{\tau}}(\underline{v})$

$\underline{\underline{\tau}} = ???$



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Non-Newtonian Constitutive Equations

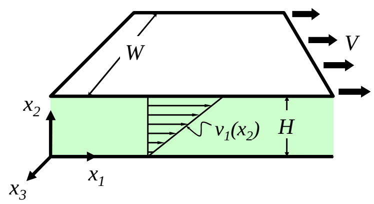
Need measurements

We know we need to make measurements to know more,

For non-Newtonian fluids, measurements are **not easy**:

- shear flow (not the only choice)
- Four stresses in shear, $\tau_{21}, \tau_{11}, \tau_{22}, \tau_{33}$
- Unknown number of material constants in $\underline{\underline{\tau}}(\underline{v})$
- Unknown number of material *functions* in $\underline{\underline{\tau}}(\underline{v})$

$\underline{\underline{\tau}} = ???$



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Non-Newtonian Constitutive Equations

Need measurements

For non-Newtonian fluids, measurements are **not easy**:

- shear flow (not the only choice)
- Four stresses in shear, $\tau_{21}, \tau_{11}, \tau_{22}, \tau_{33}$
- Unknown number of material constants in $\underline{\underline{\tau}}(\underline{v})$
- Unknown number of material *functions* in $\underline{\underline{\tau}}(\underline{v})$

We know we need to make measurements to know more,

$\underline{\underline{\tau}} = ???$

But, because we do not know the functional form of $\underline{\underline{\tau}}(\underline{v})$, we don't know what we need to measure to know more!

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Non-Newtonian Constitutive Equations

What should we do?

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Non-Newtonian Constitutive Equations

What should we do?

1. Pick a small number of simple flows — Chapter 4: Standard flows
   - Standardize the flows
   - Make them easy to calculate with
   - Make them easy to produce in the lab

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Non-Newtonian Constitutive Equations

What should we do?

1. Pick a small number of simple flows — Chapter 4: Standard flows
   - Standardize the flows
   - Make them easy to calculate with
   - Make them easy to produce in the lab
2. Make calculations
3. Make measurements

Chapter 5: Material Functions
Chapter 6: Experimental Data

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Non-Newtonian Constitutive Equations

What should we do?

1. Pick a small number of simple flows — Chapter 4: Standard flows
    - Standardize the flows
    - Make them easy to calculate with
    - Make them easy to produce in the lab
2. Make calculations
3. Make measurements

Chapter 5: Material Functions
Chapter 6: Experimental Data

4. Try to deduce $\underline{\underline{\tau}}(\underline{v})$

Chapter 7: GNF
Chapter 8: GLVE
Chapter 9: Advanced

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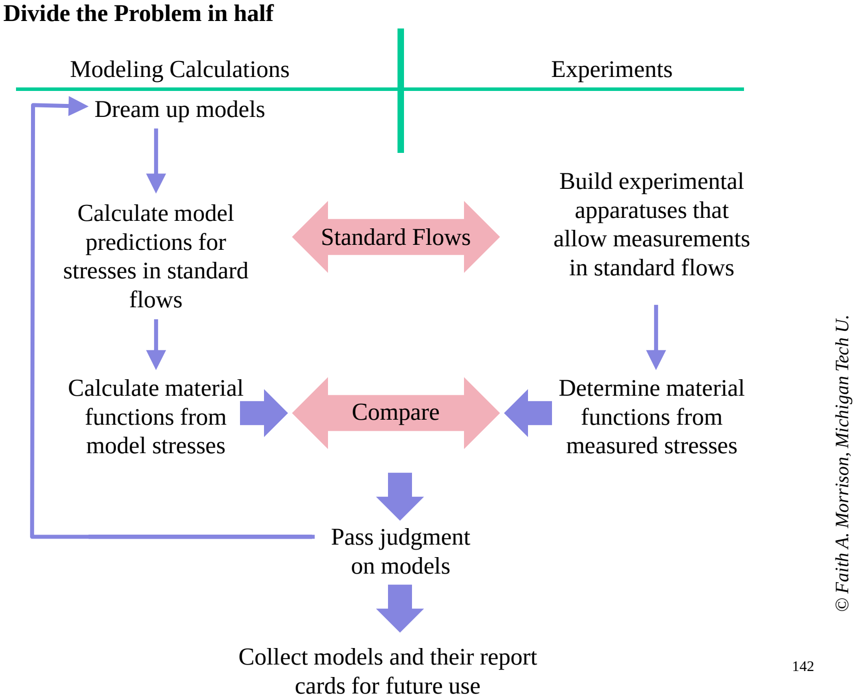
Tactic: Divide the Problem in half



Standard flows – choose a velocity field (not an apparatus or a procedure)

- For model predictions, calculations are straightforward
- For experiments, design can be optimized for accuracy and fluid variety

Material functions – choose a common vocabulary of stress and kinematics to report results

- Make it easier to compare model/experiment
- Record an "inventory" of fluid behavior (expertise)

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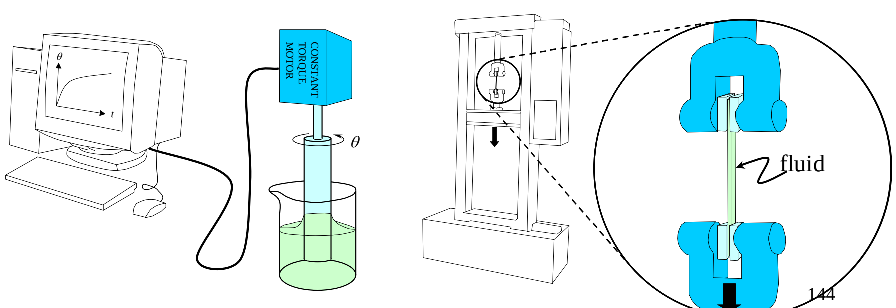
Newtonian fluids: $\underline{\underline{\tau}} = -\mu \dot{\underline{\underline{\gamma}}}$

vs.

non-Newtonian fluids: $\underline{\underline{\tau}} \neq -\mu \dot{\underline{\underline{\gamma}}}$

How can we investigate non-Newtonian behavior?



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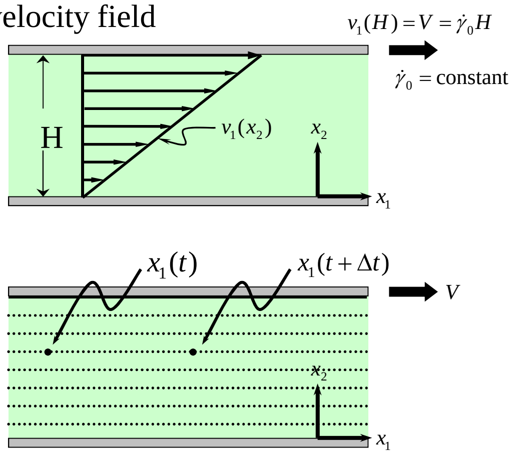
Simple Shear Flow

velocity field



$$\underline{v} \equiv \begin{pmatrix} \dot{\varsigma}(t)x_2 \\ 0 \\ 0 \end{pmatrix}_{123}$$

path lines

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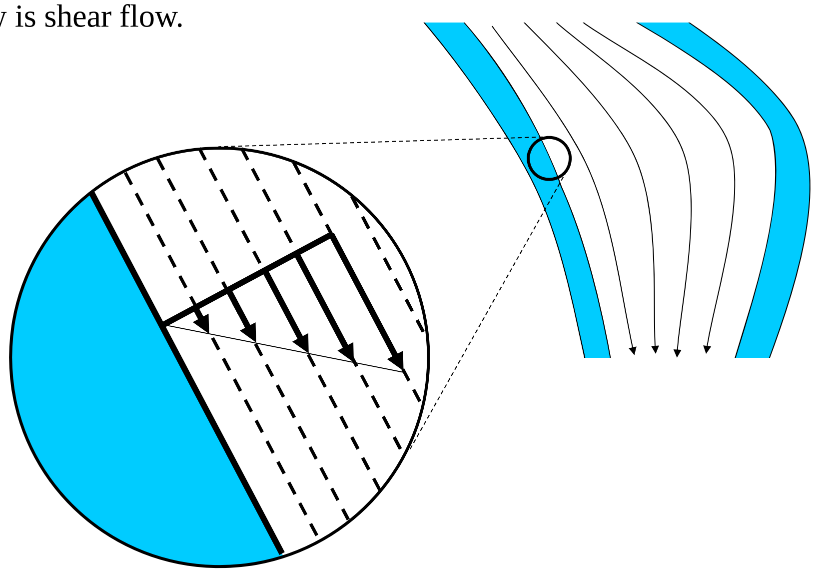
Near solid surfaces, the flow is shear flow.



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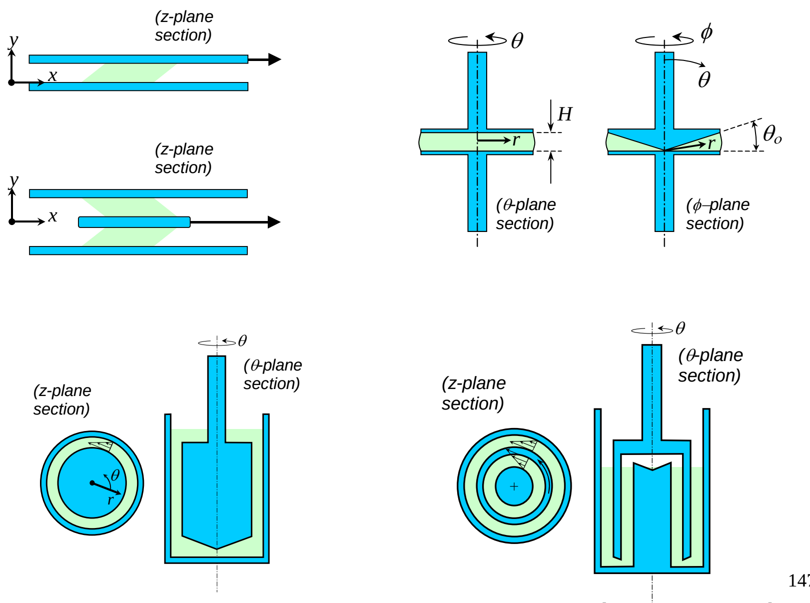
Experimental Shear Geometries



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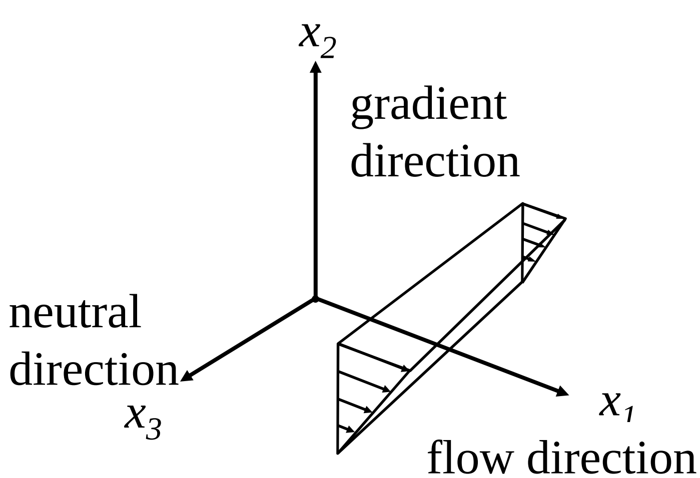
Standard Nomenclature for Shear Flow



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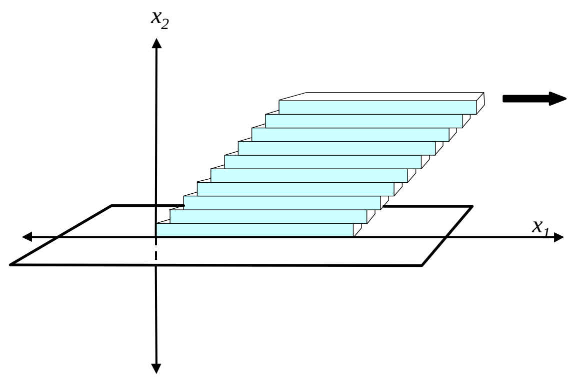
Why is shear a standard flow?

- simple velocity field
- represents all sliding flows
- simple stress tensor



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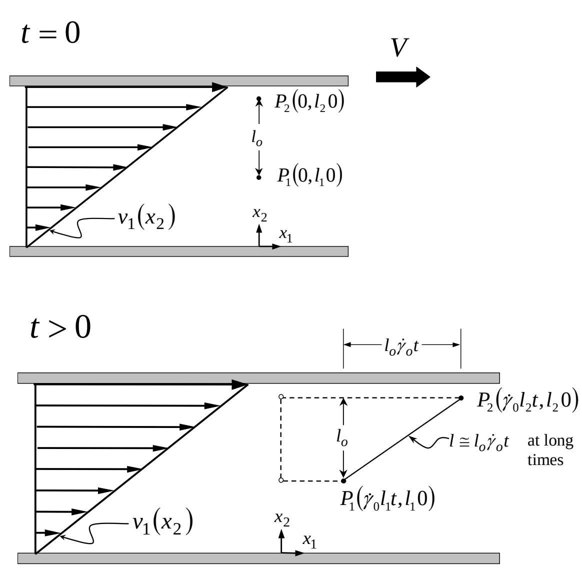
How do particles move apart in shear flow?

Consider two particles in the same x_1-x_2 plane, initially along the x_2 axis.



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How do particles move apart in shear flow?

$$\underline{v} = \begin{pmatrix} \dot{\gamma}_0 x_2 \\ 0 \\ 0 \end{pmatrix}_{123}$$

Each particle has a different velocity depending on its x_2 position:

$$v_1 = \dot{\gamma}_0 x_2$$

Consider two particles in the same x_1-x_2 plane, initially along the x_2 axis ($x_1=0$).

$$P_1: \quad v_1 = \dot{\gamma}_0 l_1$$
$$P_2: \quad v_1 = \dot{\gamma}_0 l_2$$

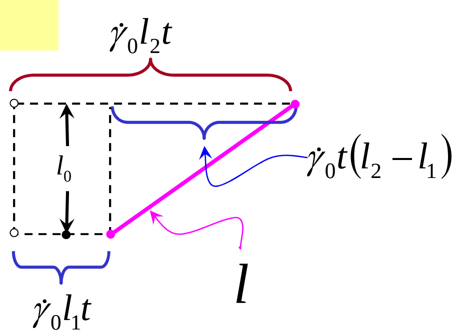
The initial x_1 position of each particle is $x_1=0$. After t seconds, the two particles are at the following positions:

$$P_1(t): \quad x_1 = \dot{\gamma}_0 l_1 t$$
$$P_2(t): \quad x_1 = \underbrace{\dot{\gamma}_0 l_2}\underbrace{t}$$

$$location = initial + \left(\frac{length}{time}\right)(time)$$

151

What is the separation of the particles after time t?



$$l^2 = l_0^2 + \left[\dot{\gamma}_0 t (l_2 - l_1)\right]^2$$
$$= l_0^2 + \dot{\gamma}_0^2 t^2 l_0^2$$
$$= l_0^2\left(1 + \dot{\gamma}_0^2 t^2\right)$$
$$l = l_0\sqrt{1 + \dot{\gamma}_0^2 t^2} \quad \approx l_0 \dot{\gamma}_0 t$$

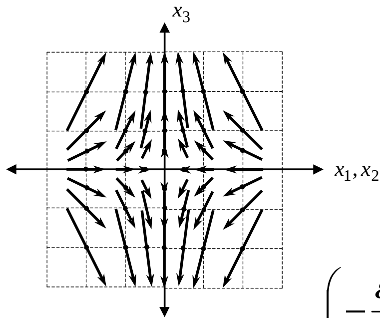
negligible as $t \rightarrow \infty$

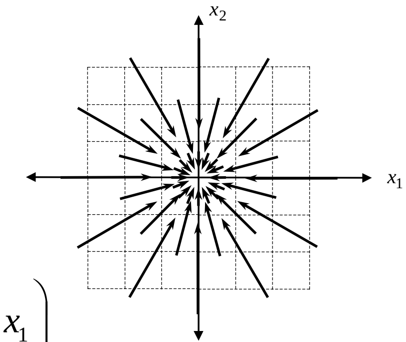
$$\boxed{l \approx l_0 \dot{\gamma}_0 t}$$

In shear the distance between points is directly proportional to time

152

Uniaxial Elongational Flow



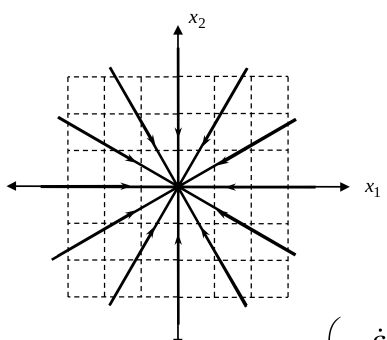


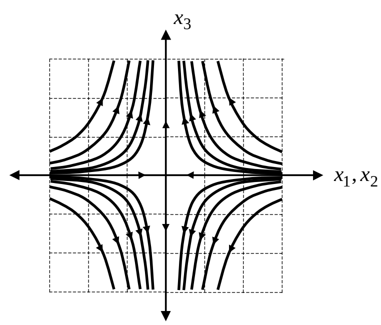
$$\underline{v} \equiv \begin{pmatrix} -\frac{\dot{\varepsilon}(t)}{2} x_1 \\ -\frac{\dot{\varepsilon}(t)}{2} x_2 \\ \dot{\varepsilon}(t) x_3 \end{pmatrix}_{123} \qquad \dot{\varepsilon}(t) > 0$$

velocity field

153

Uniaxial Elongational Flow



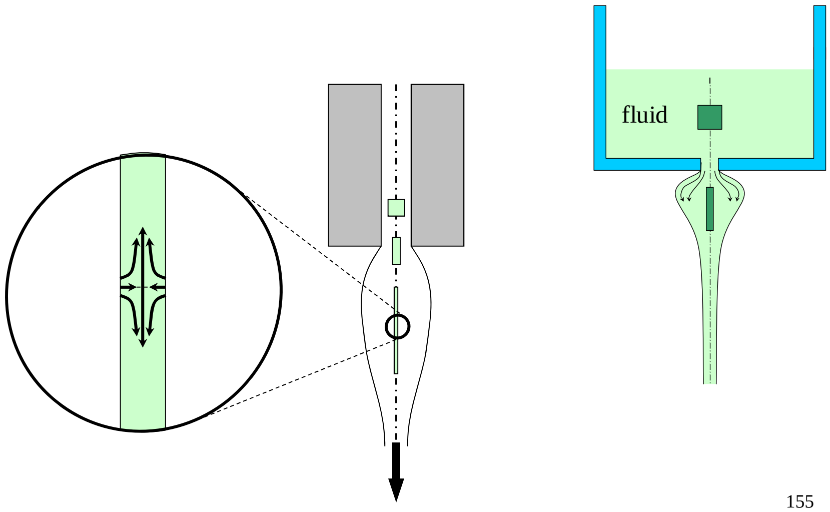


$$\underline{v} \equiv \begin{pmatrix} -\frac{\dot{\varepsilon}(t)}{2} x_1 \\ -\frac{\dot{\varepsilon}(t)}{2} x_2 \\ \dot{\varepsilon}(t) x_3 \end{pmatrix}_{123} \qquad \dot{\varepsilon}(t) > 0$$

path lines

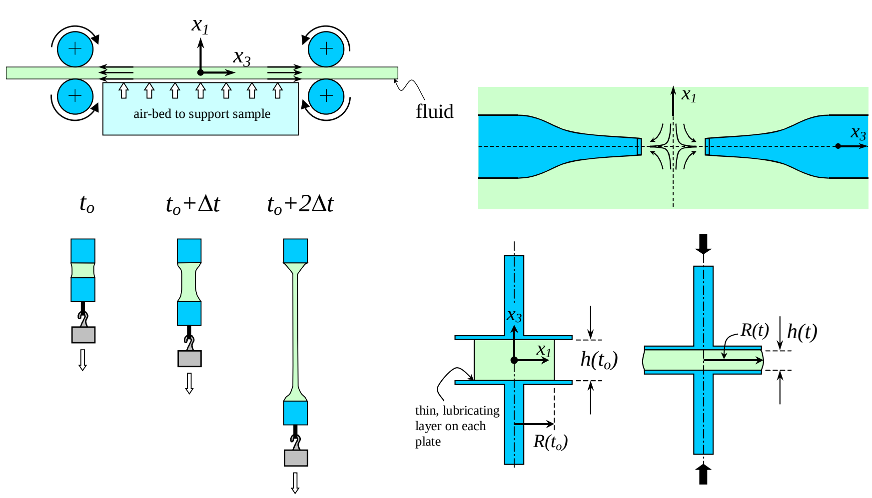
154

Elongational flow occurs when there is stretching - die exit, flow through contractions



155

Experimental Elongational Geometries



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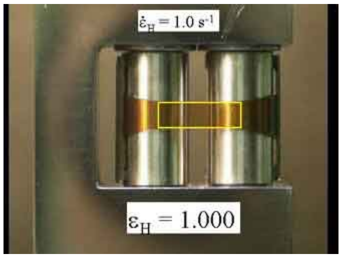
Sentmanat Extension Rheometer (2005)

- •Originally developed for rubbers, good for melts
- •Measures elongational viscosity, startup, other material functions
- •Two counter-rotating drums
- •Easy to load; reproducible



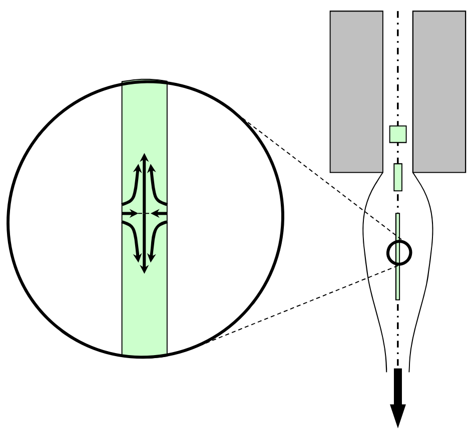
www.xpansioninstruments.com

http://www.xpansioninstruments.com/rheo-optics.htm

157

Why is elongation a standard flow?

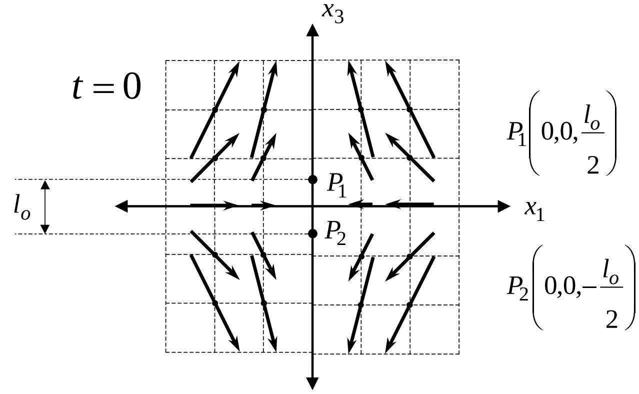
- •simple velocity field
- •represents all stretching flows
- •simple stress tensor



158

How do particles move apart in elongational flow?

Consider two particles in the same x_1-x_3 plane, initially along the x_3 axis.



159

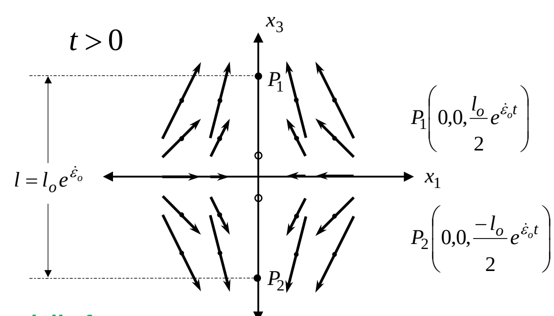
How do particles move apart in elongational flow?

Consider two particles in the same x_1-x_3 plane, initially along the x_3 axis.

$$x_1 = 0$$
$$x_2 = 0$$
$$x_3 \text{ varies}$$

$$\underline{v} = \begin{pmatrix} -\frac{\dot{\varepsilon}_0}{2}x_1 \\ -\frac{\dot{\varepsilon}_0}{2}x_2 \\ \dot{\varepsilon}_0 x_3 \end{pmatrix}_{123} = \begin{pmatrix} 0 \\ 0 \\ \dot{\varepsilon}_0 x_3 \end{pmatrix}_{123}$$

$$v_3 = \frac{dx_3}{dt} = \dot{\varepsilon}_0 x_3$$
$$\frac{dx_3}{x_3} = \dot{\varepsilon}_0 dt$$
$$\ln x_3 = \dot{\varepsilon}_0 t + C_1$$
$$x_3 = x_3(0)e^{\dot{\varepsilon}_0 t}$$



$$l = l_0 e^{\dot{\varepsilon}_0 t}$$

Particles move apart exponentially fast.

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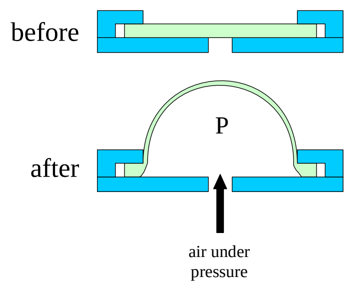
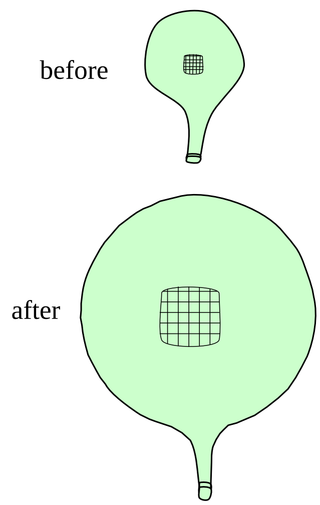
A second type of shear-free flow: Biaxial Stretching



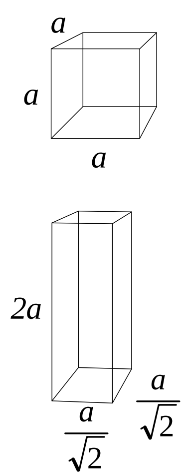


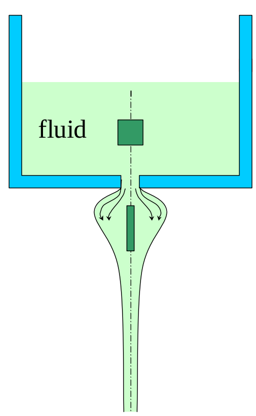
$$\underline{v} \equiv \begin{pmatrix} -\frac{\dot{\varepsilon}(t)}{2} x_1 \\ -\frac{\dot{\varepsilon}(t)}{2} x_2 \\ \dot{\varepsilon}(t) x_3 \end{pmatrix}_{123} \qquad \dot{\varepsilon}(t) < 0$$

161

How do uniaxial and biaxial deformations differ?

Consider a uniaxial flow in which a particle is doubled in length in the flow direction.

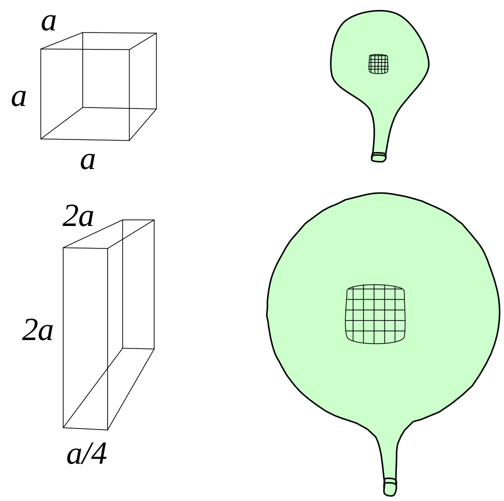




162

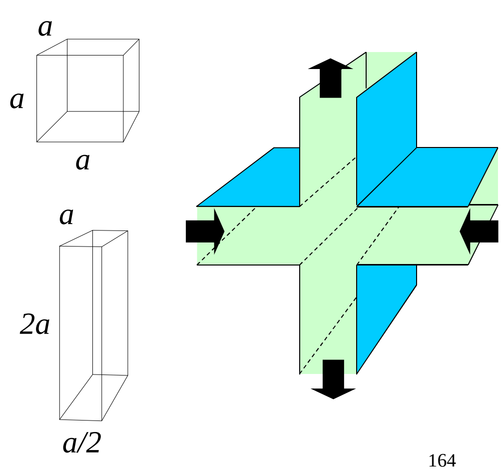
How do uniaxial and biaxial deformations differ?

Consider a biaxial flow in which a particle is doubled in length in the flow direction.



A third type of shear-free flow:
Planar Elongational Flow

$$\underline{v} \equiv \begin{pmatrix} -\dot{\varepsilon}(t)x_1 \\ 0 \\ \dot{\varepsilon}(t)x_3 \end{pmatrix}_{123} \qquad \dot{\varepsilon}(t) > 0$$



All three shear-free flows can be written together as:

$$\underline{v} = \begin{pmatrix} -\frac{1}{2}\dot{\varepsilon}(t)(1+b)x_1 \\ -\frac{1}{2}\dot{\varepsilon}(t)(1-b)x_2 \\ \dot{\varepsilon}(t)x_3 \end{pmatrix}_{123}$$

Elongational flow: b=0, $\dot{\varepsilon}(t) > 0$

Biaxial stretching: b=0, $\dot{\varepsilon}(t) < 0$

Planar elongation: b=1, $\dot{\varepsilon}(t) > 0$

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Why have we chosen these flows?

ANSWER: Because these simple flows have **symmetry**.

And symmetry allows us to draw conclusions about the stress tensor that is associated with these flows ***for any fluid*** subjected to that flow.

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In general:

$$\underline{\underline{\tau}} = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}_{123}$$

But the stress tensor is symmetric – leaving 6 independent stress components.

Can we choose a flow to use in which there are fewer than 6 independent stress components?

Yes we can – **symmetric flows**

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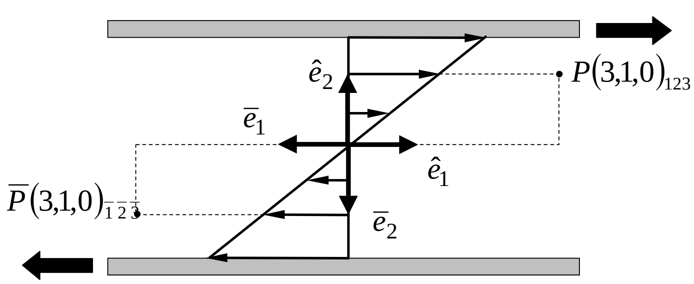
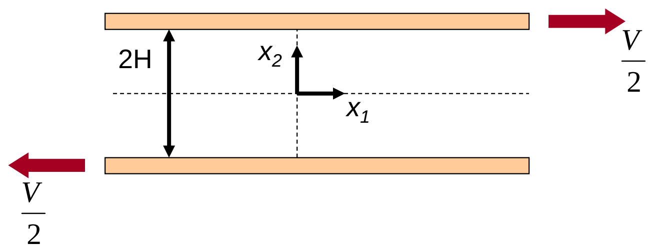
How does the stress tensor simplify for shear (*and later, elongational*) flow?



168

What would the velocity function be for a Newtonian fluid in this coordinate system?

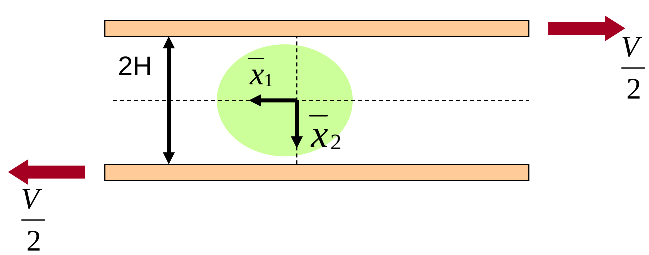
$$\underline{v} = \begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix}_{123}$$



169

What would the velocity function be for a Newtonian fluid in **this** coordinate system?

$$\underline{v} = \begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix}_{123}$$



170

Vectors are independent of coordinate system, but in general the coefficients will be different when the same vector is written in two different coordinate systems:

$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}_{123} = \begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \bar{v}_3 \end{pmatrix}_{\bar{1}\bar{2}\bar{3}}$$

For shear flow and the two particular coordinate systems we have just examined, however:

$$\underline{v} = \begin{pmatrix} \frac{V}{2H} x_2 \\ 0 \\ 0 \end{pmatrix}_{123} = \begin{pmatrix} \frac{V}{2H} \bar{x}_2 \\ 0 \\ 0 \end{pmatrix}_{\bar{1}\bar{2}\bar{3}}$$

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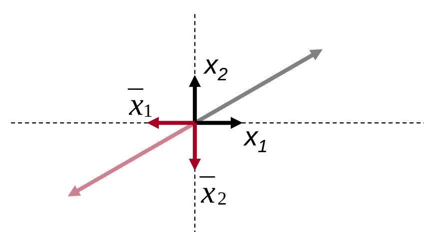
$$\underline{v} = \begin{pmatrix} \frac{V}{2H} x_2 \\ 0 \\ 0 \end{pmatrix}_{123} = \begin{pmatrix} \frac{V}{2H} \bar{x}_2 \\ 0 \\ 0 \end{pmatrix}_{\bar{1}\bar{2}\bar{3}}$$



If we plug in the **same number** in for x_2 and $\bar{x}_2$, we will NOT be asking about the same point in space, but we WILL get the same exact velocity vector.

Since stress is calculated from the velocity field, we will get the **same exact stress components** when we calculate them from either vector representation.

$$\boxed{\begin{aligned} v_n &= \bar{v}_n \\ \tau_{pk} &= \bar{\tau}_{pk} \end{aligned}}$$

This is an unusual circumstance only true for the particular coordinate systems chosen.

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What do we learn if we formally transform $\underline{V}$ from one coordinate system to the other?

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What do we learn if we formally transform $\underline{\underline{\tau}}$ from one coordinate system to the other?

$$\boxed{\begin{gathered}\hat{e}_1 = -\bar{e}_1 \\ \hat{e}_2 = -\bar{e}_2 \\ \hat{e}_3 = \bar{e}_3\end{gathered}}$$

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What do we learn if we formally transform $\underline{V}$ from one coordinate system to the other?

$$\underline{\underline{\tau}} = \tau_{ms}\hat{e}_m\hat{e}_s = \bar{\tau}_{ms}\bar{e}_m\bar{e}_s$$

(now, substitute from previous slide and simplify)

You try.

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Conclusion:

Because of symmetry, there are only 5 nonzero components of the extra stress tensor in shear flow.

SHEAR:

$$\underline{\underline{\tau}} = \begin{pmatrix} \tau_{11} & \tau_{12} & 0 \\ \tau_{21} & \tau_{22} & 0 \\ 0 & 0 & \tau_{33} \end{pmatrix}_{123}$$

This greatly simplifies the experimentalists tasks as only four stress components must be measured instead of 6 (recall $\tau_{21} = \tau_{12}$).

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Summary:

We have found a coordinate system (the shear coordinate system) in which there are only 5 non-zero coefficients of the stress tensor. In addition, $\tau_{21} = \tau_{12}$.

This leaves only four stress components to be measured for this flow, expressed in this coordinate system.

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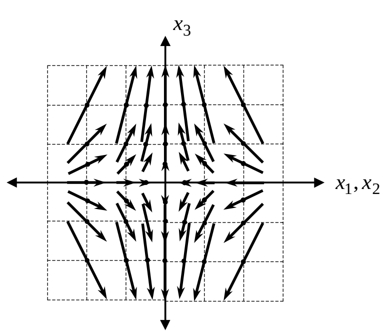
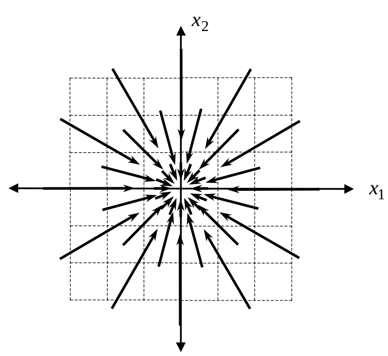
How does the stress tensor simplify for elongational flow?





There is 180º of symmetry around all three coordinate axes.

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Because of symmetry, there are only 3 nonzero components of the extra stress tensor in elongational flows.

ELONGATION:

$$\underline{\underline{\tau}} = \begin{pmatrix} \tau_{11} & 0 & 0 \\ 0 & \tau_{22} & 0 \\ 0 & 0 & \tau_{33} \end{pmatrix}_{123}$$

This greatly simplifies the experimentalists tasks as only three stress components must be measured instead of 6.

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Standard Flows Summary

Choose velocity field:

$$\underline{v} \equiv \begin{pmatrix} \dot{\varsigma}(t) x_2 \\ 0 \\ 0 \end{pmatrix}_{123}$$

$$\underline{v} = \begin{pmatrix} -\frac{1}{2}\dot{\varepsilon}(t)(1+b)x_1 \\ -\frac{1}{2}\dot{\varepsilon}(t)(1-b)x_2 \\ \dot{\varepsilon}(t)x_3 \end{pmatrix}_{123}$$

Symmetry alone implies:
(no constitutive equation needed yet)

$$\underline{\underline{\tau}} = \begin{pmatrix} \tau_{11} & \tau_{12} & 0 \\ \tau_{21} & \tau_{22} & 0 \\ 0 & 0 & \tau_{33} \end{pmatrix}_{123}$$

$$\underline{\underline{\tau}} = \begin{pmatrix} \tau_{11} & 0 & 0 \\ 0 & \tau_{22} & 0 \\ 0 & 0 & \tau_{33} \end{pmatrix}_{123}$$

By choosing these symmetric flows, we have reduced the number of stress components that we need to measure.

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Tactic: Divide the Problem in half

Modeling Calculations

Experiments

Dream up models

Calculate model predictions for stresses in standard flows

Standard Flows

Build experimental apparatuses that allow measurements in standard flows

Calculate material functions from model stresses

Compare

Determine material functions from measured stresses

Pass judgment on models

Collect models and their report cards for future use

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Next, build and assume this

Choose velocity field:

$$\underline{v} \equiv \begin{pmatrix} \dot{\varsigma}(t) x_2 \\ 0 \\ 0 \end{pmatrix}_{123}$$

$$\underline{v} = \begin{pmatrix} -\frac{1}{2}\dot{\varepsilon}(t)(1+b)x_1 \\ -\frac{1}{2}\dot{\varepsilon}(t)(1-b)x_2 \\ \dot{\varepsilon}(t)x_3 \end{pmatrix}_{123}$$

Symmetry alone implies:

(no constitutive equation needed yet)

$$\underline{\underline{\tau}} = \begin{pmatrix} \tau_{11} & \tau_{12} & 0 \\ \tau_{21} & \tau_{22} & 0 \\ 0 & 0 & \tau_{33} \end{pmatrix}_{123}$$

$$\underline{\underline{\tau}} = \begin{pmatrix} \tau_{11} & 0 & 0 \\ 0 & \tau_{22} & 0 \\ 0 & 0 & \tau_{33} \end{pmatrix}_{123}$$

Measure and predict this

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One final comment on measuring stresses. . .

What is measured is the total stress, $\underline{\underline{\Pi}}$:

$$\underline{\underline{\Pi}} = \begin{pmatrix} p+\tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & p+\tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & p+\tau_{33} \end{pmatrix}_{123}$$

For the normal stresses we are faced with the difficulty of separating p from τ_{ii}.

Compressible fluids:

$$\boxed{p = \frac{nRT}{V}}$$

Get p from measurements of T and V.

Incompressible fluids:

?

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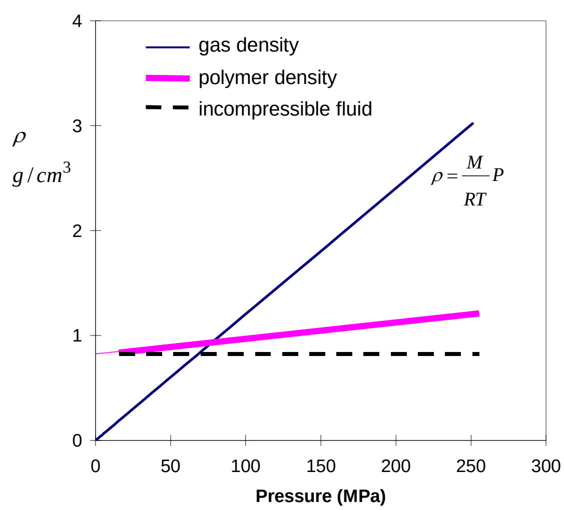
Density does not vary (much) with pressure for polymeric fluids.



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For incompressible fluids it is not possible to separate *p* from τ_{ii}.

Luckily, this is not a problem since we only need $\nabla \cdot \underline{\underline{\Pi}} = \nabla p + \nabla \cdot \underline{\underline{\tau}}$

Equation of motion

$$\boxed{\begin{aligned}\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} &= -\nabla \underline{\underline{\Pi}} + \rho \underline{g} \\ &= -\nabla P - \nabla \cdot \underline{\underline{\tau}} + \rho \underline{g}\end{aligned}}$$

We do not need τ_{ii} directly to solve for velocities

Solution? *Normal stress differences*

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Normal Stress Differences

First normal stress difference $N_1 \equiv \Pi_{11} - \Pi_{22} = \tau_{11} - \tau_{22}$

Second normal stress difference $N_2 \equiv \Pi_{22} - \Pi_{33} = \tau_{22} - \tau_{33}$

In shear flow, three stress quantities are measured $\boxed{\tau_{21}, N_1, N_2}$

In elongational flow, two stress quantities are measured $\boxed{\tau_{33} - \tau_{11}, \tau_{22} - \tau_{11}}$

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Normal Stress Differences

First normal stress difference
$$N_1 \equiv \Pi_{11} - \Pi_{22} = \tau_{11} - \tau_{22}$$

Second normal stress difference
$$N_2 \equiv \Pi_{22} - \Pi_{33} = \tau_{22} - \tau_{33}$$

In shear flow, three stress quantities are measured

$$\boxed{\tau_{21}, N_1, N_2}$$

Are shear normal stress differences real?

In elongational flow, two stress quantities are measured

$$\boxed{\tau_{33} - \tau_{11}, \tau_{22} - \tau_{11}}$$

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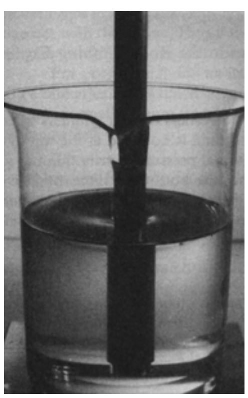
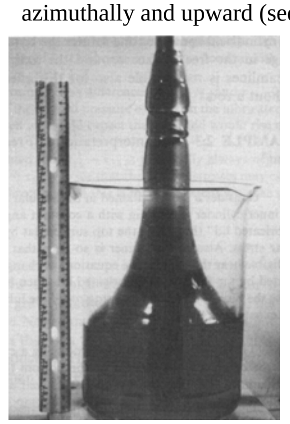
First normal stress effects: rod climbing

$$\tau_{11} - \tau_{22} < 0$$

Extra tension in the 1-direction pulls azimuthally and upward (see DPL p65).



Newtonian - glycerin



Viscoelastic - solution of polyacrylamide in glycerin

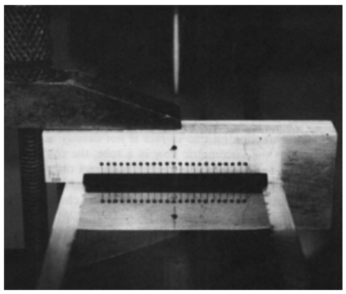
Bird, et al., *Dynamics of Polymeric Fluids*, vol. 1, Wiley, 1987, Figure 2.3-1 page 63. (DPL)

188

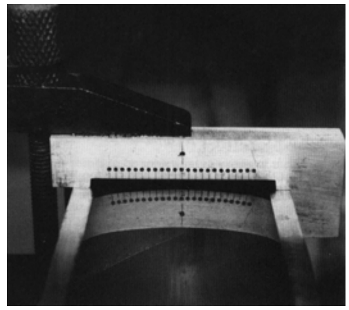
Second normal stress effects: inclined open-channel flow

$$\tau_{22} - \tau_{33} > 0$$

Extra tension in the 2-direction pulls down the free surface where dv_1/dx_2 is greatest (see DPL p65).



Newtonian - glycerin

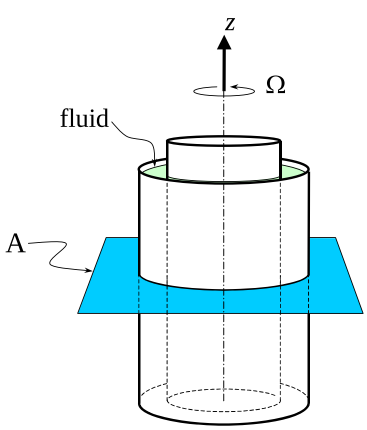


Viscoelastic - 1% soln of polyethylene oxide in water

$$\boxed{N_2 \simeq -N_1/10}$$

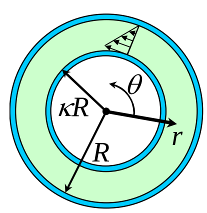
R. I. Tanner, *Engineering Rheology*, Oxford 1985, Figure 3.6 page 104

Example: Can the equation of motion predict rod climbing for typical values of N_1, N_2?



$$\underline{v} = \begin{pmatrix} 0 \\ v_\theta \\ 0 \end{pmatrix}_{r\theta z}$$

cross-section A:



What is $\dfrac{d\Pi_{zz}}{dr}$?

www.chem.mtu.edu/~fmorriso/cm4650/rod_climb.pdf

Bird et al. p64

What's next?

Shear	Shear-free (elongational, extensional)
$\underline{v} \equiv \begin{pmatrix} \dot{\varsigma}(t) x_2 \\ 0 \\ 0 \end{pmatrix}_{123}$	$\underline{v} = \begin{pmatrix} -\frac{1}{2}\dot{\varepsilon}(t)(1+b)x_1 \\ -\frac{1}{2}\dot{\varepsilon}(t)(1-b)x_2 \\ \dot{\varepsilon}(t)x_3 \end{pmatrix}_{123}$

Even with just these 2 (or 4) standard flows, we can still generate an *infinite* number of flows by varying $\dot{\varsigma}(t)$ and $\dot{\varepsilon}(t)$.

Elongational flow: b=0, $\dot{\varepsilon}(t) > 0$

Biaxial stretching: b=0, $\dot{\varepsilon}(t) < 0$

Planar elongation: b=1, $\dot{\varepsilon}(t) > 0$

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We seek to quantify the behavior of non-Newtonian fluids

Procedure:

1. Choose a flow type (shear or a type of elongation).
2. Specify $\dot{\varsigma}(t)$ or $\dot{\varepsilon}(t)$ as appropriate.
3. Impose the flow on a fluid of interest.
4. Measure stresses.

shear	τ_{21}, N_1, N_2
elongation	$\tau_{33}-\tau_{11}, \tau_{22}-\tau_{11}$

5. Report stresses in terms of material functions.

6a. Compare measured material functions with predictions of these material functions (from proposed constitutive equations).

7a. Choose the most appropriate constitutive equation for use in numerical modeling.

6b. Compare measured material functions with those measured on other materials.

7a. Draw conclusions on the likely properties of the unknown material based on the comparison.

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Chapter 5: Material Functions

CM4650
Polymer Rheology
Michigan Tech

Steady Shear Flow Material Functions

Kinematics:

$$\underline{v} \equiv \begin{pmatrix} \dot{\varsigma}(t) x_2 \\ 0 \\ 0 \end{pmatrix}_{123} \qquad \dot{\varsigma}(t) = \dot{\gamma}_0 = \text{constant}$$

Material Functions:

$$\eta \equiv \frac{-\tau_{21}}{\dot{\gamma}_0}$$

Viscosity

First normal-stress coefficient

$$\Psi_1 \equiv \frac{-(\tau_{11} - \tau_{22})}{\dot{\gamma}_0^2}$$

Second normal-stress coefficient

$$\Psi_2 \equiv \frac{-(\tau_{22} - \tau_{33})}{\dot{\gamma}_0^2}$$

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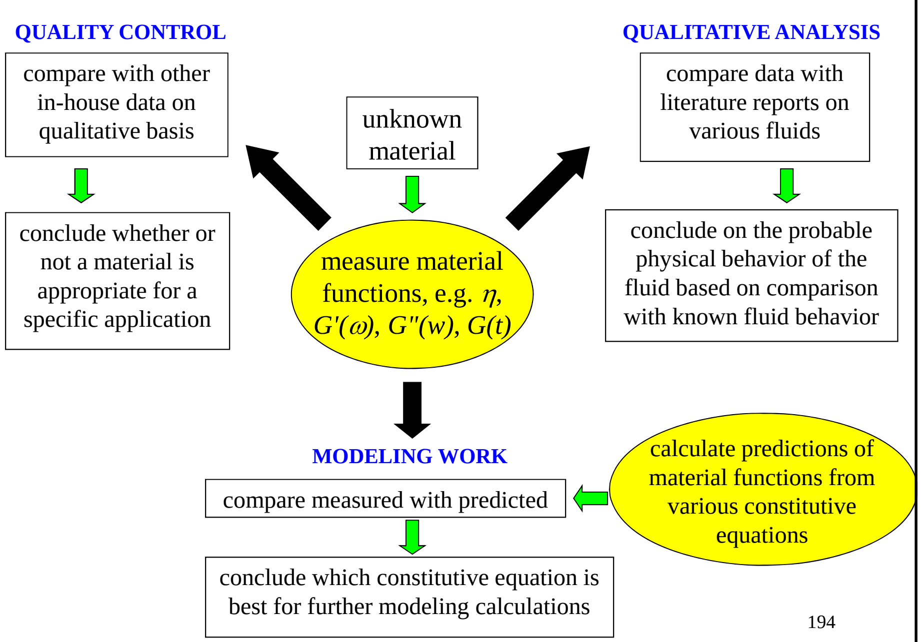
Role of Material Functions in Rheological Analysis



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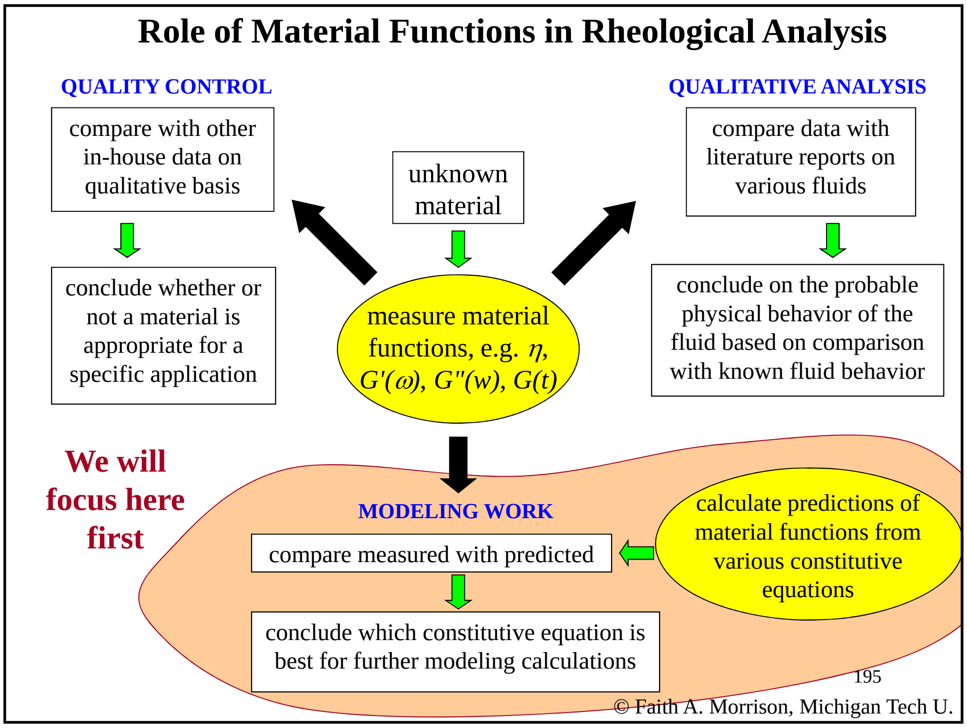

Material function definitions

kinematics

1. Choice of flow (shear or elongation)

$$\underline{v} \equiv \begin{pmatrix} \dot{\varsigma}(t) x_2 \\ 0 \\ 0 \end{pmatrix}_{123} \qquad \underline{v} = \begin{pmatrix} -\frac{1}{2}\dot{\varepsilon}(t)(1+b)x_1 \\ -\frac{1}{2}\dot{\varepsilon}(t)(1-b)x_2 \\ \dot{\varepsilon}(t)x_3 \end{pmatrix}_{123}$$

Elongational flow: b=0, $\dot{\varepsilon}(t) > 0$

Biaxial stretching: b=0, $\dot{\varepsilon}(t) < 0$

Planar elongation: b=1, $\dot{\varepsilon}(t) > 0$

2. Choice of details of $\dot{\varsigma}(t)$ or $\dot{\varepsilon}(t)$.

3. Material functions definitions: will be based on

   τ_{21}, N_1, N_2 in shear or $\tau_{33} - \tau_{11}, \tau_{22} - \tau_{11}$

   in elongational flows.

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(I call these my "recipe cards")

Steady Shear Flow Material Functions

Kinematics:

$$\underline{v} \equiv \begin{pmatrix} \dot{\varsigma}(t) x_2 \\ 0 \\ 0 \end{pmatrix}_{123} \qquad \dot{\varsigma}(t) = \dot{\gamma}_0 = \text{constant}$$

Material Functions:

$$\eta \equiv \frac{-\tau_{21}}{\dot{\gamma}_0}$$

Viscosity

First normal-stress coefficient

$$\Psi_1 \equiv \frac{-(\tau_{11} - \tau_{22})}{\dot{\gamma}_0^2}$$

Second normal-stress coefficient

$$\Psi_2 \equiv \frac{-(\tau_{22} - \tau_{33})}{\dot{\gamma}_0^2}$$

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How do we predict material functions?

ANSWER: From the constitutive equation.

$$\underline{\underline{\tau}} = f(\underline{v})$$

What does the **Newtonian** Fluid model predict in steady shearing?

$$\underline{\underline{\tau}} = -\mu \dot{\underline{\underline{\gamma}}} = -\mu \left[\nabla \underline{v} + (\nabla \underline{v})^T \right]$$

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What does the **Newtonian** Fluid model predict in steady shearing?

$$\underline{\underline{\tau}} = -\mu \dot{\underline{\underline{\gamma}}} = -\mu \left[\nabla \underline{v} + \left(\nabla \underline{v} \right)^T \right]$$

You try.

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*What do we **measure** for these material functions?*

(for polymer solutions, for example)

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Steady shear viscosity and first normal stress coefficient

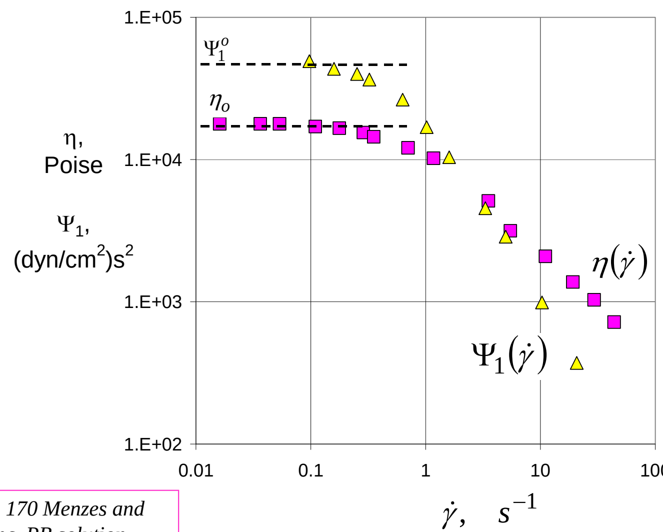

Figure 6.1, p. 170 Menzes and Graessley conc. PB solution

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Steady shear viscosity and first normal stress coefficient

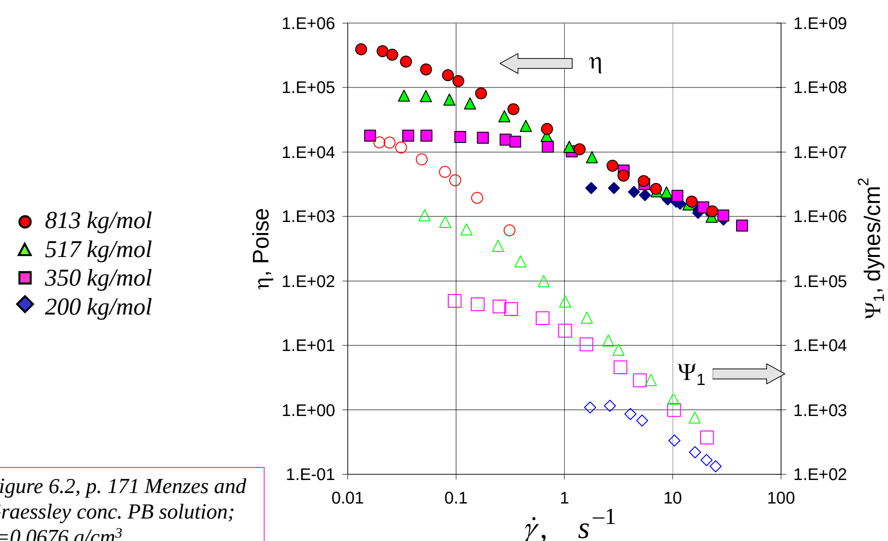

Figure 6.2, p. 171 Menzes and Graessley conc. PB solution; c=0.0676 g/cm³

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Steady shear viscosity for linear and branched PDMS

+ linear 131 kg/mole
▲ branched 156 kg/mole
□ linear 418 kg/mol
◆ branched 428 kg/mol

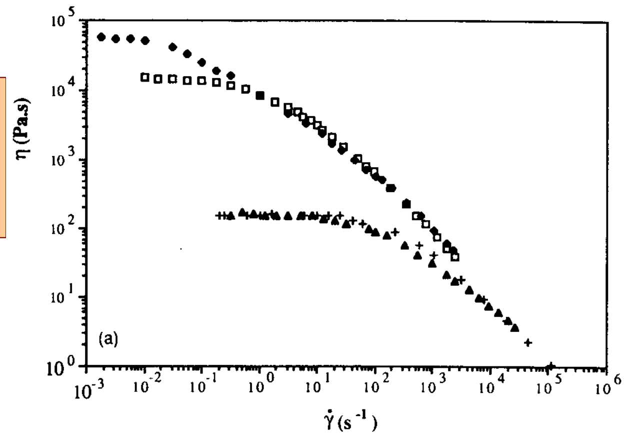

Figure 6.3, p. 172 Piau et al., linear and branched PDMS

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What have material functions taught us so far?

•Newtonian constitutive equation is inadequate

1. Predicts constant shear viscosity (not always true)
2. Predicts no shear normal stresses (these stresses are generated for many fluids)

•Behavior depends on the material (chemical structure, molecular weight, concentration)

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Can we fix the Newtonian Constitutive Equation?

$$\underline{\underline{\tau}} = -\mu\left[\nabla \underline{v} + (\nabla \underline{v})^T\right]$$

Let's replace μ with a function of shear rate because we want to predict a non-constant viscosity in shear

$$\underline{\underline{\tau}} = -M(\dot{\gamma}_0)\left[\nabla \underline{v} + (\nabla \underline{v})^T\right]$$

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What does this model predict for steady shear viscosity?

$$\underline{\underline{\tau}} = -M(\dot{\gamma}_0)\left[\nabla \underline{v} + (\nabla \underline{v})^T\right]$$

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What does this model predict for steady shear viscosity?

$$\underline{\underline{\tau}} = -M(\dot{\gamma}_0)\left[\nabla \underline{v} + (\nabla \underline{v})^T\right]$$

You try.

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What does this model predict for steady shear viscosity?

$$\underline{\underline{\tau}} = -M(\dot{\gamma}_0)\left[\nabla \underline{v} + (\nabla \underline{v})^T\right]$$

Answer: $\eta = M(\dot{\gamma}_0)$

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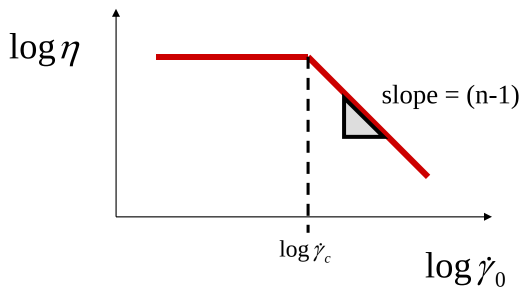
If we choose:

$$M(\dot{\gamma}_0)=\begin{cases} M_0 & \dot{\gamma}_0<\dot{\gamma}_c \\ m\dot{\gamma}_0^{n-1} & \dot{\gamma}_0\geq\dot{\gamma}_c \end{cases}$$



Problem solved!

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But what about the normal stresses?

$$\underline{\underline{\tau}}=-M(\dot{\gamma}_0)\left[\nabla\underline{v}+(\nabla\underline{v})^T\right]$$

$$\nabla\underline{v}=\begin{pmatrix}0&0&0\\ \dot{\gamma}_0&0&0\\ 0&0&0\end{pmatrix}_{123} \qquad \underline{\underline{\dot{\gamma}}}=\begin{pmatrix}0&\dot{\gamma}_0&0\\ \dot{\gamma}_0&0&0\\ 0&0&0\end{pmatrix}_{123}$$

It appears that $\underline{\underline{\tau}}$ should not be simply proportional to $\underline{\underline{\dot{\gamma}}}$

Try something else . . .

$$\underline{\underline{\tau}}=-\mu\underline{\underline{\dot{\gamma}}}+\underline{\underline{I}}\ f(\underline{v})$$

$$\underline{\underline{\tau}}=f(\underline{v})\ \nabla v\cdot(\nabla v)^T$$

$$\underline{\underline{\tau}}=A\left[\nabla v\cdot(\nabla v)^T\right]+B\nabla v+C(\nabla v)^T$$

$$\ldots$$

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But which ones?

To sort out how to fix the Newtonian equation, we need more observations (to give us ideas).

Let's try another material function that's not a steady flow (but stick to shear).

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Start-up of Steady Shear Flow Material Functions

Kinematics:

$$\underline{v} \equiv \begin{pmatrix} \dot{\varsigma}(t) x_2 \\ 0 \\ 0 \end{pmatrix}_{123} \qquad \dot{\varsigma}(t) = \begin{cases} 0 & t < 0 \\ \dot{\gamma}_0 & t \geq 0 \end{cases}$$

Material Functions:

$$\eta^+ \equiv \frac{-\tau_{21}(t)}{\dot{\gamma}_0}$$

Shear stress growth function

First normal-stress growth function

$$\Psi_1^+ \equiv \frac{-(\tau_{11} - \tau_{22})}{\dot{\gamma}_0^2}$$

Second normal-stress growth function

$$\Psi_2^+ \equiv \frac{-(\tau_{22} - \tau_{33})}{\dot{\gamma}_0^2}$$

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What does the **Newtonian** Fluid model predict in start-up of steady shearing?

$$\underline{\underline{\tau}} = -\mu \dot{\underline{\underline{\gamma}}} = -\mu\left[\nabla \underline{v} + \left(\nabla \underline{v}\right)^T\right]$$

Again, since we know $\underline{\boldsymbol{v}}$, we can just plug it in and calculate the stresses.

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What does the **Newtonian** Fluid model predict in start-up of steady shearing?

$$\underline{\underline{\tau}} = -\mu \dot{\underline{\underline{\gamma}}} = -\mu\left[\nabla \underline{v} + \left(\nabla \underline{v}\right)^T\right]$$

You try.

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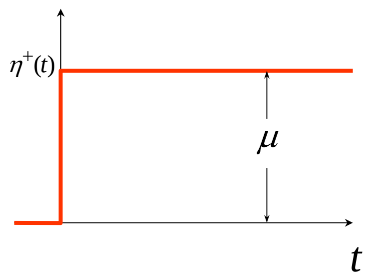
Material functions predicted for *start-up of steady shearing* of a Newtonian fluid

$$\eta^+(t) = \begin{cases} 0 & t<0 \\ \mu & t \ge 0 \end{cases}$$



$$\Psi_1^+ \equiv \frac{-(\tau_{11}-\tau_{22})}{\dot\gamma_0^2} = 0$$

$$\Psi_2^+ \equiv \frac{-(\tau_{22}-\tau_{33})}{\dot\gamma_0^2} = 0$$

Do these predictions match observations?

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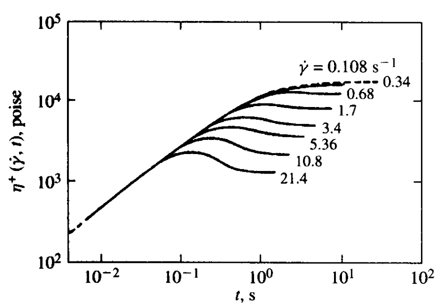
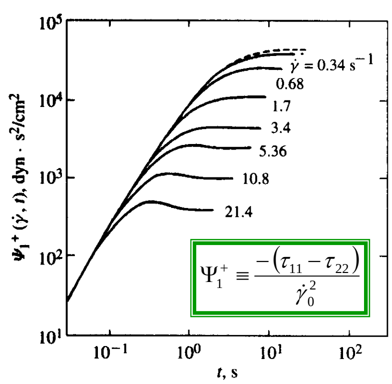
Startup of Steady Shearing

$$\underline{v} \equiv \begin{pmatrix} \dot\varsigma(t) x_2 \\ 0 \\ 0 \end{pmatrix}_{123} \qquad \dot\varsigma(t) = \begin{cases} 0 & t<0 \\ \dot\gamma_0 & t \ge 0 \end{cases}$$



$$\boxed{\eta^+ \equiv \frac{-\tau_{21}(t)}{\dot\gamma_0}}$$



Figures 6.49, 6.50, p. 208
Menezes and Graessley, PB soln

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What about other non-steady flows?

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Cessation of Steady Shear Flow Material Functions

Kinematics:

$$\underline{v} \equiv \begin{pmatrix} \dot{\varsigma}(t) x_2 \\ 0 \\ 0 \end{pmatrix}_{123} \qquad \dot{\varsigma}(t) = \begin{cases} \dot{\gamma}_0 & t < 0 \\ 0 & t \geq 0 \end{cases}$$

Material Functions:

$$\eta^{-} \equiv \frac{-\tau_{21}(t)}{\dot{\gamma}_0}$$

Shear stress decay function

First normal-stress decay function

$$\Psi_1^{-} \equiv \frac{-\left(\tau_{11} - \tau_{22}\right)}{\dot{\gamma}_0^2}$$

Second normal-stress decay function

$$\Psi_2^{-} \equiv \frac{-\left(\tau_{22} - \tau_{33}\right)}{\dot{\gamma}_0^2}$$

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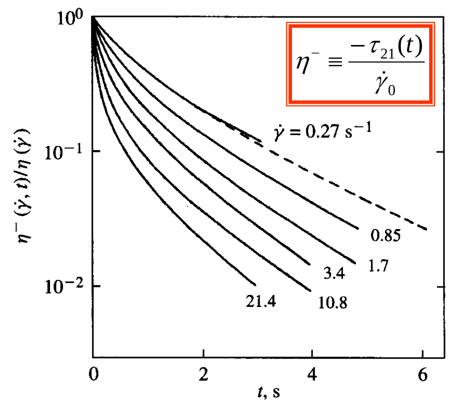
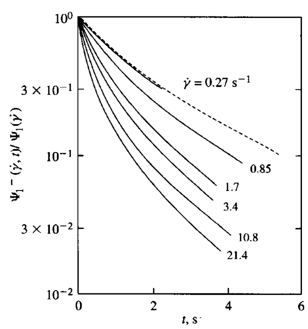
Cessation of Steady Shearing

$$\underline{v} \equiv \begin{pmatrix} \dot{\varsigma}(t) x_2 \\ 0 \\ 0 \end{pmatrix}_{123} \qquad \dot{\varsigma}(t) = \begin{cases} \dot{\gamma}_0 & t < 0 \\ 0 & t \geq 0 \end{cases}$$



$$\eta^- \equiv \frac{-\tau_{21}(t)}{\dot{\gamma}_0}$$



$$\Psi_1^- \equiv \frac{-(\tau_{11} - \tau_{22})}{\dot{\gamma}_0^2}$$

Figures 6.51, 6.52, p. 209 Menezes and Graessley, PB soln

What does the model we guessed at predict for start-up and cessation of shear?

$$\underline{\underline{\tau}} = -M(\dot{\gamma}_0)\left[\nabla \underline{v} + (\nabla \underline{v})^T\right]$$

$$M(\dot{\gamma}_0) = \begin{cases} M_0 & \dot{\gamma}_0 < \dot{\gamma}_c \\ m\dot{\gamma}_0^{n-1} & \dot{\gamma}_0 \geq \dot{\gamma}_c \end{cases}$$

What does the model we guessed at predict for start-up and cessation of shear?

$$\underline{\underline{\tau}} = -M(\dot{\gamma}_0)\left[\nabla \underline{v} + (\nabla \underline{v})^T\right]$$

You try.

$$M(\dot{\gamma}_0) = \begin{cases} M_0 & \dot{\gamma}_0 < \dot{\gamma}_c \\ m\dot{\gamma}_0^{n-1} & \dot{\gamma}_0 \geq \dot{\gamma}_c \end{cases}$$

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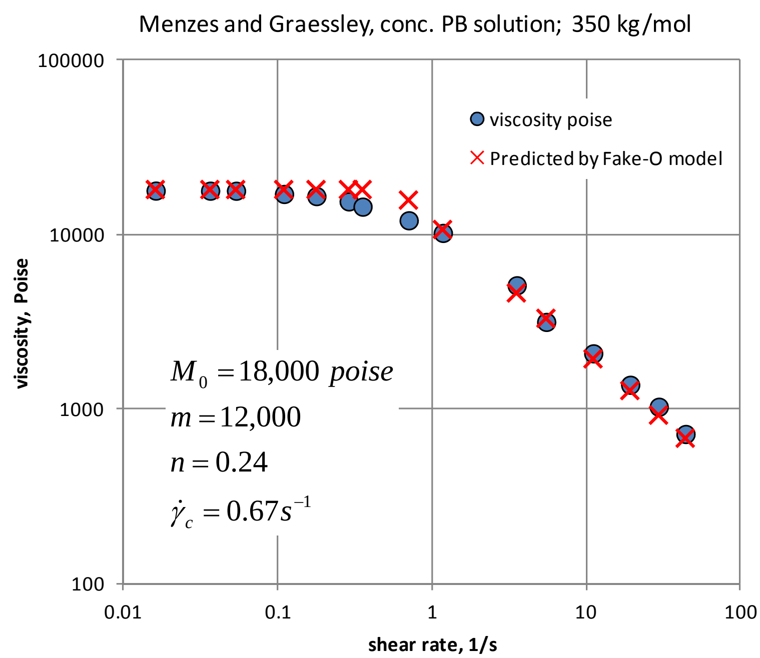

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$$\underline{\underline{\tau}} = -M(\dot{\gamma}_0)\left[\nabla \underline{v} + (\nabla \underline{v})^T\right]$$

$$M(\dot{\gamma}_0) = \begin{cases} M_0 & \dot{\gamma}_0 < \dot{\gamma}_c \\ m\dot{\gamma}_0^{n-1} & \dot{\gamma}_0 \geq \dot{\gamma}_c \end{cases}$$

Observations

•The model predicts an instantaneous stress response, and this is not what is observed for polymers

•The predicted unsteady material functions depend on the shear rate, which is observed for polymers

$$\eta^+ = \eta^+(t, \dot{\gamma}_0)$$ ⟵ **Progress here**

•No normal stresses are predicted

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$$\underline{\underline{\tau}} = -M(\dot{\gamma}_0)\left[\nabla \underline{v} + (\nabla \underline{v})^T\right]$$

$$M(\dot{\gamma}_0) = \begin{cases} M_0 & \dot{\gamma}_0 < \dot{\gamma}_c \\ m\dot{\gamma}_0^{n-1} & \dot{\gamma}_0 \geq \dot{\gamma}_c \end{cases}$$

Observations

•The model predicts an instantaneous stress response, and this is not what is observed for polymers ⟵ **Lacks memory**

•The predicted unsteady material functions depend on the shear rate, which is observed for polymers

$$\eta^+ = \eta^+(t, \dot{\gamma}_0)$$ ⟵ **Progress here**

•No normal stresses are predicted ⟵ **Related to nonlinearities**

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To proceed to better-designed constitutive equations, we need to know more about material behavior, i.e. we need to know more material functions to predict, and we need measurements of these material functions.

•More non-steady material functions (material functions that tell us about memory)

•Material functions that tell us about nonlinearity (strain)

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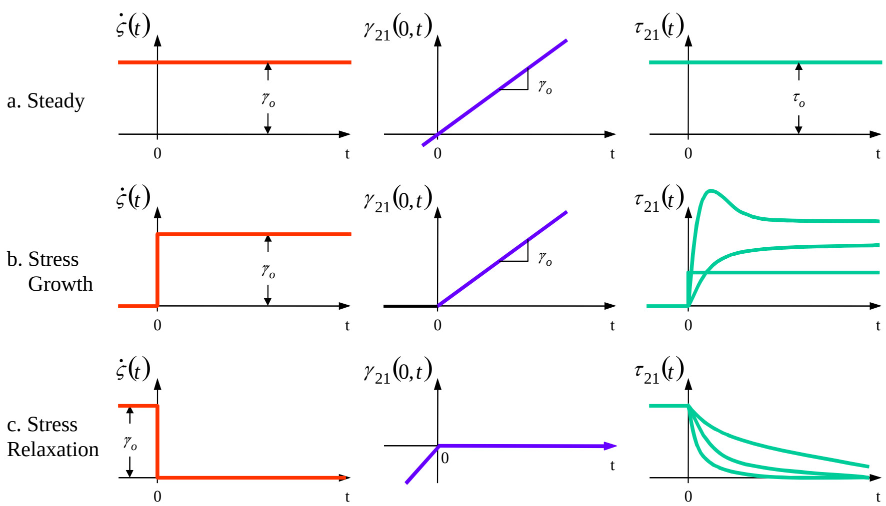
Summary of shear rate kinematics *(part 1)*



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The next three families of material functions incorporate the concept of strain.

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