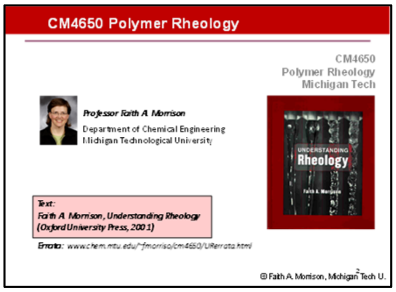
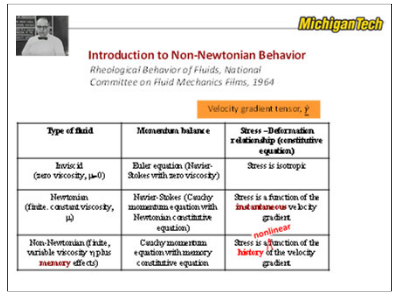
Review:





We:

- Defined rheology
- Contrasted with Newtonian and non-Newtonian behavior
- Saw demonstrations (film)

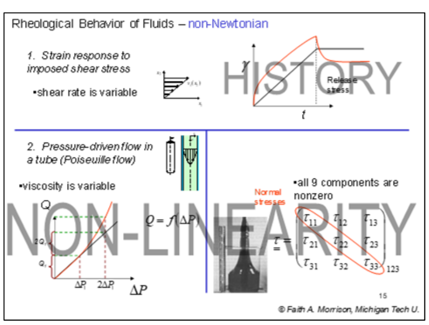
Now…

Key to deformation and flow is the momentum balance:

$$\rho\left(\frac{\partial \underline{v}}{\partial t}+\underline{v}\cdot\nabla\underline{v}\right)=-\nabla p-\nabla\cdot\underline{\underline{\tau}}+\rho\underline{g}$$

Newtonian fluids:
- Linear
- Instantaneous
- $\underline{\underline{\tau}}(t)=-\mu\underline{\underline{\dot{\gamma}}}(t)$

Non-Newtonian fluids:



- Non-linear
- Non-instantaneous
- $\underline{\underline{\tau}}(t)=?$ (missing piece)

2

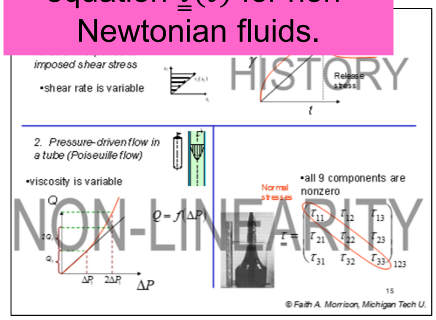
Key to deformation and flow is the momentum balance:

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\right) = -\nabla p - \nabla\cdot\underline{\underline{\tau}} + \rho\underline{g}$$

Newtonian fluids:
- Linear
- Instantaneous

$) = -\mu\underline{\underline{\dot{\gamma}}}(t)$

We're going to be trying to identify the constitutive equation $\underline{\underline{\tau}}(t)$ for non-Newtonian fluids.



- Non-linear
- Non-instantaneous
- $\underline{\underline{\tau}}(t) = ?$ (missing piece)

3

Key to deformation and flow is the momentum balance:

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\right) = -\nabla p - \nabla\cdot\underline{\underline{\tau}} + \rho\underline{g}$$

Newtonian fluids:
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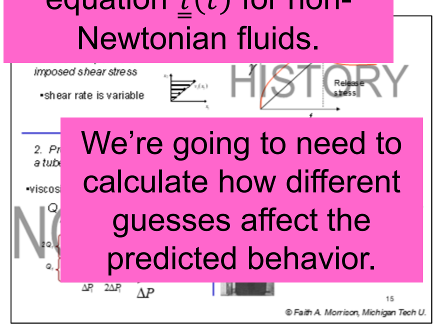


We're going to need to calculate how different guesses affect the predicted behavior.

- Non-linear
- Non-instantaneous
- $\underline{\underline{\tau}}(t) = ?$ (missing piece)

4

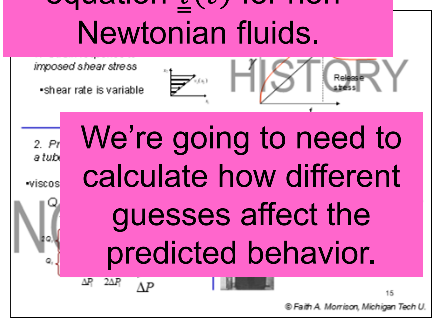
Key to deformation and flow is the momentum balance:

$$\rho\left(\frac{\partial \underline{v}}{\partial t}+\underline{v}\cdot\nabla\underline{v}\right)=-\nabla p-\nabla\cdot\underline{\underline{\tau}}+\rho\underline{g}$$

Newtonian fluids:
- Linear
- Instantaneous

$) = -\mu\dot{\underline{\underline{\gamma}}}(t)$

We're going to be trying to identify the constitutive equation $\underline{\underline{\tau}}(t)$ for non-Newtonian fluids.



We're going to need to calculate how different guesses affect the predicted behavior.

- Non-linear
- Non-instantaneous
- $\underline{\underline{\tau}}(t) = ?$

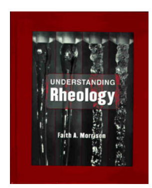
(missing piece)

We need to understand and be able to manipulate this mathematical notation.

Tech U.

Chapter 2: Mathematics Review

CM4650
Polymer Rheology
Michigan Tech



1. Vector review
2. Einstein notation
3. Tensors

Professor Faith A. Morrison
Department of Chemical Engineering
Michigan Technological University

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Chapter 2: Mathematics Review

1. Scalar – *a mathematical entity that has magnitude only*

e.g.: temperature T
speed *v*
time *t*
density r

– *scalars may be constant or may be variable*

Laws of Algebra for Scalars:		
	yes commutative	ab = ba
	yes associative	a(bc) = (ab)c
	yes distributive	a(b+c) = ab+ac

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2. Vector – *a mathematical entity that has magnitude and direction*

e.g.: force on a surface $\underline{f}$
velocity $\underline{v}$

– *vectors may be constant or may be variable*

Definitions

magnitude of a vector – a scalar associated with a vector

$$|\underline{v}| = v \quad |\underline{f}| = f$$

unit vector – a vector of unit length

$$\frac{\underline{v}}{|\underline{v}|} = \hat{v}$$

a unit vector in the direction of $\underline{v}$

This notation $(\underline{v}, \hat{v}, \underline{f})$ is called *Gibbs* notation.

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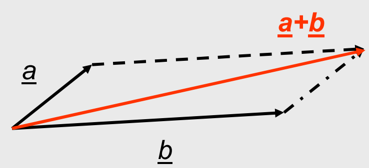
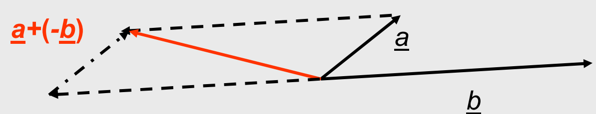
Laws of Algebra for Vectors:

1. Addition



2. Subtraction



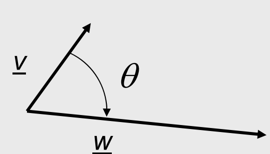
Laws of Algebra for Vectors (continued):

3. Multiplication by scalar $\alpha \underline{v}$

yes commutative	$\alpha \underline{v} = \underline{v} \alpha$
yes associative	$\alpha(\beta \underline{v}) = (\alpha\beta)\underline{v} = \alpha\beta\underline{v}$
yes distributive	$\alpha(\underline{v} + \underline{w}) = \alpha\underline{v} + \alpha\underline{w}$

4. Multiplication of vector by vector
   4a. scalar (dot) (inner) product

$$\underline{v} \cdot \underline{w} = vw\cos\theta$$



Note: we can find magnitude with dot product

$$\underline{v} \cdot \underline{v} = vv\cos 0 = v^2$$

$$v = |\underline{v}| = \sqrt{\underline{v} \cdot \underline{v}}$$

Laws of Algebra for Vectors (continued):

4a. scalar (dot) (inner) product (con't)

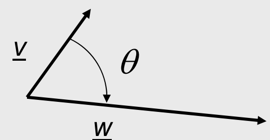
yes commutative $\underline{v}\cdot\underline{w}=\underline{w}\cdot\underline{v}$

NO associative $\cancel{\underline{v}\cdot\underline{w}\cdot\underline{z}}$ no such operation

yes distributive $\underline{z}\cdot\left(\underline{v}+\underline{w}\right)=\underline{z}\cdot\underline{v}+\underline{z}\cdot\underline{w}$

4b. vector (cross) (outer) product

$$\underline{v}\times\underline{w}=vw\sin\theta\,\hat{e}$$



$\hat{e}$ is a unit vector perpendicular to both $\underline{v}$ and $\underline{w}$ following the right-hand rule

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Laws of Algebra for Vectors (continued):

4b. vector (cross) (outer) product (con't)

NO commutative $\underline{v}\times\underline{w}\neq\underline{w}\times\underline{v}$

NO associative $\underline{v}\times\underline{w}\times\underline{z}\neq\left(\underline{v}\times\underline{w}\right)\times\underline{z}\neq\underline{v}\times\left(\underline{w}\times\underline{z}\right)$

yes distributive $\underline{z}\times\left(\underline{v}+\underline{w}\right)=\left(\underline{z}\times\underline{v}\right)+\left(\underline{z}\times\underline{w}\right)$

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Coordinate Systems

•Allow us to make actual calculations with vectors

Rule: any three vectors that are *non-zero* and *linearly independent* (non-coplanar) may form a coordinate basis

Three vectors are linearly dependent if a, b, and g can be found such that:

$$\alpha \underline{a} + \beta \underline{b} + \gamma \underline{c} = \underline{0}$$
$$for \quad \alpha, \beta, \gamma \neq 0$$

If α, β, and γ are found to be zero, the vectors are linearly independent.

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How can we do actual calculations with vectors?

Rule: *any* vector may be expressed as the linear combination of three, non-zero, non-coplanar basis vectors

coefficient of a in the $\hat{e}_y$ direction

any vector

$$\underline{a} = a_x \hat{e}_x + a_y \hat{e}_y + a_z \hat{e}_z = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}_{xyz}$$

$$= a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3$$

$$= \sum_{j=1}^{3} a_j \hat{e}_j$$

This notation is called *matrix* notation.

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Trial calculation: dot product of two vectors

$$
\begin{aligned}
\underline{a}\cdot\underline{b} &= \left(a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3\right)\cdot\left(b_1\hat{e}_1 + b_2\hat{e}_2 + b_3\hat{e}_3\right) \\
&= a_1\hat{e}_1\cdot\left(b_1\hat{e}_1 + b_2\hat{e}_2 + b_3\hat{e}_3\right) + \\
&\quad a_2\hat{e}_2\cdot\left(b_1\hat{e}_1 + b_2\hat{e}_2 + b_3\hat{e}_3\right) + \\
&\quad a_3\hat{e}_3\cdot\left(b_1\hat{e}_1 + b_2\hat{e}_2 + b_3\hat{e}_3\right) \\
&= a_1\hat{e}_1\cdot b_1\hat{e}_1 + a_1\hat{e}_1\cdot b_2\hat{e}_2 + a_1\hat{e}_1\cdot b_3\hat{e}_3 + \\
&\quad a_2\hat{e}_2\cdot b_1\hat{e}_1 + a_2\hat{e}_2\cdot b_2\hat{e}_2 + a_2\hat{e}_2\cdot b_3\hat{e}_3 + \\
&\quad a_3\hat{e}_3\cdot b_1\hat{e}_1 + a_3\hat{e}_3\cdot b_2\hat{e}_2 + a_3\hat{e}_3\cdot b_3\hat{e}_3
\end{aligned}
$$

If we choose the basis to be orthonormal - mutually perpendicular and of unit length - then we can simplify.

If we choose the basis to be orthonormal - mutually perpendicular and of unit length, then we can simplify.

$$
\begin{aligned}
&\hat{e}_1\cdot\hat{e}_1 = 1 \\
&\hat{e}_1\cdot\hat{e}_2 = 0 \\
&\hat{e}_1\cdot\hat{e}_3 = 0 \\
&\cdots
\end{aligned}
$$

$$
\begin{aligned}
\underline{a}\cdot\underline{b} &= a_1\hat{e}_1\cdot b_1\hat{e}_1 + a_1\hat{e}_1\cdot b_2\hat{e}_2 + a_1\hat{e}_1\cdot b_3\hat{e}_3 + \\
&\quad a_2\hat{e}_2\cdot b_1\hat{e}_1 + a_2\hat{e}_2\cdot b_2\hat{e}_2 + a_2\hat{e}_2\cdot b_3\hat{e}_3 + \\
&\quad a_3\hat{e}_3\cdot b_1\hat{e}_1 + a_3\hat{e}_3\cdot b_2\hat{e}_2 + a_3\hat{e}_3\cdot b_3\hat{e}_3 \\
&= a_1b_1 + a_2b_2 + a_3b_3
\end{aligned}
$$

We can generalize this operation with a technique called Einstein notation.

Einstein Notation

a system of notation for vectors and tensors that allows for the calculation of results in Cartesian coordinate systems.

$$\underline{a} = a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3$$
$$= \sum_{j=1}^{3} a_j\hat{e}_j$$
$$= a_j\hat{e}_j = a_m\hat{e}_m$$

This notation called *Einstein* notation.

- the initial choice of subscript letter is *arbitrary*
- the presence of a pair of like subscripts implies a missing summation sign

Einstein Notation (con't)

The result of the dot products of basis vectors can be summarized by the Kronecker delta function

$$\hat{e}_1 \cdot \hat{e}_1 = 1$$
$$\hat{e}_1 \cdot \hat{e}_2 = 0$$
$$\hat{e}_1 \cdot \hat{e}_3 = 0$$
$$\cdots$$

$$\underbrace{\hat{e}_i \cdot \hat{e}_p = \delta_{ip} = \begin{cases} 1 & i = p \\ 0 & i \neq p \end{cases}}_{\text{Kronecker delta}}$$

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Einstein Notation (con't)

To carry out a dot product of two arbitrary vectors . . .

Detailed Notation

$$\begin{aligned}\underline{a}\cdot\underline{b} &= (a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3)\cdot(b_1\hat{e}_1 + b_2\hat{e}_2 + b_3\hat{e}_3)\\ &= a_1\hat{e}_1\cdot b_1\hat{e}_1 + a_1\hat{e}_1\cdot b_2\hat{e}_2 + a_1\hat{e}_1\cdot b_3\hat{e}_3 +\\ &\quad a_2\hat{e}_2\cdot b_1\hat{e}_1 + a_2\hat{e}_2\cdot b_2\hat{e}_2 + a_2\hat{e}_2\cdot b_3\hat{e}_3 +\\ &\quad a_3\hat{e}_3\cdot b_1\hat{e}_1 + a_3\hat{e}_3\cdot b_2\hat{e}_2 + a_3\hat{e}_3\cdot b_3\hat{e}_3\\ &= a_1b_1 + a_2b_2 + a_3b_3\end{aligned}$$

Einstein Notation

$$\begin{aligned}\underline{a}\cdot\underline{b} &= a_j\hat{e}_j\cdot b_m\hat{e}_m\\ &= a_j\delta_{jm}b_m\\ &= a_jb_j\end{aligned}$$

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3. Tensor – *the indeterminate vector product of two (or more) vectors*

e.g.: stress $\underline{\underline{\tau}}$
velocity gradient $\underline{\underline{\dot{\gamma}}}$

– *tensors may be constant or may be variable*

Definitions

dyad or dyadic product – a tensor written explicitly as the indeterminate vector product of two vectors

$\underline{a}\,\underline{d}$ dyad

$\underline{\underline{A}}$ general representation of a tensor

This notation ($\underline{a}\ \underline{d}$, $\underline{\underline{A}}$)is also part of *Gibbs* notation.

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Laws of Algebra for Indeterminate Product of Vectors:

NO commutative $\underline{a}\ \underline{v} \neq \underline{v}\ \underline{a}$

yes associative $\underline{b}\left(\underline{a}\ \underline{v}\right) = \left(\underline{b}\ \underline{a}\right)\underline{v} = \underline{b}\ \underline{a}\ \underline{v}$

yes distributive $\underline{a}\left(\underline{v} + \underline{w}\right) = \underline{a}\ \underline{v} + \underline{a}\ \underline{w}$

How can we represent tensors with respect to a chosen coordinate system?

Just follow the rules of tensor algebra

$$
\begin{aligned}
\underline{a}\ \underline{m} &= \left(a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3\right)\left(m_1\hat{e}_1 + m_2\hat{e}_2 + m_3\hat{e}_3\right) \\
&= a_1\hat{e}_1 m_1\hat{e}_1 + a_1\hat{e}_1 m_2\hat{e}_2 + a_1\hat{e}_1 m_3\hat{e}_3 + \\
&\quad a_2\hat{e}_2 m_1\hat{e}_1 + a_2\hat{e}_2 m_2\hat{e}_2 + a_2\hat{e}_2 m_3\hat{e}_3 + \\
&\quad a_3\hat{e}_3 m_1\hat{e}_1 + a_3\hat{e}_3 m_2\hat{e}_2 + a_3\hat{e}_3 m_3\hat{e}_3 \\
&= \sum_{k=1}^{3}\sum_{w=1}^{3} a_k\hat{e}_k m_w\hat{e}_w \\
&= \sum_{k=1}^{3}\sum_{w=1}^{3} a_k m_w\ \hat{e}_k\hat{e}_w
\end{aligned}
$$

Any tensor may be written as the sum of 9 dyadic products of basis vectors

What about $\underline{\underline{A}}$? Same.

$$\underline{\underline{A}} = \sum_{i=1}^{3} \sum_{j=1}^{3} A_{ij}\, \hat{e}_i \hat{e}_j$$

Einstein notation for tensors: *drop the summation sign; every double index implies a summation sign has been dropped.*

$$\underline{\underline{A}} = A_{ij}\, \hat{e}_i \hat{e}_j = A_{pk}\, \hat{e}_p \hat{e}_k$$

Reminder: the initial choice of subscript letters is arbitrary

How can we use Einstein Notation to calculate dot products between vectors and tensors?

It's the same as between vectors.

$$\underline{a} \cdot \underline{b} =$$
$$\underline{a} \cdot \underline{u}\; \underline{v} =$$
$$\underline{b} \cdot \underline{\underline{A}} =$$

Summary of Einstein Notation

1. Express vectors, tensors, (later, vector operators) in a Cartesian coordinate system as the sums of coefficients multiplying basis vectors - each separate summation has a different index
2. Drop the summation signs
3. Dot products between basis vectors result in the Kronecker delta function because the Cartesian system is orthonormal.

Note:

•In Einstein notation, the presence of repeated indices implies a missing summation sign

•The choice of initial index (i, m, p, etc.) is *arbitrary* - it merely indicates which indices change together

3. Tensor – *(continued)*

Definitions

Scalar product of two tensors

$$\underline{\underline{A}} : \underline{\underline{M}} = A_{ip}\hat{e}_i\hat{e}_p : M_{km}\hat{e}_k\hat{e}_m$$

$$= A_{ip}M_{km}\hat{e}_i\hat{e}_p : \hat{e}_k\hat{e}_m$$

carry out the dot products indicated

$$= A_{ip}M_{km} \left(\hat{e}_p \cdot \hat{e}_k\right)\left(\hat{e}_i \cdot \hat{e}_m\right)$$

$$= A_{ip}M_{km} \quad \delta_{pk}\delta_{im}$$

"p" becomes "k"
"i" becomes "m"

$$= A_{mk}M_{km}$$

But, what is a tensor really?

A tensor is a handy representation of a *Linear Vector Function*

scalar function: $y = f(x) = x^2 + 2x + 3$

a mapping of values of *x* onto values of *y*

vector function: $\underline{w} = f(\underline{v})$

a mapping of vectors of $\underline{v}$ into vectors $\underline{w}$

How do we express a vector function?

What is a linear function?

Linear, in this usage, has a precise, mathematical definition.

Linear functions (scalar and vector) have the following two properties:

$$f(\lambda x) = \lambda f(x)$$
$$f(x + w) = f(x) + f(w)$$

It turns out . . .

Multiplying vectors and tensors is a convenient way of representing the actions of a linear vector function (as we will now show).

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Tensors are *Linear Vector Functions*

Let $f(\underline{a}) = \underline{b}$ be a linear vector function.

We can write $\underline{a}$ in Cartesian coordinates.

$$\underline{a} = a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3$$

$$f(\underline{a}) = f(a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3) = \underline{b}$$

Using the linear properties of *f*, we can distribute the function action:

$$f(\underline{a}) = a_1\underbrace{f(\hat{e}_1)} + a_2\underbrace{f(\hat{e}_2)} + a_3\underbrace{f(\hat{e}_3)} = \underline{b}$$

These results are just vectors, we will name them $\underline{v}$, $\underline{w}$, and $\underline{m}$.

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Tensors are *Linear Vector Functions* (continued)

$$f(\underline{a}) = a_1\underbrace{f(\hat{e}_1)}_{\underline{v}} + a_2\underbrace{f(\hat{e}_2)}_{\underline{w}} + a_3\underbrace{f(\hat{e}_3)}_{\underline{m}} = \underline{b}$$

$$f(\underline{a}) = a_1\underline{v} + a_2\underline{w} + a_3\underline{m} = \underline{b}$$

Now we note that the coefficients a_i may be written as,

$$a_1 = \underline{a}\cdot\hat{e}_1 \qquad a_2 = \underline{a}\cdot\hat{e}_2 \qquad a_3 = \underline{a}\cdot\hat{e}_3$$

Substituting,

$$f(\underline{a}) = \underline{a}\cdot\hat{e}_1\ \underline{v} + \underline{a}\cdot\hat{e}_2\ \underline{w} + \underline{a}\cdot\hat{e}_3\ \underline{m} = \underline{b}$$

The indeterminate vector product has appeared!

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Using the distributive law, we can factor out the dot product with $\underline{a}$:

$$f(\underline{a}) = \underline{a} \cdot \underbrace{\left(\hat{e}_1\, \underline{v} + \hat{e}_2\, \underline{w} + \hat{e}_3\, \underline{m}\right)} = \underline{b}$$

This is just a tensor (the sum of dyadic products of vectors)

$$\left(\hat{e}_1\, \underline{v} + \hat{e}_2\, \underline{w} + \hat{e}_3\, \underline{m}\right) \equiv \underline{\underline{M}}$$

$$\boxed{f(\underline{a}) = \underline{a} \cdot \underline{\underline{M}} = \underline{b}}$$

CONCLUSION: Tensor operations are convenient to use to express linear vector functions.

3. *Tensor* – *(continued)*

More Definitions

Identity Tensor

$$\underline{\underline{I}} = \hat{e}_i \hat{e}_i = \hat{e}_1 \hat{e}_1 + \hat{e}_2 \hat{e}_2 + \hat{e}_3 \hat{e}_3$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123}$$

$$\begin{aligned} \underline{\underline{A}} \cdot \underline{\underline{I}} &= A_{ip} \hat{e}_i \hat{e}_p \cdot \hat{e}_k \hat{e}_k \\ &= A_{ip} \hat{e}_i \delta_{pk} \hat{e}_k \\ &= A_{ik} \hat{e}_i \hat{e}_k \\ &= \underline{\underline{A}} \end{aligned}$$

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3. *Tensor* – *(continued)* More Definitions

Zero Tensor

$$\underline{\underline{0}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{123}$$

Magnitude of a Tensor

$$\left|\underline{\underline{A}}\right| \equiv +\sqrt{\frac{\underline{\underline{A}}:\underline{\underline{A}}}{2}}$$

Note that the book has a typo on this equation: the "2" is under the square root.

$$\underline{\underline{A}}:\underline{\underline{A}} = A_{ip}\hat{e}_i\hat{e}_p : A_{km}\hat{e}_k\hat{e}_m$$
$$= A_{ip}A_{km}\left(\hat{e}_p\cdot\hat{e}_k\right)\left(\hat{e}_i\cdot\hat{e}_m\right)$$
$$= A_{mk}A_{km}$$

products across the diagonal

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3. *Tensor* – *(continued)* More Definitions

Tensor Transpose

$$\underline{\underline{M}}^T = \left(M_{ik}\hat{e}_i\hat{e}_k\right)^T = M_{ik}\hat{e}_k\hat{e}_i$$

Exchange the coefficients across the diagonal

CAUTION:

$$\left(\underline{\underline{A}}\cdot\underline{\underline{C}}\right)^T = \left(A_{ik}\hat{e}_i\hat{e}_k \cdot C_{pj}\hat{e}_p\hat{e}_j\right)^T = \left(A_{ik}C_{pj}\,\hat{e}_i\hat{e}_j\delta_{kp}\right)^T$$
$$= \left(A_{ip}C_{pj}\,\hat{e}_i\hat{e}_j\right)^T$$
$$= A_{ip}C_{pj}\,\hat{e}_j\hat{e}_i$$

It is **not** equal to: $\left(\underline{\underline{A}}\cdot\underline{\underline{C}}\right)^T = \left(A_{ip}C_{pj}\,\hat{e}_i\hat{e}_j\right)^T \neq \cancel{A_{pi}C_{jp}\,\hat{e}_i\hat{e}_j}$

I recommend you always interchange the indices on the **basis vectors** rather than on the coefficients.

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3. *Tensor – (continued)* More Definitions

Symmetric Tensor e.g.

$$\underline{\underline{M}} = \underline{\underline{M}}^T \qquad M_{ik} = M_{ki}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}_{123}$$

Antisymmetric Tensor e.g.

$$\underline{\underline{M}} = -\underline{\underline{M}}^T \qquad M_{ik} = -M_{ki}$$

$$\begin{pmatrix} 0 & -2 & -3 \\ 2 & 0 & -5 \\ 3 & 5 & 0 \end{pmatrix}_{123}$$

3. *Tensor – (continued)* More Definitions

Tensor order

Scalars, vectors, and tensors may all be considered to be tensors (entities that exist independent of coordinate system). They are tensors of different orders, however.

order = degree of complexity

scalars	0th -order tensors	3^0
vectors	1st -order tensors	3^1
tensors	2nd -order tensors	3^2
higher-order tensors	3rd -order tensors	3^3

Number of coefficients needed to express the tensor in 3D space

3. *Tensor – (continued)*

More Definitions

Tensor Invariants

Scalars that are associated with tensors; these are numbers that are independent of coordinate system.

vectors: $|\underline{v}| = v$

The magnitude of a vector is a scalar associated with the vector

It is independent of coordinate system, i.e. it is an invariant.

tensors: $\underline{\underline{A}}$

There are three invariants associated with a second-order tensor.

Tensor Invariants

$$I_{\underline{\underline{A}}} \equiv trace\underline{\underline{A}} = tr\underline{\underline{A}}$$

For the tensor written in Cartesian coordinates:

$$trace\underline{\underline{A}} = A_{pp} = A_{11} + A_{22} + A_{33}$$

$$II_{\underline{\underline{A}}} \equiv trace\left(\underline{\underline{A}} \cdot \underline{\underline{A}}\right) = \underline{\underline{A}} : \underline{\underline{A}} = A_{pk}A_{kp}$$

$$III_{\underline{\underline{A}}} \equiv trace\left(\underline{\underline{A}} \cdot \underline{\underline{A}} \cdot \underline{\underline{A}}\right) = A_{pj}A_{jh}A_{hp}$$

Note: the definitions of invariants written in terms of coefficients are only valid when the tensor is written in Cartesian coordinates.

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4. Differential Operations with Vectors, Tensors

Scalars, vectors, and tensors are differentiated to determine rates of change (with respect to time, position)

- To carryout the differentiation with respect to a *single variable*, differentiate each coefficient individually.
- There is no change in order (vectors remain vectors, scalars remain scalars, etc.

$$\frac{\partial \alpha}{\partial t} \qquad \frac{\partial \underline{w}}{\partial t} = \begin{pmatrix} \frac{\partial w_1}{\partial t} \\ \frac{\partial w_2}{\partial t} \\ \frac{\partial w_3}{\partial t} \end{pmatrix}_{123} \qquad \frac{\partial \underline{\underline{B}}}{\partial t} = \begin{pmatrix} \frac{\partial B_{11}}{\partial t} & \frac{\partial B_{21}}{\partial t} & \frac{\partial B_{31}}{\partial t} \\ \frac{\partial B_{21}}{\partial t} & \frac{\partial B_{22}}{\partial t} & \frac{\partial B_{23}}{\partial t} \\ \frac{\partial B_{31}}{\partial t} & \frac{\partial B_{32}}{\partial t} & \frac{\partial B_{33}}{\partial t} \end{pmatrix}_{123}$$

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4. Differential Operations with Vectors, Tensors (continued)

- To carryout the differentiation with respect to *3D spatial variation*, use the del (nabla) operator.
- This is a vector operator
- Del may be applied in three different ways
- Del may operate on scalars, vectors, or tensors

Del Operator

This is written in Cartesian coordinates

$$\nabla \equiv \hat{e}_1 \frac{\partial}{\partial x_1} + \hat{e}_2 \frac{\partial}{\partial x_2} + \hat{e}_3 \frac{\partial}{\partial x_3} = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix}_{123}$$

$$= \sum_{p=1}^{3} \hat{e}_p \frac{\partial}{\partial x_p} = \underbrace{\hat{e}_p \frac{\partial}{\partial x_p}}_{\text{Einstein notation for del}}$$

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4. *Differential Operations with Vectors, Tensors (continued)*

A. Scalars - gradient

Gibbs notation

$$\nabla\beta \equiv \hat{e}_1 \frac{\partial}{\partial x_1}\beta + \hat{e}_2 \frac{\partial}{\partial x_2}\beta + \hat{e}_3 \frac{\partial}{\partial x_3}\beta = \begin{pmatrix} \frac{\partial \beta}{\partial x_1} \\ \frac{\partial \beta}{\partial x_2} \\ \frac{\partial \beta}{\partial x_3} \end{pmatrix}_{123}$$

This is written in Cartesian coordinates

Gradient of a scalar field $= \hat{e}_p \frac{\partial \beta}{\partial x_p}$

The gradient of a scalar field is a *vector*

The gradient operation captures the total spatial variation of a scalar, vector, or tensor field.

•gradient operation <u>increases</u> the order of the entity operated upon

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4. *Differential Operations with Vectors, Tensors (continued)*

B. Vectors - gradient

$$\nabla \underline{w} \equiv \hat{e}_1 \frac{\partial}{\partial x_1}\underline{w} + \hat{e}_2 \frac{\partial}{\partial x_2}\underline{w} + \hat{e}_3 \frac{\partial}{\partial x_3}\underline{w}$$

$$= \hat{e}_1 \frac{\partial}{\partial x_1}\left(w_1\hat{e}_1 + w_2\hat{e}_2 + w_3\hat{e}_3\right)$$
$$+ \hat{e}_2 \frac{\partial}{\partial x_2}\left(w_1\hat{e}_1 + w_2\hat{e}_2 + w_3\hat{e}_3\right)$$
$$+ \hat{e}_3 \frac{\partial}{\partial x_3}\left(w_1\hat{e}_1 + w_2\hat{e}_2 + w_3\hat{e}_3\right)$$

$$= \hat{e}_1\hat{e}_1 \frac{\partial w_1}{\partial x_1} + \hat{e}_1\hat{e}_2 \frac{\partial w_2}{\partial x_1} + \hat{e}_1\hat{e}_3 \frac{\partial w_3}{\partial x_1} + \hat{e}_2\hat{e}_1 \frac{\partial w_1}{\partial x_2} +$$
$$\hat{e}_2\hat{e}_2 \frac{\partial w_2}{\partial x_2} + \hat{e}_2\hat{e}_3 \frac{\partial w_3}{\partial x_2} + \hat{e}_3\hat{e}_1 \frac{\partial w_1}{\partial x_3} + \hat{e}_3\hat{e}_2 \frac{\partial w_2}{\partial x_3} + \hat{e}_3\hat{e}_3 \frac{\partial w_3}{\partial x_3}$$

The basis vectors can move out of the derivatives because they are <u>constant</u> (do not change with position)

This is all written in Cartesian coordinates (basis vectors are <u>constant</u>)

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4. *Differential Operations with Vectors, Tensors (continued)*

B. Vectors - gradient *(continued)*

constants may appear on either side of the differential operator

Gradient of a vector field

Gibbs notation

$$\nabla \underline{w} \equiv \sum_{j=1}^{3}\sum_{k=1}^{3} \hat{e}_j \hat{e}_k \frac{\partial w_k}{\partial x_j} = \hat{e}_j \hat{e}_k \frac{\partial w_k}{\partial x_j} = \underbrace{\frac{\partial w_k}{\partial x_j} \overbrace{\hat{e}_j \hat{e}_k}}$$

Einstein notation for gradient of a vector

The gradient of a vector field is a *tensor*

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4. *Differential Operations with Vectors, Tensors (continued)*

C. Vectors - divergence

Divergence of a vector field

Gibbs notation

$$\nabla \cdot \underline{w} \equiv \left(\hat{e}_1 \frac{\partial}{\partial x_1} + \hat{e}_2 \frac{\partial}{\partial x_2} + \hat{e}_3 \frac{\partial}{\partial x_3} \right) \cdot w_1 \hat{e}_1 + w_2 \hat{e}_2 + w_3 \hat{e}_3$$

$$= \frac{\partial w_1}{\partial x_1} + \frac{\partial w_2}{\partial x_2} + \frac{\partial w_3}{\partial x_3}$$

$$= \sum_{i=1}^{3} \frac{\partial w_i}{\partial x_i} = \underbrace{\frac{\partial w_i}{\partial x_i}}$$

Einstein notation for divergence of a vector

The Divergence of a vector field is a *scalar*

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4. *Differential Operations with Vectors, Tensors (continued)*

C. Vectors - divergence *(continued)*

This is all written in Cartesian coordinates (basis vectors are constant)

constants may appear on either side of the differential operator

Using Einstein notation

$$\nabla \cdot \underline{w} \equiv \hat{e}_m \frac{\partial}{\partial x_m} \cdot w_j \hat{e}_j = \frac{\partial w_j}{\partial x_m} \hat{e}_m \cdot \hat{e}_j = \frac{\partial w_j}{\partial x_m} \delta_{mj}$$

$$= \frac{\partial w_j}{\partial x_j}$$

•divergence operation decreases the order of the entity operated upon

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4. *Differential Operations with Vectors, Tensors (continued)*

D. Vectors - Laplacian

Using Einstein notation:

Gibbs notation

$$\nabla \cdot \nabla \underline{w} \equiv \hat{e}_m \frac{\partial}{\partial x_m} \cdot \hat{e}_p \frac{\partial}{\partial x_p} w_j \hat{e}_j = \frac{\partial}{\partial x_m} \frac{\partial}{\partial x_p} w_j \left(\hat{e}_m \cdot \hat{e}_p\right) \hat{e}_j$$

$$= \frac{\partial}{\partial x_m} \frac{\partial}{\partial x_p} w_j \left(\delta_{mp}\right) \hat{e}_j$$

$$= \frac{\partial}{\partial x_p} \frac{\partial}{\partial x_p} w_j \, \hat{e}_j$$

The Laplacian of a vector field is a *vector*

Einstein notation

$$= \begin{pmatrix} \frac{\partial^2 w_1}{\partial x_1^2} + \frac{\partial^2 w_1}{\partial x_2^2} + \frac{\partial^2 w_1}{\partial x_3^2} \\ \frac{\partial^2 w_2}{\partial x_1^2} + \frac{\partial^2 w_2}{\partial x_2^2} + \frac{\partial^2 w_2}{\partial x_3^2} \\ \frac{\partial^2 w_3}{\partial x_1^2} + \frac{\partial^2 w_3}{\partial x_2^2} + \frac{\partial^2 w_3}{\partial x_3^2} \end{pmatrix}_{123}$$

Column **vector** notation

•Laplacian operation does not change the order of the entity operated upon

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4. Differential Operations with Vectors, Tensors (continued)

E. Scalar - divergence $\cancel{\nabla \cdot} \alpha$ *(impossible; cannot decrease order of a scalar)*

F. Scalar - Laplacian $\nabla \cdot \nabla \alpha$

G. Tensor - gradient $\nabla \underline{\underline{A}}$

H. Tensor - divergence $\nabla \cdot \underline{\underline{A}}$

I. Tensor - Laplacian $\nabla \cdot \nabla \underline{\underline{A}}$

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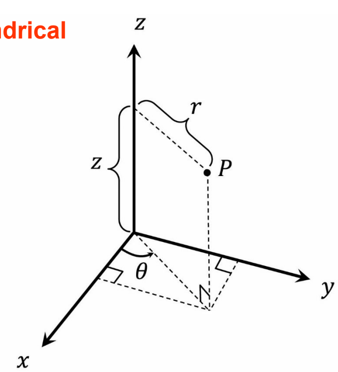
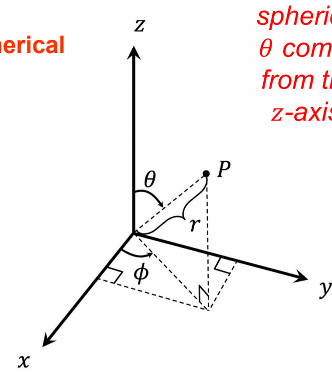
5. Curvilinear Coordinates

Cylindrical



$\boxed{\bar{r}, \theta, z \quad \hat{e}_{\bar{r}}, \hat{e}_\theta, \hat{e}_z}$

Spherical



Note: my spherical θ comes from the z-axis.

$\boxed{r, \theta, \phi \quad \hat{e}_r, \hat{e}_\theta, \hat{e}_\phi}$

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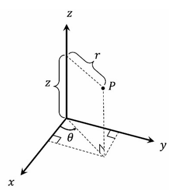
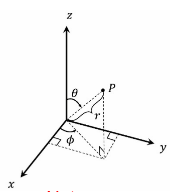


Cylindrical Coordinates

System	Coordinates	Basis vectors
Cylindrical	$r=\sqrt{x^2+y^2}$	$\hat{e}_r=\cos\theta\,\hat{e}_x+\sin\theta\,\hat{e}_y$
Cylindrical	$\theta=\tan^{-1}\left(\frac{y}{x}\right)$	$\hat{e}_\theta=(-\sin\theta)\hat{e}_x+\cos\theta\,\hat{e}_y$
Cylindrical	$z=z$	$\hat{e}_z=\hat{e}_z$
Cylindrical	$x=r\cos\theta$	$\hat{e}_x=\cos\theta\,\hat{e}_r+(-\sin\theta)\hat{e}_\theta$
Cylindrical	$y=r\sin\theta$	$\hat{e}_y=\sin\theta\,\hat{e}_r+\cos\theta\,\hat{e}_\theta$
Cylindrical	$z=z$	$\hat{e}_z=\hat{e}_z$



Note: my spherical θ comes from the z-axis.

Spherical Coordinates

System	Coordinates	Basis vectors
Spherical	$x=r\sin\theta\cos\phi$	$\hat{e}_x=(\sin\theta\cos\phi)\hat{e}_r+(\cos\theta\cos\phi)\hat{e}_\theta+(-\sin\phi)\,\hat{e}_\phi$
Spherical	$y=r\sin\theta\sin\phi$	$\hat{e}_y=(\sin\theta\sin\phi)\hat{e}_r+(\cos\theta\sin\phi)\hat{e}_\theta+\cos\phi\,\hat{e}_\phi$
Spherical	$z=r\cos\theta$	$\hat{e}_z=\cos\theta\ \hat{e}_r+(-\sin\theta)\hat{e}_\theta$
Spherical	$r=\sqrt{x^2+y^2+z^2}$	$\hat{e}_r=(\sin\theta\cos\phi)\hat{e}_x+(\sin\theta\sin\phi)\hat{e}_y+\cos\theta\hat{e}_z$
Spherical	$\theta=\tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$	$\hat{e}_\theta=(\cos\theta\cos\phi)\hat{e}_x+(\cos\theta\sin\phi)\hat{e}_y+(-\sin\theta)\hat{e}_z$
Spherical	$\phi=\tan^{-1}\left(\frac{y}{x}\right)$	$\hat{e}_\phi=(-\sin\phi)\hat{e}_x+\cos\phi\,\hat{e}_y$

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5. Curvilinear Coordinates

Cylindrical	$\bar{r},\theta,z$	$\hat{e}_{\bar{r}},\hat{e}_\theta,\hat{e}_z$
Spherical	r,θ,ϕ	$\hat{e}_r,\hat{e}_\theta,\hat{e}_\phi$

See text figures 2.11 and 2.12

These coordinate systems are ortho-normal, *but they are not constant* (they vary with position).

This causes some non-intuitive effects when derivatives are taken.

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5. Curvilinear Coordinates (continued)

$$\underline{v} = v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_z \hat{e}_z$$

$$\nabla \cdot \underline{v} = \nabla \cdot \left(v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_z \hat{e}_z \right)$$

$$= \underbrace{\left(\frac{\partial}{\partial x} \hat{e}_x + \frac{\partial}{\partial y} \hat{e}_y + \frac{\partial}{\partial z} \hat{e}_z \right)} \cdot \left(v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_z \hat{e}_z \right)$$

First, we need to write this in cylindrical coordinates.

solve for Cartesian basis vectors and substitute above

$$\begin{cases} \hat{e}_r = \cos\theta \, \hat{e}_x + \sin\theta \, \hat{e}_y \\ \hat{e}_\theta = -\sin\theta \, \hat{e}_x + \cos\theta \, \hat{e}_y \\ \hat{e}_z = \hat{e}_z \end{cases}$$

$$\left. \begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned} \right\}$$

substitute above using chain rule (see next slide for details)

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$$\hat{e}_x = \cos\theta \, \hat{e}_r - \sin\theta \, \hat{e}_\theta$$
$$\hat{e}_y = \sin\theta \, \hat{e}_r + \cos\theta \, \hat{e}_\theta$$

$$x = r\cos\theta \qquad r = \sqrt{x^2 + y^2}$$
$$y = r\sin\theta \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$
$$z = z$$

$$\boxed{\nabla \psi = \left(\frac{\partial \psi}{\partial x} \hat{e}_x + \frac{\partial \psi}{\partial y} \hat{e}_y + \frac{\partial \psi}{\partial z} \hat{e}_z \right)}$$

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial \psi}{\partial \theta} \frac{\partial \theta}{\partial x} + \frac{\partial \psi}{\partial z} \frac{\partial z}{\partial x} = \frac{\partial \psi}{\partial r} \cos\theta + \frac{\partial \psi}{\partial \theta} \left(\frac{-\sin\theta}{r} \right)$$

$$\frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial \psi}{\partial \theta} \frac{\partial \theta}{\partial y} + \frac{\partial \psi}{\partial z} \frac{\partial z}{\partial y} = \frac{\partial \psi}{\partial r} \sin\theta + \frac{\partial \psi}{\partial \theta} \left(\frac{\cos\theta}{r} \right)$$

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5. Curvilinear Coordinates (continued)

Result:
$$\nabla = \left(\frac{\partial}{\partial x}\hat{e}_x + \frac{\partial}{\partial y}\hat{e}_y + \frac{\partial}{\partial z}\hat{e}_z\right)$$
$$= \hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_z\frac{\partial}{\partial z}$$

Now, proceed:
$$\nabla\cdot\underline{v} = \left(\hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_z\frac{\partial}{\partial z}\right)\cdot\left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z\right)$$

(We <u>cannot</u> use Einstein notation because these are not Cartesian coordinates)

$$= \hat{e}_r\frac{\partial}{\partial r}\cdot\left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z\right) +$$
$$\hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta}\cdot\left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z\right) +$$
$$\hat{e}_z\frac{\partial}{\partial z}\cdot\left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z\right)$$

Curvilinear coordinate notation

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5. Curvilinear Coordinates (continued)

$$\nabla\cdot\underline{v} = \hat{e}_r\frac{\partial}{\partial r}\cdot\left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z\right) +$$
$$\hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta}\cdot\left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z\right) +$$
$$\hat{e}_z\frac{\partial}{\partial z}\cdot\left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z\right)$$

$$\hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta}\cdot v_r\hat{e}_r = \hat{e}_\theta\cdot\frac{1}{r}\frac{\partial v_r\hat{e}_r}{\partial \theta}$$
$$= \hat{e}_\theta\cdot\frac{1}{r}\left(v_r\frac{\partial \hat{e}_r}{\partial \theta} + \hat{e}_r\frac{\partial v_r}{\partial \theta}\right)$$

$$\frac{\partial \hat{e}_r}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\cos\theta\,\hat{e}_x + \sin\theta\,\hat{e}_y\right)$$
$$= -\sin\theta\,\hat{e}_x + \cos\theta\,\hat{e}_y$$
$$= \hat{e}_\theta$$

Curvilinear coordinate notation

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5. Curvilinear Coordinates (continued)

$$\hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}\cdot v_r \hat{e}_r = \hat{e}_\theta \cdot \frac{1}{r}\frac{\partial v_r \hat{e}_r}{\partial \theta}$$

$$= \hat{e}_\theta \cdot \frac{1}{r}\left(v_r \frac{\partial \hat{e}_r}{\partial \theta} + \hat{e}_r \frac{\partial v_r}{\partial \theta} \right)$$

$$= \hat{e}_\theta \cdot \frac{1}{r}\left(v_r \hat{e}_\theta + \hat{e}_r \frac{\partial v_r}{\partial \theta} \right)$$

$$= \frac{1}{r} v_r$$

This term is not intuitive, and appears because the basis vectors in the curvilinear coordinate systems vary with position.

Curvilinear coordinate notation

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5. Curvilinear Coordinates (continued)

Final result for divergence of a vector in cylindrical coordinates:

$$\nabla \cdot \underline{v} = \hat{e}_r \frac{\partial}{\partial r}\cdot\left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z\right) +$$
$$\hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}\cdot\left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z\right) +$$
$$\hat{e}_z \frac{\partial}{\partial z}\cdot\left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z\right)$$

$$\boxed{\nabla \cdot \underline{v} = \frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z}}$$

Curvilinear coordinate notation

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5. Curvilinear Coordinates (continued)

Curvilinear Coordinates (summary)

- The basis vectors are ortho-normal
- The basis vectors are **non-constant** (vary with position)
- These systems are convenient when the flow system mimics the coordinate surfaces in curvilinear coordinate systems.
- *We **cannot** use Einstein notation* – must use Tables in Appendix C2 (pp464-468).

Curvilinear coordinate notation

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6. Vector and Tensor Theorems and definitions

In Chapter 3 we review Newtonian fluid mechanics using the vector/tensor vocabulary we have learned thus far. We just need a few more theorems to prepare us for those studies. These are presented without proof.

Gauss Divergence Theorem

Gibbs notation

$$\iiint_V \nabla \cdot \underline{b}\, dV = \iint_S \hat{n} \cdot \underline{b}\, dS$$

__outwardly__ directed unit normal

This theorem establishes the utility of the divergence operation. The integral of the divergence of a vector field over a volume is equal to the net outward flow of that property through the bounding surface.

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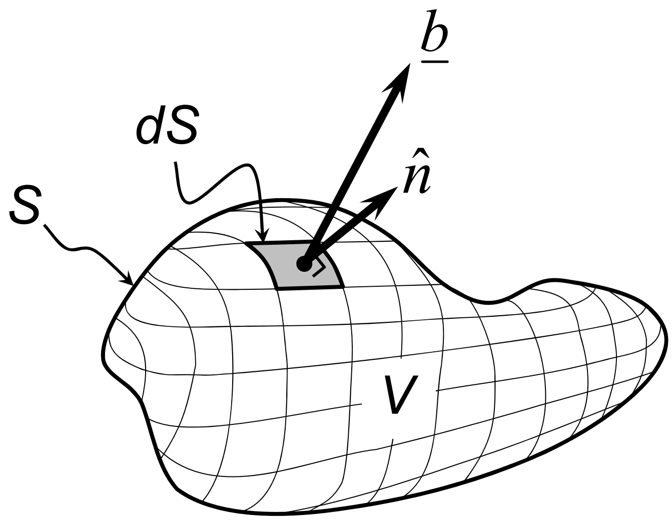

6. Vector and Tensor Theorems *(continued)*

Leibnitz Rule for differentiating integrals

constant limits

$$I = \int_{\alpha}^{\beta} f(x,t)\, dx$$

$$\frac{dI}{dt} = \frac{d}{dt}\int_{\alpha}^{\beta} f(x,t)\, dx$$

$$= \int_{\alpha}^{\beta} \frac{\partial f(x,t)}{\partial t}\, dx$$

one dimension, constant limits

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6. Vector and Tensor Theorems *(continued)*

Leibnitz Rule for differentiating integrals

$$J = \int_{\alpha(t)}^{\beta(t)} f(x,t)\,dx$$

variable limits

$$\frac{dJ}{dt} = \frac{d}{dt}\int_{\alpha(t)}^{\beta(t)} f(x,t)\,dx$$

$$= \int_{\alpha(t)}^{\beta(t)} \frac{\partial f(x,t)}{\partial t}\,dx + \boxed{\frac{d\beta}{dt} f(\beta,t) - \frac{d\alpha}{dt} f(\alpha,t)}$$

one dimension, *variable* limits

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6. Vector and Tensor Theorems *(continued)*

Leibnitz Rule for differentiating integrals

$$J = \iiint_{V(t)} f(x,y,z,t)\,dV$$

$$\frac{dJ}{dt} = \frac{d}{dt}\iiint_{V(t)} f(x,y,z,t)\,dV$$

$$= \iiint_{V(t)} \frac{\partial f(x,y,z,t)}{\partial t}\,dV + \iint_{S(t)} f\left(\underline{v}_{surface} \cdot \hat{n}\right) dS$$

velocity of the surface element *dS*

three dimensions, *variable* limits

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6. Vector and Tensor Theorems *(continued)*

Substantial Derivative

Consider a function $f(x,y,z,t)$

true for any path:

$$df \equiv \left(\frac{\partial f}{\partial x}\right)_{yzt} dx + \left(\frac{\partial f}{\partial y}\right)_{xzt} dy + \left(\frac{\partial f}{\partial z}\right)_{xyt} dz + \left(\frac{\partial f}{\partial t}\right)_{xyz} dt$$

choose special path:

$$\frac{df}{dt} \equiv \left(\frac{\partial f}{\partial x}\right)_{yzt} \frac{dx}{dt} + \left(\frac{\partial f}{\partial y}\right)_{xzt} \frac{dy}{dt} + \left(\frac{\partial f}{\partial z}\right)_{xyt} \frac{dz}{dt} + \left(\frac{\partial f}{\partial t}\right)_{xyz}$$

time rate of change of *f* along a chosen path

x-component of velocity along that path

When the chosen path is the path of a fluid particle, then these are the components of the particle velocities.

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6. Vector and Tensor Theorems *(continued)*

Substantial Derivative

When the chosen path is the path of a fluid particle, then the space derivatives are the components of the particle velocities.

$$\frac{df}{dt} \equiv \left(\frac{\partial f}{\partial x}\right)_{yzt} \frac{dx}{dt} + \left(\frac{\partial f}{\partial y}\right)_{xzt} \frac{dy}{dt} + \left(\frac{\partial f}{\partial z}\right)_{xyt} \frac{dz}{dt} + \left(\frac{\partial f}{\partial t}\right)_{xyz}$$

$$\left(\frac{df}{dt}\right)_{\substack{along\\ a\ particle\\ path}} \equiv \underbrace{\left(\frac{\partial f}{\partial x}\right)_{yzt} v_x + \left(\frac{\partial f}{\partial y}\right)_{xzt} v_y + \left(\frac{\partial f}{\partial z}\right)_{xyt} v_z}_{\underline{v}\cdot\nabla f} + \left(\frac{\partial f}{\partial t}\right)_{xyz}$$

Substantial Derivative

$$\boxed{\left(\frac{df}{dt}\right)_{\substack{along\\ a\ particle\\ path}} \equiv \frac{Df}{Dt} = \frac{\partial f}{\partial t} + \underline{v}\cdot\nabla f}$$

Gibbs notation

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Notation Summary:

Gibbs—no reference to coordinate system $(\underline{a}, \underline{\underline{A}}, \nabla\rho, \nabla\cdot\underline{a})$

Einstein—references to Cartesian coordinate system (ortho-normal, constant) $(a_i\hat{e}_i, A_{pk}\hat{e}_p\hat{e}_k)$

Matrix—uses column or row vectors for vectors and 3×3 matrix of coefficients for tensors

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{123}, \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}_{123}$$

Curvilinear coordinate—references to curvilinear coordinate system (ortho-normal, vary with position)

$$\begin{pmatrix} a_r \\ a_\theta \\ a_z \end{pmatrix}_{r\theta z}, \begin{pmatrix} A_{rr} & A_{r\theta} & A_{rz} \\ A_{\theta r} & A_{\theta\theta} & A_{\theta z} \\ A_{zr} & A_{z\theta} & A_{zz} \end{pmatrix}_{r\theta z}$$

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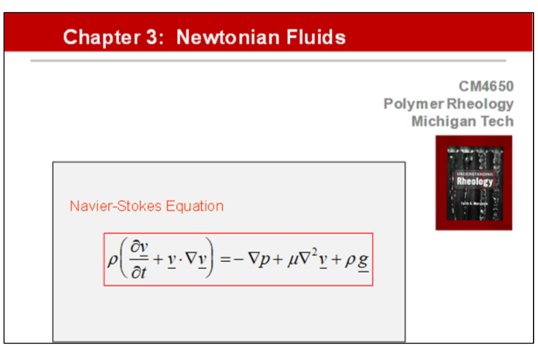
Done with Math background.

Let's use it with Newtonian fluids



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