Chapter 3: Newtonian Fluids

CM4650
Polymer Rheology
Michigan Tech

Navier-Stokes Equation

$$\rho\left(\frac{\partial \underline{v}}{\partial t}+\underline{v}\cdot\nabla\underline{v}\right)=-\nabla p+\mu\nabla^2\underline{v}+\rho\underline{g}$$

1

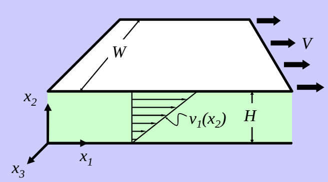
Chapter 3: Newtonian Fluid Mechanics

TWO GOALS

- •Derive governing equations (mass and momentum balances
- •Solve governing equations for velocity and stress fields

QUICK START

First, before we get deep into derivation, let's do a Navier-Stokes problem to get you started in the mechanics of this type of problem solving.

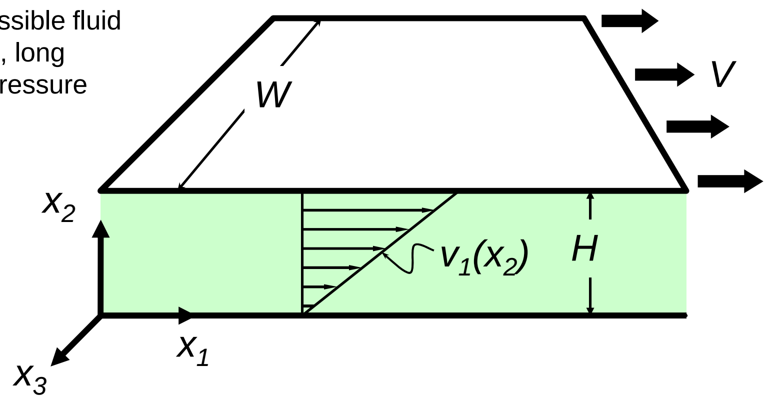


2

EXAMPLE: Drag flow between infinite parallel plates

$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}_{123}$$

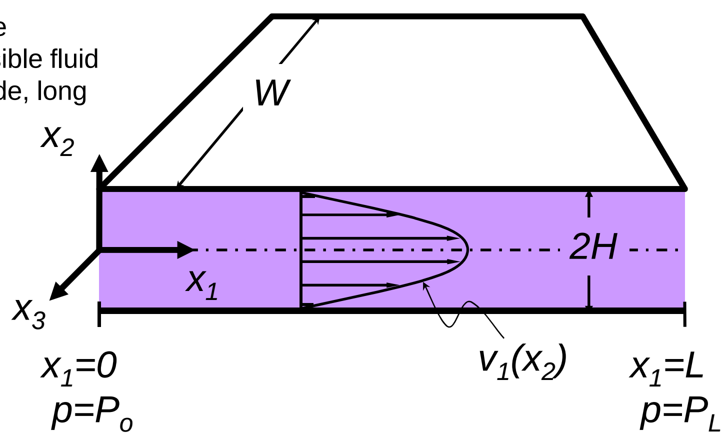
- Newtonian
- steady state
- incompressible fluid
- very wide, long
- uniform pressure



3

EXAMPLE: Poiseuille flow between infinite parallel plates

- Newtonian
- steady state
- Incompressible fluid
- infinitely wide, long



4

Engineering Quantities of Interest

(any flow)

In more complex flows, we can use general expressions that work in all cases.

volumetric flow rate

$$Q = \iint_S \left(\hat{n}\cdot\underline{v}\right)\Big|_{surface}\, dS$$

average velocity

$$\langle v_z \rangle = \frac{\iint_S \left(\hat{n}\cdot\underline{v}\right)\big|_{surface} dS}{\iint_S \, dS}$$

Using the general formulas will help prevent errors.

Here, $\hat{n}$ is the outwardly pointing unit normal of dS; it points in the direction "through" S

5

The stress tensor was invented to make the calculation of fluid stress easier.

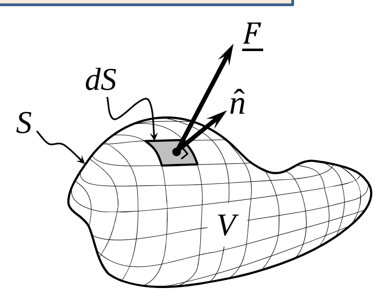
Total stress tensor, $\underline{\underline{\Pi}}$:

$$\underline{\underline{\Pi}} \equiv p\underline{\underline{I}} + \underline{\underline{\tau}}$$

(any flow, small surface)

$$\text{Force on the surface } dS = \hat{n}\cdot\left(-\underline{\underline{\Pi}}\right)\, dS$$

(using the stress convention of *Understanding Rheology*)



Here, $\hat{n}$ is the outwardly pointing unit normal of dS; it points in the direction "through" S

6

To get the total force on the macroscopic surface S, we integrate over the entire surface of interest.

Fluid force on the surface S

$$\underline{F} = \iint_S \left[\hat{n} \cdot \overbrace{\left(-p\underline{\underline{I}} - \underline{\underline{\tau}}\right)}^{-\underline{\underline{\Pi}}}\right]_{surface} dS$$

$$\begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}_{xyz} = \underbrace{\iint_S (\hat{n}_x \quad \hat{n}_y \quad \hat{n}_z) \cdot \begin{pmatrix} -p - \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & -p - \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & -p - \tau_{zz} \end{pmatrix}_{xyz}}_{\hat{n},\ \underline{\underline{\tau}} \text{ and } p \text{ evaluated at the surface } dS} dS$$

(using the stress convention of *Understanding Rheology*)

7

Engineering Quantities of Interest

(any flow)

Using the general formulas will help prevent errors (like forgetting the pressure).

force **on** the surface, S

$$\underline{F} = \iint_S \left[\hat{n} \cdot \left(-p\underline{\underline{I}} - \underline{\underline{\tau}}\right)\right]_{surface} dS$$

z-component of force on the surface, S

$$F_z = \hat{e}_z \cdot \iint_S \left[\hat{n} \cdot \left(-p\underline{\underline{I}} - \underline{\underline{\tau}}\right)\right]_{surface} dS$$

(using the stress convention of *Understanding Rheology*)

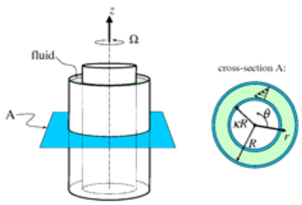
8

Engineering Quantities of Interest

(any flow)

Total Fluid Torque on a surface, S

$$\underline{T} = \iint_S \left[\underline{R} \times (\hat{n} \cdot (-\underline{\underline{\Pi}})) \right]_{surface} dS$$



$\underline{R}$ is the vector from the axis of rotation to dS

(using the stress convention of *Understanding Rheology*)

Common surface shapes:

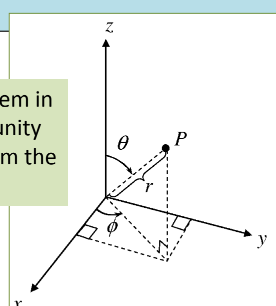
rectangular : $dS = dxdy$

circular top : $dS = r\, dr d\theta$

surface of cylinder : $dS = R d\theta dz$

sphere : $dS = (R d\theta)(r \sin\theta\, d\phi) = R^2 \sin\theta\, d\theta d\phi$

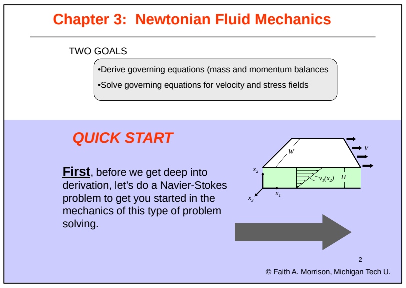
Note: spherical coordinate system in use by fluid mechanics community uses $0 < \theta < \pi$ as the angle from the z=axis to the point.



For more areas, see Exam 1 formula handout:

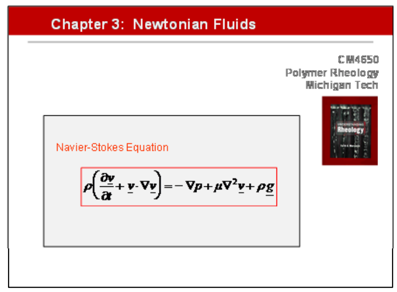
pages.mtu.edu/~fmorriso/cm4650/formula_sheet_for_exam1_2018.pdf

Review:



We got a *Quick Start* with Newtonian problem solving…

Now…
Back to exploring the origin of the equations (so we can adapt to non-Newtonian)



Chapter 3: Newtonian Fluid Mechanics

TWO GOALS

- •Derive governing equations (mass and momentum balances
- •Solve governing equations for velocity and stress fields

Mass Balance

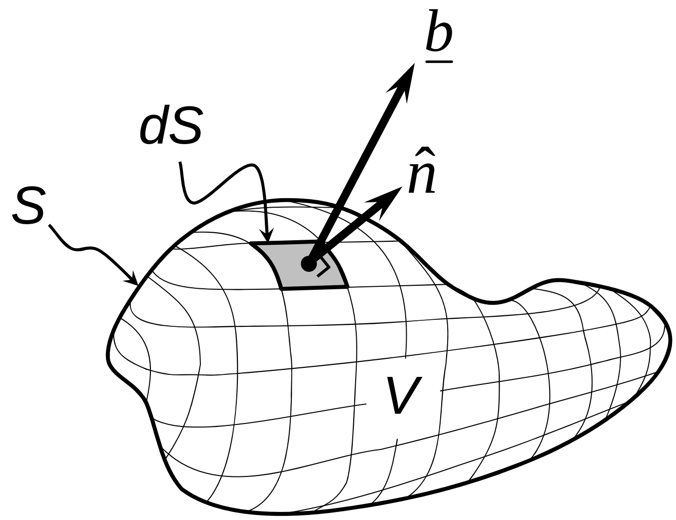
Consider an arbitrary control volume *V* enclosed by a surface *S*

$$\begin{pmatrix} \text{rate of increase} \\ \text{of mass in } CV \end{pmatrix} = \begin{pmatrix} \text{net flux of} \\ \text{mass into } CV \end{pmatrix}$$

12

Mathematics Review

Polymer Rheology



13

Chapter 3: Newtonian Fluid Mechanics

Polymer Rheology

Mass Balance *(continued)*

Consider an arbitrary volume V enclosed by a surface S

$$\begin{pmatrix} \text{rate of increase} \\ \text{of mass in } V \end{pmatrix} = \frac{d}{dt}\left(\iiint_V \rho \, dV \right)$$

outwardly pointing unit normal

$$\begin{pmatrix} \text{net flux of} \\ \text{mass into } V \\ \text{through surface } S \end{pmatrix} = -\iint_S \rho \, \hat{n} \cdot \underline{v} \, dS$$

14

Chapter 3: Newtonian Fluid Mechanics

Polymer Rheology

Mass Balance (continued)

Leibnitz rule

$$\frac{d}{dt}\left(\iiint_V \rho \, dV\right) = -\iint_S \rho \, \hat{n}\cdot\underline{v}\, dS$$

$$\iiint_V \frac{\partial\rho}{\partial t} dV = -\iint_S \hat{n}\cdot\left(\rho\underline{v}\right) dS$$

Gauss Divergence Theorem

$$= -\iiint_V \nabla\cdot\left(\rho\underline{v}\right) dV$$

$$\iiint_V \left(\frac{\partial\rho}{\partial t} + \nabla\cdot\left(\rho\underline{v}\right)\right) dV = 0$$

15

Chapter 3: Newtonian Fluid Mechanics

Polymer Rheology

Mass Balance (continued)

$$\iiint_V \left(\frac{\partial\rho}{\partial t} + \nabla\cdot\left(\rho\underline{v}\right)\right) dV = 0$$

Since V is *arbitrary*,

Continuity equation:
microscopic mass balance

$$\boxed{\frac{\partial\rho}{\partial t} + \nabla\cdot\left(\rho\underline{v}\right) = 0}$$

16

Mass Balance (continued)

Continuity equation (general fluids)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho(\nabla \cdot \underline{v}) + \underline{v} \cdot \nabla \rho = 0$$

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \underline{v}) = 0$$

For ρ=constant (incompressible fluids):

$$\nabla \cdot \underline{v} = 0$$

Momentum Balance

Consider an arbitrary control volume *V* enclosed by a surface *S*

Momentum is conserved.

$$\begin{pmatrix} rate\ of\ increase \\ of\ momentum\ in\ CV \end{pmatrix} = \begin{pmatrix} net\ \ flux\ of \\ momentum\ into\ \ CV \end{pmatrix} + \begin{pmatrix} sum\ of \\ forces\ on\ \ CV \end{pmatrix}$$

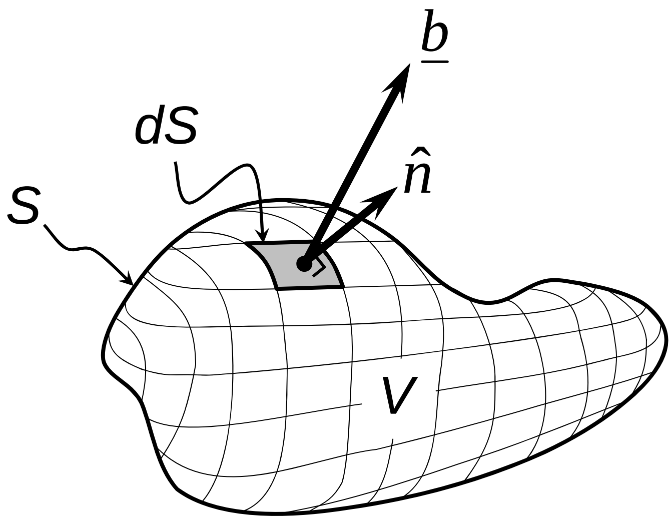
resembles the rate term in the mass balance

resembles the flux term in the mass balance

Forces: body (gravity) molecular forces

Momentum Balance

Polymer Rheology



19

Momentum Balance *(continued)*

Polymer Rheology

$$\begin{pmatrix} \textit{rate of increase} \\ \textit{of momentum in } V \end{pmatrix} = \frac{d}{dt}\left(\iiint_V \rho \underline{v}\, dV \right)$$

$$= \iiint_V \frac{\partial}{\partial t}\left(\rho \underline{v}\right) dV$$

Leibnitz rule

$$\begin{pmatrix} \textit{net flux of} \\ \textit{momentum into } V \end{pmatrix} = -\iint_S \hat{n}\cdot\left(\rho \underline{v}\,\underline{v}\right) dS$$

$$= -\iiint_V \nabla\cdot\left(\rho \underline{v}\,\underline{v}\right) dV$$

Gauss Divergence Theorem

20

Momentum Balance *(continued)*

Polymer Rheology

Forces on *V*

Body Forces (non-contact)

$$\begin{pmatrix} force\ on\ V \\ due\ to\ \underline{g} \end{pmatrix} = \iiint_V \rho \underline{g}\ dV$$

21

Chapter 3: Newtonian Fluid Mechanics

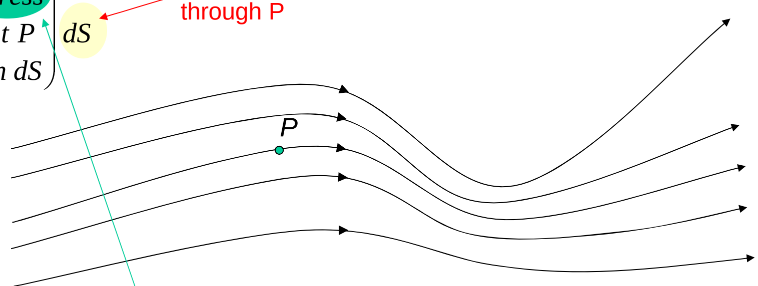
Polymer Rheology

Molecular Forces (contact) – <u>this is the tough one</u>

$$\underline{f} = \begin{pmatrix} stress \\ at\ P \\ on\ dS \end{pmatrix} dS$$

choose a surface through P

the force on that surface

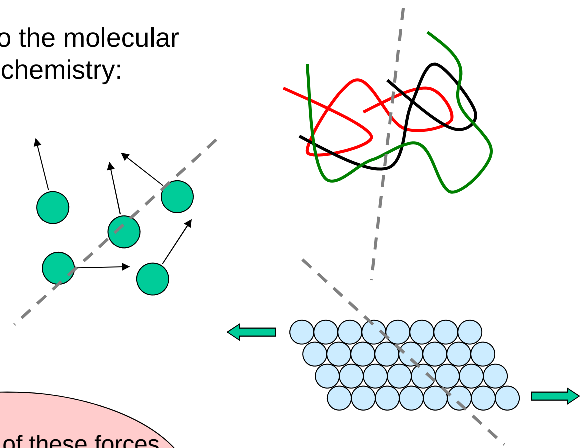


We need an expression for the state of **stress** at an arbitrary point P in a flow.

22

Molecular Forces *(continued)*

Think back to the molecular picture from chemistry:



The specifics of these forces, connections, and interactions must be captured by the molecular forces term that we seek.

23

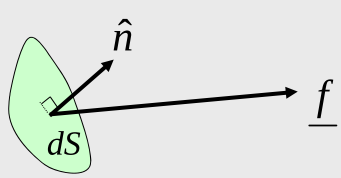
Molecular Forces *(continued)*

•We will concentrate on **expressing the molecular forces** mathematically;

•We leave to later the task of relating the resulting mathematical expression to experimental observations.

First, choose a surface:

•arbitrary shape
•small

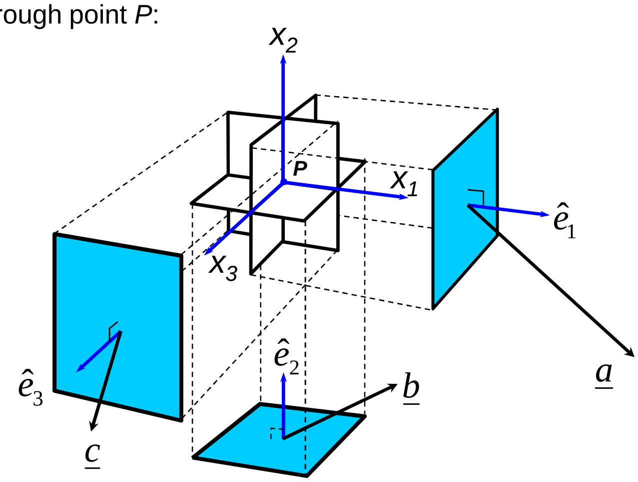


$$\begin{pmatrix} stress \\ at\ P \\ on\ dS \end{pmatrix} dS = \underline{f}$$

What is *f* ?

24

Consider the forces on three mutually perpendicular surfaces through point *P*:



25

Molecular Forces *(continued)*

$\underline{a}$ is stress on a "1" surface at P

$\underbrace{\text{"1" surface}}_{\text{a surface with unit normal } \hat{e}_1}$

a surface with unit normal $\hat{e}_1$

$\underline{b}$ is stress on a "2" surface at P

$\underline{c}$ is stress on a "3" surface at P

We can write these vectors in a Cartesian coordinate system:

$$\underline{a} = a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3$$
$$= \Pi_{11}\hat{e}_1 + \Pi_{12}\hat{e}_2 + \Pi_{13}\hat{e}_3$$

stress on a "1" surface in the 1-direction

26

Molecular Forces *(continued)*

$$\underline{a} = a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3$$
$$= \Pi_{11}\hat{e}_1 + \Pi_{12}\hat{e}_2 + \Pi_{13}\hat{e}_3$$
$$\underline{b} = b_1\hat{e}_1 + b_2\hat{e}_2 + b_3\hat{e}_3$$
$$= \Pi_{21}\hat{e}_1 + \Pi_{22}\hat{e}_2 + \Pi_{23}\hat{e}_3$$
$$\underline{c} = c_1\hat{e}_1 + c_2\hat{e}_2 + c_3\hat{e}_3$$
$$= \Pi_{31}\hat{e}_1 + \Pi_{32}\hat{e}_2 + \Pi_{33}\hat{e}_3$$

$\underline{a}$	is stress on a "1" surface at P
$\underline{b}$	is stress on a "2" surface at P
$\underline{c}$	is stress on a "3" surface at P

So far, this is nomenclature; next we relate these expressions to force on an arbitrary surface.

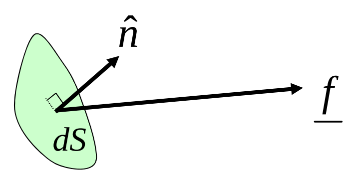
Π_{pk}

Stress on a "*p*" surface in the *k*-direction

27

Molecular Forces *(continued)*

How can we write $\underline{f}$ (the force on an arbitrary surface dS) in terms of the P_{pk}?



$$\underline{f} = f_1\hat{e}_1 + f_2\hat{e}_2 + f_3\hat{e}_3$$

f_1 is force on dS in 1-direction

f_2 is force on dS in 2-direction

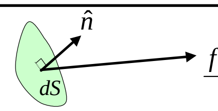
f_3 is force on dS in 3-direction

There are <u>three</u> P_{pk} that relate to forces in the 1-direction:

$$\Pi_{11}, \Pi_{21}, \Pi_{31}$$

28

Molecular Forces *(continued)*



How can we write $\underline{f}$ (the force on an arbitrary surface dS) in terms of the quantities Π_{pk}?

$$\underline{f} = f_1\hat{e}_1 + f_2\hat{e}_2 + f_3\hat{e}_3$$

f_1 , the force on dS in 1-direction, can be broken into <u>three</u> parts associated with the three stress components:

$$\Pi_{11}, \Pi_{21}, \Pi_{31}$$

first part:

$$(\Pi_{11})\underbrace{\begin{pmatrix} \text{projection of} \\ dA \text{ onto the} \\ 1-\text{surface} \end{pmatrix}}_{\hat{n}\cdot\hat{e}_1\, dS} = \Pi_{11}\hat{n}\cdot\hat{e}_1\, dS$$

$$\left(\frac{force}{area}\right) \cdot (area)$$

29

Molecular Forces *(continued)*

f_1 , the force on dS in 1-direction, is composed of THREE parts:

first part: $$(\Pi_{11})\begin{pmatrix} \text{projection of} \\ dA \text{ onto the} \\ 1-\text{surface} \end{pmatrix} = \Pi_{11}\hat{n}\cdot\hat{e}_1\, dS$$

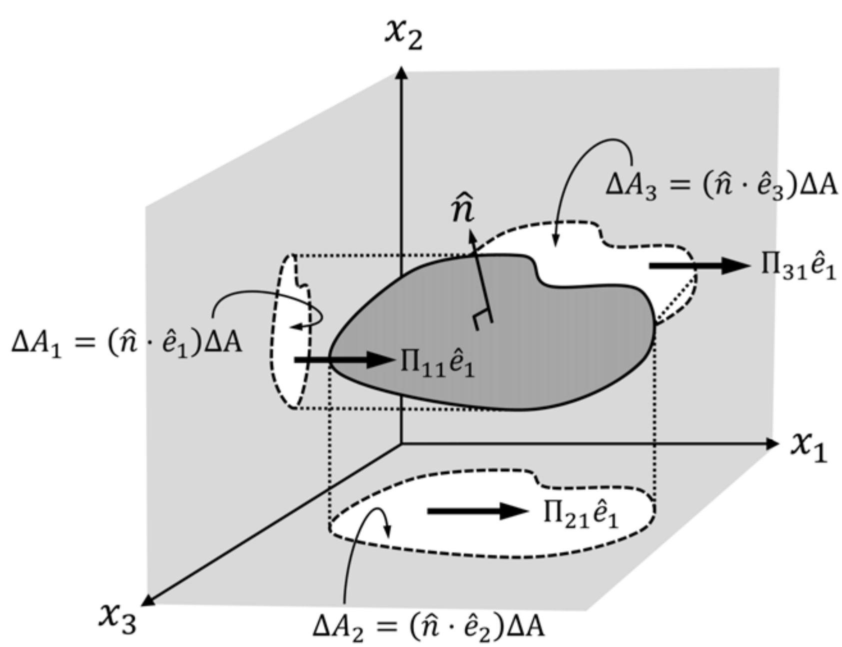
second part: $$(\Pi_{21})\begin{pmatrix} \text{projection of} \\ dA \text{ onto the} \\ 2-\text{surface} \end{pmatrix} = \Pi_{21}\hat{n}\cdot\hat{e}_2\, dS$$

stress on a 2 -surface in the 1-direction

third part: $$(\Pi_{31})\begin{pmatrix} \text{projection of} \\ dA \text{ onto the} \\ 3-\text{surface} \end{pmatrix} = \Pi_{31}\hat{n}\cdot\hat{e}_3\, dS$$

the sum of these three = f_1

30



Molecular Forces *(continued)*

f_1, the force in the 1-direction on an arbitrary surface *dS* is composed of THREE parts.

$$f_1 = \Pi_{11}\hat{n}\cdot\hat{e}_1\, dS + \underbrace{\Pi_{21}}_{\text{stress}}\underbrace{\hat{n}\cdot\hat{e}_2}_{\text{appropriate area}}\, dS + \Pi_{31}\hat{n}\cdot\hat{e}_3\, dS$$

Using the distributive law:

$$\boxed{f_1 = \hat{n}\cdot\left(\Pi_{11}\hat{e}_1 + \Pi_{21}\hat{e}_2 + \Pi_{31}\hat{e}_3\right) dS}$$

Force in the 1-direction on an arbitrary surface *dS*

32

Molecular Forces *(continued)*

The same logic applies in the 2-direction and the 3-direction

$$\boxed{\begin{aligned} f_1 &= \hat{n}\cdot\left(\Pi_{11}\hat{e}_1+\Pi_{21}\hat{e}_2+\Pi_{31}\hat{e}_3\right)dS \\ f_2 &= \hat{n}\cdot\left(\Pi_{12}\hat{e}_1+\Pi_{22}\hat{e}_2+\Pi_{32}\hat{e}_3\right)dS \\ f_3 &= \hat{n}\cdot\left(\Pi_{13}\hat{e}_1+\Pi_{23}\hat{e}_2+\Pi_{33}\hat{e}_3\right)dS \end{aligned}}$$

Assembling the force vector:

$$\begin{aligned} \underline{f} &= f_1\hat{e}_1 + f_2\hat{e}_2 + f_3\hat{e}_3 \\ &= dS\ \hat{n}\cdot\left(\Pi_{11}\hat{e}_1+\Pi_{21}\hat{e}_2+\Pi_{31}\hat{e}_3\right)\hat{e}_1 \\ &\quad + dS\ \hat{n}\cdot\left(\Pi_{12}\hat{e}_1+\Pi_{22}\hat{e}_2+\Pi_{32}\hat{e}_3\right)\hat{e}_2 \\ &\quad + dS\ \hat{n}\cdot\left(\Pi_{13}\hat{e}_1+\Pi_{23}\hat{e}_2+\Pi_{33}\hat{e}_3\right)\hat{e}_3 \end{aligned}$$

33

Molecular Forces *(continued)*

Assembling the force vector:

$$\underline{f} = f_1\hat{e}_1 + f_2\hat{e}_2 + f_3\hat{e}_3$$

$$\begin{aligned} &= dS\ \hat{n}\cdot\left(\Pi_{11}\hat{e}_1+\Pi_{21}\hat{e}_2+\Pi_{31}\hat{e}_3\right)\hat{e}_1 \\ &\quad + dS\ \hat{n}\cdot\left(\Pi_{12}\hat{e}_1+\Pi_{22}\hat{e}_2+\Pi_{32}\hat{e}_3\right)\hat{e}_2 \\ &\quad + dS\ \hat{n}\cdot\left(\Pi_{13}\hat{e}_1+\Pi_{23}\hat{e}_2+\Pi_{33}\hat{e}_3\right)\hat{e}_3 \end{aligned}$$

$$= dS\ \hat{n}\cdot\underbrace{\begin{bmatrix} \Pi_{11}\hat{e}_1\hat{e}_1+\Pi_{21}\hat{e}_2\hat{e}_1+\Pi_{31}\hat{e}_3\hat{e}_1 \\ +\Pi_{12}\hat{e}_1\hat{e}_2+\Pi_{22}\hat{e}_2\hat{e}_2+\Pi_{32}\hat{e}_3\hat{e}_2 \\ +\Pi_{13}\hat{e}_1\hat{e}_3+\Pi_{23}\hat{e}_2\hat{e}_3+\Pi_{33}\hat{e}_3\hat{e}_3 \end{bmatrix}}$$

linear combination of dyadic products = **tensor**

34

Molecular Forces *(continued)*

Assembling the force vector:

$$\underline{f} = dS\ \hat{n} \cdot \left[\Pi_{11}\hat{e}_1\hat{e}_1 + \Pi_{21}\hat{e}_2\hat{e}_1 + \Pi_{31}\hat{e}_3\hat{e}_1 + \Pi_{12}\hat{e}_1\hat{e}_2 + \Pi_{22}\hat{e}_2\hat{e}_2 + \Pi_{32}\hat{e}_3\hat{e}_2 + \Pi_{13}\hat{e}_1\hat{e}_3 + \Pi_{23}\hat{e}_2\hat{e}_3 + \Pi_{33}\hat{e}_3\hat{e}_3 \right]$$

$$= dS\ \hat{n} \cdot \sum_{p=1}^{3}\sum_{m=1}^{3} \Pi_{pm}\hat{e}_p\hat{e}_m$$

$$= dS\ \hat{n} \cdot \Pi_{pm}\hat{e}_p\hat{e}_m$$

$$\boxed{\underline{f} = dS\ \hat{n} \cdot \underline{\underline{\Pi}}}$$

Total stress tensor (molecular stresses)

35

Momentum Balance *(continued)* Polymer Rheology

$$\begin{pmatrix} \textit{rate of increase} \\ \textit{of momentum in } V \end{pmatrix} = \begin{pmatrix} \textit{net flux of} \\ \textit{momentum into } V \end{pmatrix} + \begin{pmatrix} \textit{sum of} \\ \textit{forces on } V \end{pmatrix}$$

$$\iiint_V \frac{\partial}{\partial t}(\rho \underline{v})\, dV = -\iiint_V \nabla \cdot (\rho \underline{v}\,\underline{v})\, dV + \iiint_V \rho \underline{g}\ dV + \begin{matrix} \textit{molecular} \\ \textit{forces} \end{matrix}$$

$$\begin{aligned} \begin{matrix} \textit{molecular} \\ \textit{forces} \end{matrix} &= \iint_S \begin{pmatrix} \textit{molecular} \\ \textit{forces on} \\ dS \end{pmatrix} \\ &= \iint_S \hat{n} \cdot \left(-\underline{\underline{\Pi}}\right) dS \\ &= \iiint_V \nabla \cdot \left(-\underline{\underline{\Pi}}\right) dV \end{aligned}$$

We use a stress sign convention that requires a negative sign here.

Gauss Divergence Theorem

36

Momentum Balance *(continued)*

Polymer Rheology

$$\begin{pmatrix}\textit{rate of increase}\\ \textit{of momentum in } V\end{pmatrix} = \begin{pmatrix}\textit{net flux of}\\ \textit{momentum into } V\end{pmatrix} + \begin{pmatrix}\textit{sum of}\\ \textit{forces on } V\end{pmatrix}$$

$$\iiint_V \frac{\partial}{\partial t}(\rho \underline{v})\, dV = -\iiint_V \nabla\cdot(\rho \underline{v}\,\underline{v})\, dV + \iiint_V \rho \underline{g}\, dV + \begin{matrix}\textit{molecular}\\ \textit{forces}\end{matrix}$$

$$\begin{matrix}\textit{molecular}\\ \textit{forces}\end{matrix} = \iint_S \begin{pmatrix}\textit{molecular}\\ \textit{forces on}\\ dS\end{pmatrix}$$

$$= \iint_S \hat{n}\cdot\left(-\underline{\underline{\Pi}}\right) dS$$

$$= \iiint_V \nabla\cdot\left(-\underline{\underline{\Pi}}\right) dV$$

UR/Bird choice: positive compression (pressure is positive)

Gauss Divergence Theorem

37

Momentum Balance *(continued)*

Polymer Rheology

$$\underline{F}_{\substack{on\\ surface}} = \iint_S \hat{n}\cdot\left(-\underline{\underline{\Pi}}\right) dS = \iint_S \hat{n}\cdot\left(\underline{\underline{\tilde{\Pi}}}\right) dS$$

$$\Pi_{yx} \qquad \tilde{\Pi}_{yx}$$

UR/Bird choice: fluid at lesser *y* exerts force on fluid at greater *y*

(IFM/Mechanics choice: (opposite)

38

Momentum Balance *(continued)*

Polymer Rheology

Final Assembly:

$$\begin{pmatrix} \text{rate of increase} \\ \text{of momentum in } V \end{pmatrix} = \begin{pmatrix} \text{net flux of} \\ \text{momentum into } V \end{pmatrix} + \begin{pmatrix} \text{sum of} \\ \text{forces on } V \end{pmatrix}$$

$$\iiint_V \frac{\partial}{\partial t}(\rho \underline{v})\, dV = -\iiint_V \nabla \cdot (\rho \underline{v}\underline{v})\, dV + \iiint_V \rho \underline{g}\, dV - \iiint_V \nabla \cdot \underline{\underline{\Pi}}\, dV$$

$$\iiint_V \left[\frac{\partial \rho \underline{v}}{\partial t} + \nabla \cdot (\rho \underline{v}\underline{v}) - \rho \underline{g} + \nabla \cdot \underline{\underline{\Pi}} \right] dV = 0$$

Because *V* is *arbitrary*, we may conclude:

$$\boxed{\frac{\partial \rho \underline{v}}{\partial t} + \nabla \cdot (\rho \underline{v}\underline{v}) - \rho \underline{g} + \nabla \cdot \underline{\underline{\Pi}} = 0}$$

Microscopic momentum balance

Momentum Balance *(continued)*

Polymer Rheology

Microscopic momentum balance

$$\boxed{\frac{\partial \rho \underline{v}}{\partial t} + \nabla \cdot (\rho \underline{v}\underline{v}) - \rho \underline{g} + \nabla \cdot \underline{\underline{\Pi}} = 0}$$

After some rearrangement:

$$\rho \left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} \right) = -\nabla \cdot \underline{\underline{\Pi}} + \rho \underline{g}$$

$$\rho \frac{D\underline{v}}{Dt} = -\nabla \cdot \underline{\underline{\Pi}} + \rho \underline{g}$$

Equation of Motion

Now, what to do with $\underline{\underline{\Pi}}$?

Momentum Balance *(continued)* Polymer Rheology

Now, what to do with $\underline{\underline{\Pi}}$? Pressure is part of it.

Pressure

definition: An isotropic force/area of molecular origin. Pressure is the same on any surface drawn through a point and acts normally to the chosen surface.

$$pressure = p\,\underline{\underline{I}} = p\,\hat{e}_1\hat{e}_1 + p\,\hat{e}_2\hat{e}_2 + p\,\hat{e}_3\hat{e}_3 = \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}_{123}$$

Test: what is the force on a surface with unit normal $\hat{n}$?

41

Momentum Balance *(continued)* Polymer Rheology

back to our question,

Now, what to do with $\underline{\underline{\Pi}}$? Pressure is part of it.

There are other, nonisotropic stresses

Extra Molecular Stresses

definition: The extra stresses are the molecular stresses that are not isotropic

$$\underbrace{\underline{\underline{\tau}}}_{\text{Extra stress tensor,}} \equiv \underline{\underline{\Pi}} - p\,\underline{\underline{I}}$$

(other sign convention: $\underline{\underline{\tilde{\tau}}} = \underline{\underline{\tilde{\Pi}}} + p\underline{\underline{I}}$*)*

i.e. everything complicated in molecular deformation

Now, what to do with $\underline{\underline{\tau}}$?

This becomes the central question of rheological study

42

Momentum Balance *(continued)* Polymer Rheology

Stress sign convention affects any expressions with $\underline{\underline{\Pi}}, \underline{\underline{\tilde{\Pi}}}$ or $\underline{\underline{\tau}}, \underline{\underline{\tilde{\tau}}}$

$$\underline{\underline{\Pi}} \equiv \underline{\underline{\tau}} + p\,\underline{\underline{I}}$$

UR/Bird choice: fluid at lesser *y* exerts force on fluid at greater *y*

$$\underline{\underline{\tilde{\Pi}}} \equiv \underline{\underline{\tilde{\tau}}} - p\,\underline{\underline{I}}$$

(IFM/Mechanics choice: (opposite)

43

Momentum Balance *(continued)* Polymer Rheology

Constitutive equations for Stress

- are tensor equations
- relate the velocity field to the stresses generated by molecular forces
- are based on observations (empirical) or are based on molecular models (theoretical)
- are typically found by trial-and-error
- are justified by how well they work for a system of interest
- are observed to be symmetric

$$\underline{\underline{\tau}} = f(\nabla \underline{v}, \quad material\ properties)$$

Observation: the stress tensor is symmetric

44

Momentum Balance *(continued)*
Polymer Rheology

Microscopic momentum balance

$$\rho\left(\frac{\partial \underline{v}}{\partial t}+\underline{v}\cdot\nabla\underline{v}\right)=-\nabla\cdot\underline{\underline{\Pi}}+\rho\underline{g}$$

Equation of Motion

In terms of the extra stress tensor:

$$\rho\left(\frac{\partial \underline{v}}{\partial t}+\underline{v}\cdot\nabla\underline{v}\right)=-\nabla p-\nabla\cdot\underline{\underline{\tau}}+\rho\underline{g}$$

Equation of Motion

Cauchy Momentum Equation

Components in three coordinate systems (our sign convention):
http://www.chem.mtu.edu/~fmorriso/cm310/Navier2007.pdf

45

Momentum Balance *(continued)*
Polymer Rheology

Newtonian Constitutive equation

$$\underline{\underline{\tau}}=-\mu\left(\nabla\underline{v}+\left(\nabla\underline{v}\right)^T\right)$$

•for incompressible fluids (see text for compressible fluids)

•is empirical

•may be justified for some systems with molecular modeling calculations

Note: $\tilde{\underline{\underline{\tau}}}=+\mu(\nabla\underline{v}+(\nabla\underline{v})^T)$

(IFM choice: (opposite)

46

Momentum Balance *(continued)* Polymer Rheology

How is the Newtonian Constitutive equation related to Newton's Law of Viscosity?

$$\boxed{\underline{\underline{\tau}} = -\mu\left(\nabla\underline{v} + (\nabla\underline{v})^T\right)}$$

- incompressible fluids

$$\boxed{\tau_{21} = -\mu\frac{\partial v_1}{\partial x_2}}$$

- incompressible fluids
- rectilinear flow (straight lines)
- no variation in x_3-direction

47

Momentum Balance *(continued)* Polymer Rheology

Back to the momentum balance . . .

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\right) = -\nabla p - \nabla\cdot\underline{\underline{\tau}} + \rho\underline{g}$$

Equation of Motion

$$\underline{\underline{\tau}} = -\mu\left(\nabla\underline{v} + (\nabla\underline{v})^T\right)$$

We can incorporate the Newtonian constitutive equation into the momentum balance to obtain a momentum-balance equation that is specific to incompressible, Newtonian fluids

48

Momentum Balance *(continued)*

Polymer Rheology

Navier-Stokes Equation

$$\rho\left(\frac{\partial \underline{v}}{\partial t}+\underline{v}\cdot\nabla\underline{v}\right)=-\nabla p+\mu\nabla^2\underline{v}+\rho\underline{g}$$

- incompressible fluids
- Newtonian fluids

Note: The Navier-Stokes is unaffected by the stress sign convention because neither $\underline{\underline{\tau}}$ nor $\underline{\underline{\tilde{\tau}}}$ appear.

49

Momentum Balance *(continued)*

Polymer Rheology

Next?

Navier-Stokes Equation

$$\rho\left(\frac{\partial \underline{v}}{\partial t}+\underline{v}\cdot\nabla\underline{v}\right)=-\nabla p+\mu\nabla^2\underline{v}+\rho\underline{g}$$

Newtonian Problem Solving

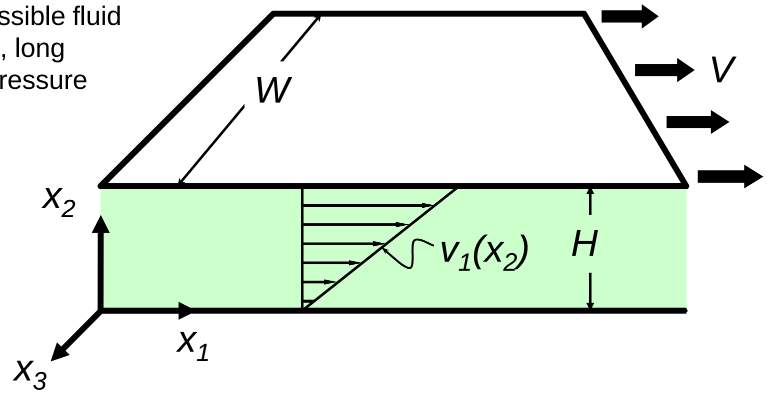
50

from QUICK START

EXAMPLE: Drag flow between infinite parallel plates

$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}_{123}$$

- Newtonian
- steady state
- incompressible fluid
- very wide, long
- uniform pressure

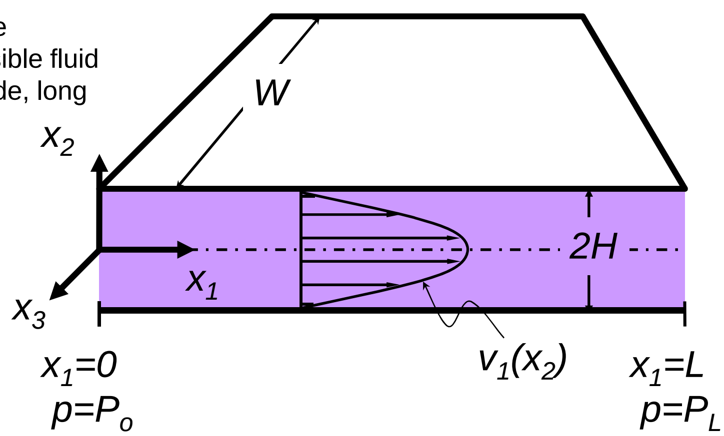


51

from QUICK START

EXAMPLE: Poiseuille flow between infinite parallel plates

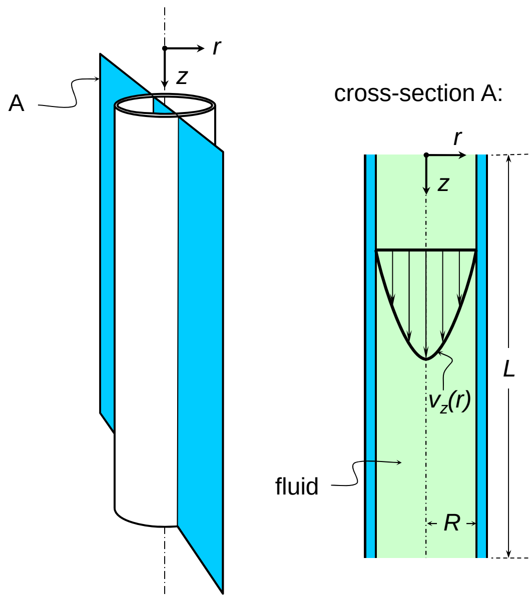
- Newtonian
- steady state
- Incompressible fluid
- infinitely wide, long



52

EXAMPLE: Poiseuille flow in a tube

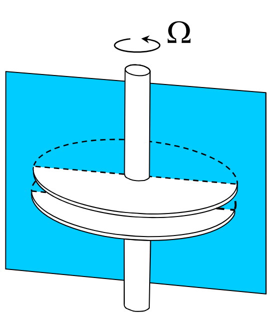
- Newtonian
- Steady state
- incompressible fluid
- long tube

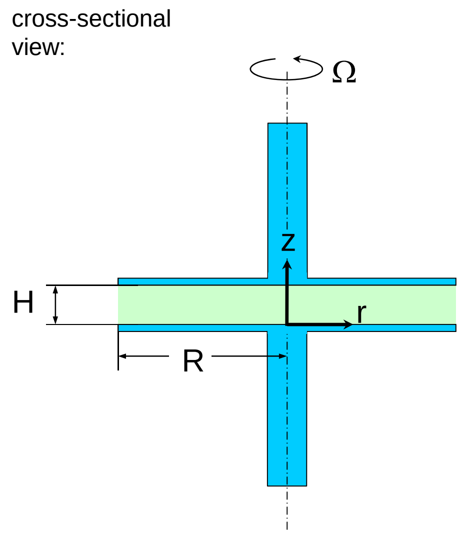


53

EXAMPLE: Torsional flow between parallel plates

- Newtonian
- Steady state
- incompressible fluid
- $v_\theta = zf(r)$





54

Done with
Newtonian Fluids.

Chapter 3: Newtonian Fluids

CM4650
Polymer Rheology
Michigan Tech

Navier-Stokes Equation

$$\rho\left(\frac{\partial \underline{v}}{\partial t}+\underline{v}\cdot\nabla\underline{v}\right)=-\nabla p+\mu\nabla^2\underline{v}+\rho\underline{g}$$

1

Let's move on to
Standard Flows

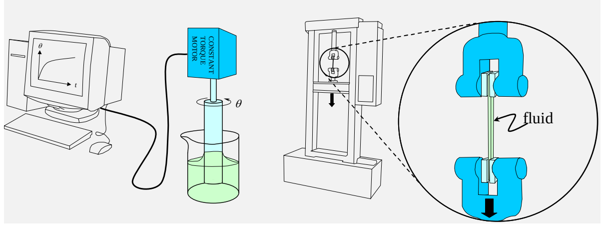
55

Chapter 4: Standard Flows

CM4650
Polymer Rheology
Michigan Tech

Newtonian fluids: $\underline{\underline{\tau}}=-\mu\dot{\underline{\underline{\gamma}}}$ *vs.* non-Newtonian fluids: $\underline{\underline{\tau}}\neq-\mu\dot{\underline{\underline{\gamma}}}$

How can we investigate non-Newtonian behavior?



56