On the Hardness of Adding Nonmasking Fault Tolerance

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Abstract—This paper investigates the complexity of adding nonmasking fault tolerance, where a nonmasking fault-tolerant program guarantees recovery from states reached due to the occurrence of faults to states from where its specifications are satisfied. We first demonstrate that adding nonmasking fault tolerance to low atomicity programs—where processes have read/write restrictions with respect to the variables of other processes—is NP-complete (in the size of the state space) on an unfair or weakly fair scheduler. Then, we establish a surprising result that even under strong fairness, addition of nonmasking fault tolerance remains NP-hard! The NP-hardness of adding nonmasking fault tolerance is based on a polynomial-time reduction from the 3-SAT problem to the problem of designing self-stabilizing programs from their non-stabilizing versions, which is a special case of adding nonmasking fault tolerance. While it is known that designing self-stabilization under the assumption of strong fairness is polynomial, we demonstrate that adding self-stabilization to non-stabilizing programs is NP-hard under weak fairness.

Index Terms—Fault tolerance, distributed programs, NP-hardness

1 Introduction

Today's distributed programs are subject to a variety of types of faults (e.g., node crash, process restart, transient faults, message loss) due to their inherent complexity, human errors and environmental factors (e.g., soft errors). Such programs should guarantee service availability even in the presence of faults. Nonetheless, designing and verifying recovery of distributed programs is a difficult task in part due to the limitations of distribution and the need for global recovery by a coordination of local actions. This paper investigates the complexity of augmenting an existing distributed program with nonmasking fault tolerance (i.e., *adding* nonmasking fault tolerance), where a *nonmasking* program ensures recovery from a subset of states reached due to the occurrence of faults to states from where its specifications are satisfied. A special case of nonmasking tolerance is *self-stabilization* where recovery should be provided from any state.

Several researchers have investigated the problem of adding nonmasking fault tolerance to programs [1], [2], [3], [4], [5], [6]. For instance, Liu and Joseph [4] present a method for the transformation of a fault-intolerant program to a fault-tolerant version thereof by going through a set of refinement steps—where a *fault-intolerant* program provides no guarantees when faults occur. They model faults by state perturbation, where program actions are executed in an interleaving with fault actions. Arora and Gouda [2], [3] provide a unified theory for the formulation of fault tolerance functionalities in terms of closure and convergence, where *closure* means that, in the absence of faults, a fault-tolerant program remains in a set of legitimate states, called its *invariant*, and convergence specifies that the state of the program is recovered to its invariant from a superset of the invariant reached due to the occurrence of faults, called a *fault-span*. Arora and Gouda [2], [3] use the notions of closure and convergence to define three levels of fault tolerance based on the extent to which safety and liveness specifications [7] are satisfied in the presence of faults. A *failsafe* fault-tolerant program ensures its safety at all times even if faults occur, whereas, in the presence of faults, a *nonmasking* program provides recovery to its invariant; no guarantees on meeting safety during recovery. A *masking* fault-tolerant program is both failsafe and nonmasking. Arora et al. [5] design nonmasking fault tolerance by creating a dependency graph of the local constraints of program processes, and by illustrating how these constraints should be satisfied so global recovery is achieved. In a shared memory model, Kulkarni and Arora [6] demonstrate that adding failsafe/nonmasking/masking fault tolerance to high atomicity programs can be done in polynomial time in the size of the state space (under no fairness), where the processes of a *high atomicity* program can read/write all program variables in an atomic step. Nonetheless, they show that, for distributed programs, adding masking fault tolerance is NP-complete (in the size of the state space) on an unfair scheduler. The authors of [6] model distribution in a *low atomicity* shared memory model, where each process has read and write restrictions with respect to the local variables of other processes. Kulkarni and Ebnenasir [8], [9] show that adding failsafe fault tolerance to distributed programs is also an NP-complete problem. However, the complexity of adding nonmasking fault tolerance has remained an open problem for more than a decade. While this problem is known to be in NP, no polynomial-time algorithms are known for efficient design of nonmasking fault tolerance for low atomicity programs; nor has there been a proof of NP-completeness.

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	No Fairness	Weak Fairness	Strong Fairness
Nonmasking	NP-complete*	NP-complete*	NP-complete*
Self-Stabilization	NP-complete*	NP-complete*	P

Fig. 1. The complexity of adding nonmasking fault tolerance and self-stabilization under different fairness policies. (* denotes the contributions of this paper).

In this paper, we prove that adding nonmasking fault tolerance to low atomicity programs is NP-complete under no fairness, weak, and strong fairness assumptions (see Fig. 1). A *weakly fair* scheduler infinitely often executes any program action that is continuously enabled (i.e., ready for execution), whereas a *strongly fair* scheduler infinitely often executes any transition that is enabled infinitely often. Our hardness proof is based on a reduction from the 3-SAT problem [10] to the problem of adding self-stabilization to non-stabilizing programs under no fairness. Since self-stabilization is a special case of nonmasking fault tolerance, it follows that, in general, it is unlikely that adding nonmasking fault tolerance to low atomicity programs can be done efficiently (unless $P = NP$). We also show that even under weak fairness the addition of stabilization to low atomicity programs remains an NP-complete problem (see Fig. 1), which implies the NP-completeness of adding nonmasking fault tolerance under weak fairness in general. Moreover, we present a surprising result that, while adding stabilization under the assumption of strong fairness is known to be polynomial (in the state space) [11], [12], [13], the general case complexity of adding nonmasking fault tolerance under strong fairness remains NP-complete!

Contributions. We present

- a proof of NP-completeness of adding self-stabilization to distributed programs under no fairness and weak fairness assumptions;
- a proof of NP-completeness of adding nonmasking fault tolerance to distributed programs under any fairness assumption, thereby solving a decade-old problem, and
- a proof of NP-completeness of adding self-stabilization even in special cases where (i) a process can atomically read the global state of the distributed program and can update its own local state, and (ii) processes have *self-disabling* actions—where the execution of an action disables itself.

Organization. Section 2 presents the basic concepts of programs, faults and fault tolerance. Section 3 formally states the problem of adding nonmasking fault tolerance. Section 4 illustrates that adding nonmasking fault tolerance to low atomicity programs is in general NP-complete (on an unfair, weakly or strongly fair scheduler). Section 5 discusses related work. Finally, Section 6 makes concluding remarks and discusses future work.

2 Preliminaries

In this section, we present the formal definitions of programs, specifications, our distribution model (adapted from [6]), faults and fault tolerance (adapted from [1], [3], [11], [14]). For ease of presentation, we use a simplified version of Dijkstra's token ring (TR) protocol [1] as a running example.

Programs. A program in our setting is a representation of any system that can be captured by a (non-deterministic) finite-state machine (e.g., network protocols). Formally, a *program* p is a tuple $\langle V_p, \delta_p, \Pi_p, T_p\rangle$ of a finite set V_p of variables, a set δ_p of transitions, a finite set Π_p of N processes, where $N \geq 1$, and a topology T_p. Each variable $v_i \in V_p$, for $i \in \mathbb{N}_m$ where $\mathbb{N}_m = \{0, 1, \ldots, m-1\}$ and $m > 0$, has a finite non-empty domain D_i. A *state* s of p is a valuation $\langle d_0, d_1, \ldots, d_{m-1}\rangle$ of variables $\langle v_0, v_1, \ldots, v_{m-1}\rangle$, where $d_i \in D_i$. A *transition* t is an ordered pair of states, denoted (s_0, s_1), where s_0 is the source and s_1 is the target/destination state of t. A *process* $P_j \in \Pi_p$ is a triple $\langle \delta_j, r_j, w_j\rangle$, where $0 \leq j \leq N-1$ and $\delta_j \subseteq \mathcal{S}_p \times \mathcal{S}_p$ denotes the *set of transitions* of P_j. The set of transitions of the program p, denoted δ_p, is the union of the sets of transitions of its processes; i.e., $\delta_p = \bigcup_{j=0}^{N-1} \delta_j$. A *deadlock state* is a state with no outgoing transitions. For a variable v and a state s, $v(s)$ denotes the value of v in s. The *state space* of p, denoted S_p, is the set of all possible states of p, and $|S_p|$ denotes the size of S_p. A *state predicate* is any subset of S_p specified as a Boolean expression over V_p. We say a state predicate X *holds in a state* s (respectively, $s \in X$) *if and only if (iff)* X evaluates to true at s. For simplicity, we misuse the notations p and δ_p interchangeably.

To simplify the specification of δ_p for designers, we use Dijkstra's guarded commands language [15] as a shorthand for representing the set of program transitions. A *guarded command* (a.k.a. *action*) is of the form $grd \rightarrow stmt$, and includes a set of transitions (s_0, s_1) such that the predicate grd holds in s_0 and the atomic execution of the statement $stmt$ results in state s_1. An action $grd \rightarrow stmt$ is *enabled* in a state s iff grd holds at s. A process $P_j \in \Pi_p$ is *enabled* in s iff there exists an action of P_j that is enabled at s.

Computations. Intuitively, a computation of a program $p = \langle V_p, \delta_p, \Pi_p, T_p\rangle$ is an *interleaving* of its actions. Formally, a *computation* of p is a sequence $\sigma = \ll s_0, s_1, \ldots \gg$ of states that satisfies the following conditions: (1) for each transition (s_i, s_{i+1}) in σ, where $i \geq 0$, there exists an action $grd \rightarrow stmt$ in some process $P_j \in \Pi_p$ such that grd holds at s_i and the execution of $stmt$ at s_i yields s_{i+1}, and (2) σ is *maximal* in that either σ is infinite or if it is finite, then σ reaches a state s_f where no action is enabled. A *computation prefix* of a program p is a *finite* sequence $\sigma = \ll s_0, s_1, \ldots, s_z \gg$ of states, where $z > 0$, such that each transition (s_i, s_{i+1}) in σ ($i \in \mathbb{N}_z$) belongs to some action $grd \rightarrow stmt$ in some process $P_j \in \Pi_p$. The *projection* of a program p on a non-empty state predicate X, denoted as $\delta_p|X$, is the program $\langle V_p, \{(s_0, s_1) : (s_0, s_1) \in \delta_p \ \wedge \ s_0, s_1 \in X\}, \Pi_p, T_p\rangle$.

Properties and specifications. For a program p, a safety property *sprop* stipulates that nothing bad ever happens (e.g., no two processes access a shared resource simultaneously). Formally, we follow [7], [16] to specify a safety property *sprop* as a set of *bad* transitions that must not be executed by p; i.e., $sprop \in S_p \times S_p$. A computation $\sigma = \ll s_0, s_1, \ldots \gg$ of p *satisfies* its safety property *sprop* from s_0 iff no transition (s_i, s_{i+1}), where $i \geq 0$, is in *sprop*; i.e., σ includes no bad transitions. A liveness property, denoted *lprop*, specifies some good things that should eventually occur (e.g., a process will eventually access some shared resource). Formally, liveness is captured as a set of sequences of states. A computation $\sigma = \ll s_0, s_1, \cdots \gg$ of p

satisfies its liveness property *lprop* from s_0 iff σ has a suffix in *lprop*. Following Alpern and Schneider [7], we define a specification *spec* as a set of safety and liveness properties. A computation σ of p satisfies its specification *spec* from s_0 iff σ satisfies the safety and liveness of *spec* from s_0. A program satisfies its specification *spec* from a state predicate I iff every computation of p starting in I satisfies *spec*.

Read/Write model. We adopt a shared memory model [17] since reasoning in a shared memory setting is easier, and several (correctness-preserving) transformations [18], [19] exist for the refinement of shared memory programs to their message-passing versions. To model the topological constraints (denoted T_p) of a program p, we consider a subset of variables in V_p that each process P_j ($j \in \mathbb{N}_N$) can write, denoted w_j, and a subset of variables that P_j is allowed to read, denoted r_j. We assume that for each process P_j, $w_j \subseteq r_j$; i.e., if a process can write a variable, then it can also read that variable. A process P_j is not allowed to update a variable $v \notin w_j$.

Impact of read restrictions. Every transition of a process P_j belongs to a *group* of transitions due to the inability of P_j in reading variables that are not in r_j. Consider two processes P_0 and P_1 each having a Boolean variable that is not readable for the other process. That is, P_0 (respectively, P_1) can read and write x_0 (respectively, x_1), but cannot read x_1 (respectively, x_0). Let $\langle x_0, x_1\rangle$ denote a state of this program. Now, if P_0 writes x_0 in a transition $(\langle 0,0\rangle, \langle 1,0\rangle)$, then P_0 has to consider the possibility of x_1 being 1 when it updates x_0 from 0 to 1. As such, executing an action in which the value of x_0 is changed from 0 to 1 is captured by the fact that a group of two transitions $(\langle 0,0\rangle, \langle 1,0\rangle)$ and $(\langle 0,1\rangle, \langle 1,1\rangle)$ is included in P_0. In general, a transition is included in the set of transitions of a process *iff* its associated group of transitions is included. Formally, any two transitions (s_0, s_1) and (s'_0, s'_1) in a group of transitions formed due to the read restrictions of a process P_j meet the following constraints: $\forall v : v \in r_j : (v(s_0) = v(s'_0)) \wedge (v(s_1) = v(s'_1))$ and $\forall v : v \notin r_j : (v(s_0) = v(s_1)) \wedge (v(s'_0) = v(s'_1))$. (It is known that the total number of groups is polynomial in $|S_p|$ [6]).

Example: Token Ring. The token ring program (adapted from [1]) includes three processes $\{P_0, P_1, P_2\}$ each with an integer variable x_j, where $j \in \mathbb{N}_3$, with a domain $\{0, 1, 2\}$. The process P_0 has the following action ($\oplus$ and $\ominus$ respectively denote addition and subtraction in modulo 3):

$$A_0 : \quad (x_0 = x_2) \quad \longrightarrow \quad x_0 := x_2 \oplus 1$$

When the values of x_0 and x_2 are equal, P_0 increments x_0 by one. We use the following parametric action to represent the actions of P_j, for $1 \leq j \leq 2$:

$$A_j : \quad (x_j \neq x_{(j\ominus 1)}) \quad \longrightarrow \quad x_j := x_{(j\ominus 1)}$$

Each process P_j copies $x_{j\ominus 1}$ only if $x_j \neq x_{j\ominus 1}$, where $j = 1, 2$. By definition, process P_j *has a token* iff $x_j \neq x_{j\ominus 1}$. Process P_0 *has a token* iff $x_0 = x_2$. We define a state predicate I_{TR} that captures the set of states in which only one token exists, where I_{TR} is

$$((x_0 = x_1) \wedge (x_1 = x_2)) \ \vee \ ((x_1 \oplus 1 = x_0) \wedge (x_1 = x_2))$$
$$\vee \ ((x_0 = x_1) \wedge (x_2 \oplus 1 = x_1))$$

Each process P_j ($1 \leq j \leq 2$) is allowed to read variables $x_{j\ominus 1}$ and x_j, but can write only x_j. Process P_0 is permitted to read x_2 and x_0 and can write only x_0. Thus, since a process P_j is unable to read one variable (with a domain of three values), each group includes three transitions. $\lhd$

Closure and invariant. A state predicate X is *closed in an action* $grd \rightarrow stmt$ *iff* executing $stmt$ from any state $s \in (X \wedge grd)$ results in a state in X. We say a state predicate X is *closed in a program* p *iff* X is closed in every action of p. In other words, *closure* in X requires that every computation of p starting in X remains in X [11]. A state predicate I is an *invariant* of p *iff* I is closed in p and p satisfies its *spec* from I.

TR Example. Starting from a state in the state predicate I_{TR}, the TR protocol generates an infinite sequence of states, where all reached states belong to I_{TR}. $\lhd$

Faults. We capture the impact of faults on a program as state perturbations. Formally, a class of *faults* f for a program $p = \langle V_p, \delta_p, \Pi_p, T_p\rangle$ is a subset of $S_p \times S_p$. We use $p[]f$ to denote the transitions obtained by taking the union of the transitions in δ_p and the transitions in f. We say that a state predicate T is an f-span (read as *fault-span*) of p from a state predicate I *iff* $I \subseteq T$ and T is closed in $p[]f$. Observe that for all computations of p that start in I, T is a boundary in the state space of p to which (but not beyond which) the state of p may be perturbed by the occurrence of f. The same way we use guarded commands to represent program transitions, we use them to specify fault transitions. While we concentrate on *transient faults* that can perturb the state of a program without causing any permanent damage, the notion of state perturbation is appropriate for modeling other types of faults. Liu and Joseph [4] use state perturbation to model failstop failures. Chen and Kulkarni [20] show that 20 out of 31 categories of faults classified by Avizienis et al. [21] can be modeled by state perturbation.

TR Example. The TR protocol is subject to transient faults that can perturb its state to an arbitrary state. For instance, the following action captures the impact of transient faults on x_0, where $|$ denotes the non-deterministic assignment of values to x_0:

$$F_0 : \quad true \quad \longrightarrow \quad x_0 := 0|1|2$$

The impact of faults on x_1 and x_2 are captured with two actions F_1 and F_2 symmetric to F_0. $\lhd$

We say that a sequence of states, $\sigma = \langle s_0, s_1, \ldots, \rangle$ is a *computation of* p *in the presence of* f *iff* the following conditions are satisfied: (1) $\forall j : j > 0 : (s_{j-1}, s_j) \in (p[]f)$; (2) if σ is finite and terminates in state s_l, then there is no state s such that $(s_l, s) \in \delta_p$, and (3) $\exists n : n \geq 0 : (\forall j : j > n : (s_{j-1}, s_j) \in \delta_p)$. The first requirement captures that in each step, either a program transition or a fault transition is executed. The second requirement states that if the only transition that starts from s_l is a fault transition (s_l, s_f) then as far as the program is concerned, s_l is still a deadlock state because the program does not have control over the execution of (s_l, s_f); i.e., (s_l, s_f) may or may not be executed. Finally, the third requirement captures that the number of fault occurrences in a computation is finite. This requirement is the same as that made in previous work (e.g., [1], [3], [22], [23]) to ensure that eventually recovery can occur.

Masking fault tolerance. Let I be an invariant of a program p, *spec* denote the specification of p and f be a class of faults. We say that p is masking f-tolerant from I for *spec iff* (1) p satisfies *spec* from I in the absence of faults and there exists an f-span of p from I, denoted T; (2) T *converges* to I in p; i.e., from any state $s_0 \in T$, *every* computation of p that starts in s_0 reaches a state where I holds, and (3) from any state in T the computations of $p[]f$ include no bad transitions.

Nonmasking fault tolerance and self-stabilization. We say that p is nonmasking f-tolerant from I for *spec iff* the conditions (1) and (2) in the definition of masking tolerance are met. The program p is *self-stabilizing* from I *iff* the f-span of p is equal to S_p, and convergence to I is guaranteed from any state in S_p.

Failsafe fault tolerance. A program p is *failsafe* f-tolerant from I for *spec iff* the conditions (1) and (3) in the definition of masking tolerance hold.

Fairness. Let $\sigma = \ll s_i, s_{i+1}, \ldots, s_j, s_i \gg$ be a sequence of states in $T-I$, where $j \geq i$ and each state is reached from its predecessor by the transitions in δ_p. The sequence σ denotes a cycle in $T-I$. The definition of what constitutes a *non-progress* cycle (a.k.a. *livelock*) depends on the underlying fairness assumption. An unfair scheduler provides no guarantees as to how it would execute enabled actions, whereas a *weakly* fair scheduler infinitely often executes actions that are continuously enabled. Under weak fairness, the cycle σ in $(T-I)$ is a non-progress cycle *iff* there is no program action that is enabled in every state of σ and includes a transition that reaches a state $s' \notin \sigma$. Under no fairness assumption, any cycle in $(T-I)$ is a non-progress cycle. Under strong fairness (adapted from Gouda [11]), if the cycle σ in $(T-I)$ includes a state s_k ($i \leq k \leq j$) with an outgoing transition (s_k, s') where s' does not appear in σ, then a strongly fair scheduler would guarantee to eventually execute (s_k, s') because it is infinitely often enabled in the cycle σ. Thus, the program would recover from this cycle with the help of the strongly fair scheduler. A common definition of strong fairness states that any *action* that is enabled infinitely often is executed infinitely often. Under this definition of strong fairness, consider an action A that includes a transition (s_k, s_{k+1}) in σ and another transition (s_r, s') where s' does not appear in σ and $i \leq k, r \leq j$. Notice that A is enabled infinitely often because both s_k and s_r are visited infinitely often, however, the scheduler could meet its specification by infinitely often executing just (s_k, s_{k+1}), thereby not recovering from σ. That is why we adopt a more fine-grained definition for strong fairness compared with the common definition. Nonetheless, the results of this paper hold for both definitions since the instance of the problem of adding nonmasking fault tolerance built in its proof of NP-hardness does not include actions like A discussed above.

3 Problem Statement

In this section, we present the problem of adding nonmasking fault tolerance under different fairness assumptions. Consider a fault-intolerant program $p = \langle V_p, \delta_p, \Pi_p, T_p \rangle$, its invariant I, its specification *spec*, a class of faults f, and a fairness assumption $\mathcal{F} \in$ {unfair, weak, strong}. Our objective is to generate a revised version of p, denoted p', such that p' is nonmasking f-tolerant from an invariant I' under the fairness assumption $\mathcal{F}$. To separate fault tolerance from functional concerns, we would like to preserve the behaviors of p in the absence of f during the addition of fault tolerance; i.e., in the absence of faults, p' satisfies *spec*. Thus, during the synthesis of p' from p, no states (respectively, transitions) are added to I (respectively, $\delta_p|I$). As such, we have $I' \subseteq I$ and $p'|I' \subseteq p|I'$. Moreover, if p' starts in a state outside I', then only convergence to I' will be provided by p'. Thus, we formally state the problem as follows: (This is an adaptation of the problem of adding fault tolerance in [6].)

Problem 3.1. Adding Nonmasking Fault Tolerance:

- **Input**: (1) A program $p = \langle V_p, \delta_p, \Pi_p, T_p \rangle$ that satisfies its specification *spec* from an invariant I; (2) a class of faults f, and (3) a fairness assumption $\mathcal{F} \in$ {unfair, weak, strong}.
- **Output**: A program $p' = \langle V_p, \delta_{p'}, \Pi_p, T_p \rangle$ and an invariant I' such that: (1) I' is non-empty and $I' \subseteq I$; (2) $\delta_{p'}|I' \subseteq \delta_p|I'$, and (3) p' is nonmasking f-tolerant from I' for *spec* under $\mathcal{F}$ fairness.

We state the corresponding decision problem as follows:

Problem 3.2. Decision Problem of Adding Nonmasking Fault Tolerance:

- INSTANCE: (1) A program $p = \langle V_p, \delta_p, \Pi_p, T_p \rangle$ that satisfies its specification *spec* from an invariant I; (2) a class of faults f, and (3) a fairness assumption $\mathcal{F} \in$ {unfair, weak, strong}.
- QUESTION: Does there exist a program $p' = \langle V_p, \delta_{p'}, \Pi_p, T_p \rangle$ and a state predicate I' such that the constraints of Problem 3.1 are met under the fairness assumption $\mathcal{F}$?

A special case of Problem 3.1 is where (i) f denotes a class of transient faults; (ii) $I = I'$; (iii) $\delta_{p'}|I' = \delta_p|I'$, and (iv) p' is self-stabilizing from I under $\mathcal{F}$ fairness.

Problem 3.3. Decision Problem of Adding Stabilization:

- INSTANCE: (1) A program $p = \langle V_p, \delta_p, \Pi_p, T_p \rangle$ that satisfies its specification *spec* from an invariant I; (2) a class of *transient* faults f, and (3) a fairness assumption $\mathcal{F} \in$ {unfair, weak, strong}.
- QUESTION: Does there exist a program $p' = \langle V_p, \delta_{p'}, \Pi_p, T_p \rangle$ such that: (1) I remains unchanged (i.e., $I' = I$); (2) $\delta_{p'}|I = \delta_p|I$, and (3) p' is self-stabilizing from I for *spec* under $\mathcal{F}$ fairness?

Previous work [11], [12], [13] illustrates that if $\mathcal{F} =$ strong, then Problem 3.3 can be solved in polynomial time in $|S_p|$. Stabilization under strong fairness (a.k.a. *weak stabilization*) requires that from any state $s \in \neg I$, *there exists* a computation prefix that includes a state in I [11]. However, the general case complexity of adding stabilization under no fairness and weak fairness assumptions have been open problems thus far. Stabilization under no fairness (a.k.a. *strong stabilization*) stipulates that from any state $s \in \neg I$, *every* computation prefix includes a state in I [1], [11]. We have developed heuristics and software tools [13] that synthesize self-stabilizing programs in polynomial time. Moreover, previous research [24], [25] testifies the practical significance of adding nonmasking tolerance.

4 Hardness Results

In this section, we illustrate that adding nonmasking fault tolerance to low atomicity programs is NP-complete under no fairness (Section 4.2), weak (Section 4.3) and strong (Section 4.4) fairness assumptions. We first state the 3-SAT decision problem.

Problem 4.1. *The 3-SAT decision problem:*

- INSTANCE: A set $\mathcal{V}$ of n propositional variables $(v_0, \ldots, v_{n-1})$ and k clauses $(C_0, \ldots, C_{k-1})$ over $\mathcal{V}$ such that each clause is of the form $(l_q \vee l_r \vee l_s)$, where $q, r, s \in \mathbb{N}_n$ and $\mathbb{N}_n = \{0, 1, \ldots, n-1\}$. Each l_r denotes a literal, where a literal is either $\neg v_r$ or v_r for $v_r \in \mathcal{V}$.
- QUESTION: Is there a satisfying truth-value assignment for the variables in $\mathcal{V}$ such that each C_i evaluates to *true*, for all $i \in \mathbb{N}_k$?

Notation. We say l_r is a *negative* (respectively, *positive*) literal *iff* it has the form $\neg v_r$ (respectively, v_r), where $v_r \in \mathcal{V}$. Consider a clause $C_i = (l_q \vee l_r \vee l_s)$. We use a binary variable b^i_j, where $i \in \mathbb{N}_k$ and $j \in \mathbb{N}_3$, to denote the sign of the first, second and the third literal in C_i. For example, if $l_q = \neg v_q, l_r = v_r$ and $l_s = \neg v_s$, then we have $b^i_0 = 0, b^i_1 = 1$ and $b^i_2 = 0$. Let the tuple $B^i = \langle b^i_0, b^i_1, b^i_2 \rangle$ denote the values of b^i_j variables, for each clause C_i where $j \in \mathbb{N}_3$.

4.1 Intuition Behind Hardness Proofs

This section presents the intuition behind the hardness of adding nonmasking fault tolerance under different fairness assumptions.

No fairness. In Section 4.2, we show that adding stabilization under no fairness is NP-hard, thereby implying the NP-hardness of adding nonmasking fault tolerance in general. Consider a deadlock state s_d outside I. To ensure that some state in I will eventually be reached from s_d, we need to build a computation prefix from s_d to I while ensuring that non-progress cycles are not formed in $\neg I$. Let (s_d, s) be a transition included in a process P_j during the construction of some computation prefix. We can include (s_d, s) in the set of transitions of P_j *iff* we include any transition (s'_d, s') grouped with (s_d, s) (due to read restrictions of P_j), and (s'_d, s') does not create a cycle with other transitions. That is, one has to identify a *subset* of transition groups that construct a computation prefix from any state in $\neg I$ to I without creating cycles outside I. Thus, deciding whether a transition group should be included in some process resembles the assignment of a truth value to a propositional variable in the instance of 3-SAT.

Weak fairness (Section 4.3). Under weak fairness, a cycle c that has an action A enabled in every state of c is not considered a non-progress cycle because a weakly fair scheduler guarantees the execution of A, thereby exiting the cycle. Thus, an algorithm for the addition of stabilization need not resolve such cycles. One would think that this could simplify the design of stabilization under weak fairness, but Theorem 4.8 proves otherwise. The intuition behind this hardness result is that there might still be cycles for which there is no action similar to A. Thus, we have to deal with a similar combinatorial problem mentioned for stabilization under no fairness.

Strong fairness (Section 4.4). A strongly fair scheduler (as defined in Section 2) ensures that a program will eventually exit any reachable cycle c that has some outgoing transition from one of its states to a state outside c. Thus, to design self-stabilization under strong fairness, we need to ensure that from any state there exists a computation prefix that reaches the invariant; no need to resolve cycles. This problem is known to be solvable in polynomial time [11], [13]. Since the general case problem of adding nonmasking fault tolerance should deal with cases where faults may cause permanent damage (unlike transient faults), one has to ensure that permanent faults are not activated or else the system may reach an unrecoverable state. To ensure that a distributed program does not reach such states, designers might have to guarantee cycle-freedom (despite strong fairness) in certain subsets of the fault span; otherwise, an interleaving of fault and cycle transitions may perturb the program to an unrecoverable state. Thus, designing the general case nonmasking fault tolerance under strong fairness is at least as hard as designing self-stabilization under no fairness!

4.2 Hardness Under No Fairness

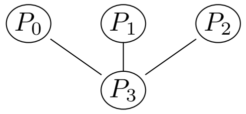
In this section, we investigate the general case complexity of Problem 3.3 under no fairness. We specifically demonstrate that, for a given intolerant program p with an invariant I, designing a revised version of p, denoted p_{ss}, such that p_{ss} is self-stabilizing from I is an NP-hard problem. Section 4.2.1 presents a polynomial-time mapping from 3-SAT to an instance of Problem 3.3. Section 4.2.2 shows that the instance of 3-SAT is satisfiable *iff* a self-stabilizing version of the instance of Problem 3.3 exists where $\mathcal{F} =$ unfair.

4.2.1 Polynomial Mapping

In this section, we present a polynomial-time mapping from an instance of 3-SAT to the instance of Problem 3.3 where $\mathcal{F} =$ unfair, denoted $p = \langle V_p, \delta_p, \Pi_p, T_p \rangle$. That is, corresponding to each propositional variable and clause, we illustrate how we construct a non-stabilizing program p, its processes Π_p, its variables V_p, its read/write restrictions, its specification *spec* and its invariant I. We shall use this mapping in Section 4.2.2 to demonstrate that the instance of 3-SAT is satisfiable *iff* a self-stabilizing version of p exists.

Processes, variables and read/write restrictions. We consider four processes, P_0, P_1, P_2, and P_3 in p. Each process P_j $(j \in \mathbb{N}_3)$ has two variables x_j and y_j, where the domain of x_j is equal to $\mathbb{N}_n$ and y_j is a binary variable. The process P_j can read both x_j and y_j, but can write only y_j. Thus, the processes P_0, P_1 and P_2 cannot read each other's variables. We also consider a fourth process P_3 that can read all variables and write to a binary variable $sat \in \mathbb{N}_2$. The variable sat can be read by processes P_0, P_1 and P_2, but cannot be written. Thus, we have $V_p = \{x_0, y_0, x_1, y_1, x_2, y_2, sat\}$, $\Pi_p = \{P_0, P_1, P_2, P_3\}$ and the topology of p is identified by the read/write restrictions of processes as depicted in Fig. 2.

Invariant. Inspired by the form of the 3-SAT instance and its requirements, we define a state predicate I_{ss} that denotes the invariant of p.



For all $i \in \mathbb{N}_3$:

$$P_i \quad : \quad \begin{cases} x_i \in \mathbb{N}_n & \text{(read-only)} \\ y_i \in \mathbb{N}_2 & \text{(read \& write)} \\ sat & \text{(read from } P_3\text{)} \end{cases}$$

$$P_3 \quad : \quad \begin{cases} sat \in \mathbb{N}_2 & \text{(read \& write)} \\ x_i, y_i & \text{(read from } P_i\text{)} \end{cases}$$

Invariant: $I_{ss} \equiv Iden \wedge Clauses \wedge (sat = 1)$

Fig. 2. Instance of Problem 3.3 under no fairness.

- Corresponding to each clause $C_i = (l_q \vee l_r \vee l_s)$, we construct a state predicate $PredC_i \equiv (x_0 = q \Longrightarrow y_0 = b_0^i) \vee (x_1 = r \Longrightarrow y_1 = b_1^i) \vee (x_2 = s \Longrightarrow y_2 = b_2^i)$. In other words, we have $PredC_i \equiv ((x_0 = q) \wedge (x_1 = r) \wedge (x_2 = s)) \Longrightarrow ((y_0 = b_0^i) \vee (y_1 = b_1^i) \vee (y_2 = b_2^i))$. This way, we construct a state predicate $Clauses \equiv (\forall i \in \mathbb{N}_k : PredC_i)$. Notice that we check the value of each x_j with respect to the index of the literal appearing in position j in C_i, where $j \in \mathbb{N}_3$. This is due to the fact that the domain of x_j is equal to the range of the indices of propositional variables (i.e., $\mathbb{N}_n$).
- A literal l_r may appear in positions i and j in distinct clauses of 3-SAT, where $i, j \in \mathbb{N}_3$ and $i \neq j$. Since each propositional variable $v_r \in \mathcal{V}$ gets a unique truth-value in 3-SAT, the truth-value of l_r is independent from its position in the 3-SAT formula. Given the way we construct the state predicate $Clauses$, it follows that, in the instance of Problem 3.3, whenever $x_i = x_j$ we should have $y_i = y_j$. Thus, we construct the state predicate $Iden \equiv (\forall i, j \in \mathbb{N}_3 : (x_i = x_j \Longrightarrow y_i = y_j))$, which is conjoined with the predicate $Clauses$.
- In the instance of Problem 3.3, we require that $(sat = 1)$ holds in all invariant states.

Based on the aforementioned reasoning, the invariant of p is equal to the state predicate I_{ss}, where

$$I_{ss} \equiv Iden \wedge Clauses \wedge (sat = 1)$$

Notice that the size of the state space of p, denoted $|S_p|$, is $2(2n)^3$, which is polynomial in the size of the 3-SAT instance.

Specification. The safety of $spec$ forbids any transition that starts in I_{ss}. That is, the instance of Problem 3.3 is *silent* in its invariant (i.e., $\delta_p | I_{ss} = \emptyset$).

Example 4.2. Example Construction

Consider the 3-SAT formula $\phi \equiv (v_0 \vee v_1 \vee v_2) \wedge (\neg v_1 \vee \neg v_1 \vee \neg v_2) \wedge (\neg v_1 \vee \neg v_1 \vee v_2) \wedge (v_1 \vee \neg v_2 \vee \neg v_0)$. Since there are three propositional variables and four clauses, we have $n = 3$ and $k = 4$. Moreover, based on the mapping described before, we have $C_0 \equiv (v_0 \vee v_1 \vee v_2)$, $C_1 \equiv (\neg v_1 \vee \neg v_1 \vee \neg v_2)$, $C_2 \equiv (\neg v_1 \vee \neg v_1 \vee v_2)$ and $C_3 \equiv (v_1 \vee \neg v_2 \vee \neg v_0)$. Thus, we have $B^0 = \langle 1, 1, 1 \rangle$, $B^1 = \langle 0, 0, 0 \rangle$, $B^2 = \langle 0, 0, 1 \rangle$ and $B^3 = \langle 1, 0, 0 \rangle$. The predicates $PredC_i$ $(i \in \mathbb{N}_4)$ have the following form:

$$\begin{aligned} PredC_0 &\equiv (x_0 = 0 \wedge x_1 = 1 \wedge x_2 = 2) \Longrightarrow \\ &\qquad (y_0 = 1 \vee y_1 = 1 \vee y_2 = 1) \\ PredC_1 &\equiv (x_0 = 1 \wedge x_1 = 1 \wedge x_2 = 2) \Longrightarrow \\ &\qquad (y_0 = 0 \vee y_1 = 0 \vee y_2 = 0) \\ PredC_2 &\equiv (x_0 = 1 \wedge x_1 = 1 \wedge x_2 = 2) \Longrightarrow \\ &\qquad (y_0 = 0 \vee y_1 = 0 \vee y_2 = 1) \\ PredC_3 &\equiv (x_0 = 1 \wedge x_1 = 2 \wedge x_2 = 0) \Longrightarrow \\ &\qquad (y_0 = 1 \vee y_1 = 0 \vee y_2 = 0) \end{aligned}$$

The state predicate $Iden$ is as defined before.

4.2.2 Correctness of Reduction

In this section, we show that the instance of 3-SAT is satisfiable *iff* convergence from S_p to I_{ss} can be added to the instance of Problem 3.3, denoted p.

Lemma 4.3. *If the instance of 3-SAT has a satisfying valuation, then stabilization can be added to the instance of Problem 3.3.*

Let there be a truth-value assignment to the propositional variables in $\mathcal{V}$ such that every clause evaluates to *true*; i.e., $\forall i : i \in \mathbb{N}_k : C_i$. Let p_{ss} denote the self-stabilizing version of p. Initially, $\delta_p = \emptyset$ and $p = p_{ss}$. Based on the value assignments to propositional variables, we include a set of transitions (represented as convergence actions) in p_{ss}. Then, we show that the following three properties hold: the invariant $I_{ss} \equiv Clauses \wedge Iden \wedge (sat = 1)$ remains closed, deadlock freedom in $\neg I_{ss}$ and livelock freedom in $p_{ss} | \neg I_{ss}$.

- If a propositional variable v_r (where $r \in \mathbb{N}_n$) is assigned *true*, then we include the following action in each process P_j, where $j \in \mathbb{N}_3$: $x_j = r \wedge y_j = 0 \wedge sat = 0 \rightarrow y_j := 1$.
- If a propositional variable v_r (where $r \in \mathbb{N}_n$) is assigned *false*, then we include the following action in each process P_j, where $j \in \mathbb{N}_3$: $x_j = r \wedge y_j = 1 \wedge sat = 0 \rightarrow y_j := 0$.
- We include the following actions in P_3: $(Iden \wedge Clauses) \wedge sat = 0 \rightarrow sat := 1$ and $\neg(Iden \wedge Clauses) \wedge sat = 1 \rightarrow sat := 0$.

Now, we illustrate that, closure, deadlock freedom and livelock freedom hold. That is, the resulting program is self-stabilizing from I_{ss}.

Closure. Since the first three processes can execute an action only in states where $sat = 0$, their actions are disabled where $sat = 1$. Thus, the first three processes exclude any transition that starts in I_{ss}; i.e., preserving the closure of I_{ss} and ensuring $p_{ss} | I_{ss} \subseteq p | I_{ss}$. Moreover, P_3 takes an action only in $\neg I_{ss}$. Thus, no action violates the closure of I_{ss}, and the second constraint of the output of Problem 3.1 holds.

Livelock freedom. To show livelock freedom, we prove that the included actions have no circular dependencies. Due to read/write restrictions, none of the first three processes executes based on the local variables of another process. Moreover, each process can update only its own y value. Once any one of the processes P_0, P_1 and P_2 updates its y value, it disables itself. Thus, the actions of one process cannot enable/disable another process. Moreover, since each action disables itself, there are no self-loops either. The guards of the actions of P_3 cannot be simultaneously *true*, and the execution of one cannot enable another (because they only update the value of sat). Only processes P_0, P_1 and P_2 can make the predicate $(Iden \wedge Clauses)$ *true* when $sat = 0$. Due to write restrictions, once P_3 sets sat to 1 from states $(Iden \wedge Clauses) \wedge (sat = 0)$, a state in I_{ss} is reached. Therefore, there are no cycles that start in $\neg I_{ss}$ and exclude any state in I_{ss}.

Deadlock Freedom. We illustrate that, in every state in $\neg I_{ss} \equiv (\neg(Iden \wedge Clauses) \vee (sat = 0))$, there is at least one action that is enabled.

- *Case 1.* $((Iden \wedge Clauses) \wedge (sat = 0))$ holds. In these states, the first action of P_3 is enabled. Thus, there are no deadlocks in this case.
- *Case 2.* $(\neg(Iden \wedge Clauses) \wedge (sat = 1))$ holds. In this case, the second action of P_3 is enabled. Thus, there are no deadlocks in this case.
- *Case 3.* $(\neg(Iden \wedge Clauses) \wedge (sat = 0))$ holds. None of the actions of P_3 are enabled in this case. Nonetheless, since $\neg(Iden \wedge Clauses)$ holds, either $\neg Iden$ or $\neg Clauses$, or both are *true*. When $\neg Clauses$ holds, there must be some state predicate $PredC_i$ $(i \in \mathbb{N}_k)$ that is *false*. (Recall that, the invariant I_{ss} includes a state predicate $PredC_i \equiv (x_0 = q \Longrightarrow y_0 = b_0^i) \vee (x_1 = r \Longrightarrow y_1 = b_1^i) \vee (x_2 = s \Longrightarrow y_2 = b_2^i)$ corresponding to each clause $C_i \equiv (l_q \vee l_r \vee l_s)$ in the instance of 3-SAT.) This means that the following three state predicates are *false*: $(x_0 = q \Longrightarrow y_0 = b_0^i), (x_1 = r \Longrightarrow y_1 = b_1^i)$ and $(x_2 = s \Longrightarrow y_2 = b_2^i)$. Since the instance of 3-SAT is satisfiable, at least one of the literals l_q, l_r or l_s must be true. As a result, based on the way we have included the actions depending on the truth-values of the propositional variables, at least one of the following actions must have been included in p_{ss}: $(x_0 = q \wedge y_0 \neq b_0^i \wedge sat = 0) \rightarrow y_0 := b_0^i$, $(x_1 = r \wedge y_1 \neq b_1^i \wedge sat = 0) \rightarrow y_1 := b_1^i$, and $(x_2 = s \wedge y_2 \neq b_2^i \wedge sat = 0) \rightarrow y_2 := b_2^i$. Thus, there is some action that is enabled when $\neg Clauses$ holds. A similar reasoning implies that there exists some action that is enabled when $\neg Iden$ holds; hence no deadlocks in Case 3.

Based on the closure of the invariant I_{ss}, deadlock freedom in $\neg I_{ss}$ and lack of non-progress cycles in $p_{ss}|\neg I_{ss}$, it follows that the resulting program p_{ss} is self-stabilizing.

Example 4.4. Example construction:

In the example discussed in this section, the formula ϕ has a satisfying assignment of $v_0 = 1$, $v_1 = 0$, $v_2 = 0$. Using this value assignment, we include the following actions in the first three processes P_j where $j \in \mathbb{N}_3$:

$$x_j = 0 \wedge y_j = 0 \wedge sat = 0 \rightarrow y_j := 1$$

$$x_j = 1 \wedge y_j = 1 \wedge sat = 0 \rightarrow y_j := 0$$

$$x_j = 2 \wedge y_j = 1 \wedge sat = 0 \rightarrow y_j := 0$$

The actions of P_3 are as follows:

$$(Iden \wedge Clauses) \wedge sat = 0 \rightarrow sat := 1$$

$$\neg(Iden \wedge Clauses) \wedge sat = 1 \rightarrow sat := 0$$

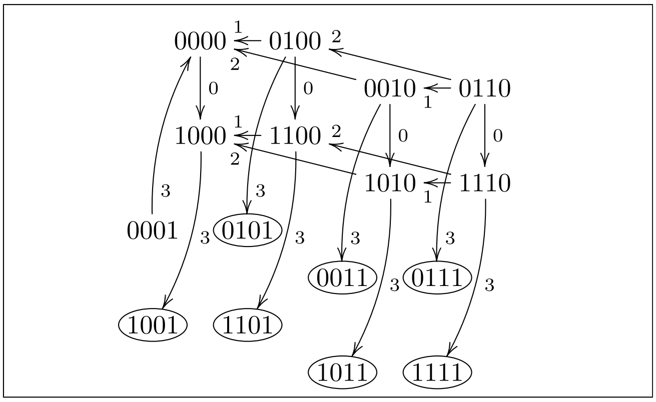


Fig. 3. $x_0 = 0$, $x_1 = 1$, $x_2 = 2$.

Fig. 3 shows the set of states where $x_0 = 0$, $x_1 = 1$, $x_2 = 2$ and transitions of the stabilizing program p_{ss}. Each state is represented by four bits which signify the values of (y_0, y_1, y_2, sat). Invariant states are depicted by ovals and the labels on the transitions denote which process executes that transition.

Lemma 4.5. *If there is a self-stabilizing version of the instance of Problem 3.3 where $\mathcal{F}$ = unfair, then the corresponding 3-SAT instance has a satisfying valuation.*

By assumption, we consider a program p_{ss} to be a self-stabilizing version of p from I_{ss}. That is, p_{ss} satisfies the requirements of Problem 3.3.

Only P_3 can correct $(sat = 0)$. Clearly, p_{ss} must preserve the closure of I_{ss}, and should not have any deadlocks or livelocks in the states in $\neg I_{ss} \equiv (\neg(Iden \wedge Clauses) \vee (sat = 0))$. Thus, p_{ss} must include actions that correct $\neg(Iden \wedge Clauses)$ and $(sat = 0)$. Since p_{ss} must adhere to the read/write restrictions of p, only P_3 can correct $(sat = 0)$ to $(sat = 1)$. For the same reason, P_3 cannot contribute to correcting $\neg(Iden \wedge Clauses)$; only P_0, P_1 and P_2 have the write permissions to do so by updating their own y values.

The rest of the reasoning is as follows: We first illustrate that P_0, P_1 and P_2 in p_{ss} must not execute in states where $(sat = 1)$. Then, we draw a correspondence between actions included in p_{ss} and how propositional variables get unique truth-values in 3-SAT and how the clauses are satisfied.

P_0, P_1 and P_2 can be enabled only when $(sat = 0)$. We observe that no process P_j $(j \in \mathbb{N}_3)$ can have a transition that starts in the invariant I_{ss}; otherwise, the constraint $\delta_{p_{ss}}|I_{ss} \subseteq \delta_p|I_{ss}$ would be violated. We also show that no recovery action of P_0, P_1 and P_2 can include a transition that starts in a state where $sat = 1$. By contradiction, assume that some P_j $(j \in \mathbb{N}_3)$ includes a transition (s_0, s_1) where $s_0 \in \neg I_{ss}$ and $sat(s_0) = 1$ for some fixed values of x_j and y_j. Since P_j cannot read x_i or y_i of other processes P_i, where $(i \in \mathbb{N}_3) \wedge (i \neq j)$, the transition (s_0, s_1) has a groupmate (s_0', s_1'), where $x_i(s_0') = x_j(s_0')$ and $y_i(s_0') = y_j(s_0')$ for all $i \in \mathbb{N}_3$ where $(i \neq j)$. Thus, $Iden$ is true at s_0'. Moreover, due to the form of the 3-SAT instance, no clause $(l_q \vee l_r \vee l_s)$ exists such that $(q = r = s)$. Thus, $Clauses$ holds at s_0' as well, thereby making s_0' an invariant state. As a result, (s_0, s_1) is grouped with a transition that starts in I_{ss}, which again violates the constraint $\delta_{p_{ss}}|I_{ss} \subseteq \delta_p|I_{ss}$. Hence, P_0, P_1 and P_2 can be enabled only when $(sat = 0)$.

Actions of P_3. We show that P_3 must set sat to 0 when $\neg(Iden \wedge Clauses) \wedge sat = 1$ and may only assign sat to 1 when $(Iden \wedge Clauses) \wedge sat = 0$. As shown above, P_0, P_1, and P_2 cannot act when $sat = 1$, forcing P_3 to execute from $\neg(Iden \wedge Clauses) \wedge (sat = 1)$. P_3 must therefore have the action $\neg(Iden \wedge Clauses) \wedge sat = 1 \rightarrow sat := 0$. Consequently, P_3 cannot assign sat to 1 when $\neg(Iden \wedge Clauses) \wedge sat = 0$; otherwise, it would create a livelock with the previous action. From states where $(Iden \wedge Clauses) \wedge sat = 0$ holds, P_3 is the only process which can change sat to 1, thereby reaching an invariant state. Thus, P_3 must include the actions $\neg(Iden \wedge Clauses) \wedge sat = 1 \rightarrow sat := 0$ and $(Iden \wedge Clauses) \wedge sat = 0 \rightarrow sat := 1$.

Each P_j*, for* $j \in \mathbb{N}_3$ *must have exactly one action for each unique value of* x_j. When $sat = 0$, fixing the value of x_j to some $a \in \mathbb{N}_n$ reduces the possible local states for process P_j to 2, where $y_j = 0$ or $y_j = 1$ for $j \in \mathbb{N}_3$. (Notice that both of these states are illegitimate since $sat = 0$.) Thus, when $(x_j = a \wedge sat = 0)$ holds, process P_j has four possible actions: $y_j = 0 \rightarrow y_j := 0$, $y_j = 0 \rightarrow y_j := 1$, $y_j = 1 \rightarrow y_j := 0$, and $y_j = 1 \rightarrow y_j := 1$. It is clear that the first and last of these actions are self-loops and cannot be included. Thus, P_j can have either action $y_j = 0 \rightarrow y_j := 1$ or $y_j = 1 \rightarrow y_j := 0$, but not both without creating a livelock. That is, P_j cannot have more than 1 action. To make $Iden$ true, P_j must include some action. By contradiction, assume that P_j has no actions. Another process P_i ($i \in \mathbb{N}_3$, $i \neq j$) can be in a state where $x_i = x_j$. There are two possibilities for the y values in this non-invariant state, $y_j = 0 \wedge y_i = 1$ or $y_j = 1 \wedge y_i = 0$. P_i can resolve either scenario with an action but cannot resolve both as this would require two actions. That is, to resolve both cases P_i needs the cooperation of P_j. Thus, P_j must have some action. Since P_j cannot have more than one action, it follows that P_j has exactly one action.

Truth-value assignment to propositional variables. Based on the above reasoning, for each value $a \in \mathbb{N}_n$, if a process P_j includes the action $x_j = a \wedge y_j = 0 \wedge sat = 0 \rightarrow y_j := 1$, then we assign *true* to the propositional variable v_a. If P_j includes the action $x_j = a \wedge y_j = 1 \wedge sat = 0 \rightarrow y_j := 0$, then we assign *false* to v_a. Let P_j include the action $x_j = a \wedge y_j = 0 \wedge sat = 0 \rightarrow y_j := 1$. By contradiction, if another process P_i, where $i \in \mathbb{N}_3 \wedge i \neq j$, includes the action $x_i = a \wedge y_i = 1 \wedge sat = 0 \rightarrow y_i := 0$, then $Iden$ would be violated and p_{ss} would never recover from the state $x_j = a \wedge x_i = a \wedge y_j = 1 \wedge y_i = 0 \wedge sat = 0$; i.e., a deadlock state, which is a contradiction with p_{ss} being self-stabilizing. Thus, each propositional variable gets a unique truth-value assignment and these value assignments are logically consistent.

Satisfying the clauses. Since p_{ss} is self-stabilizing from I_{ss}, eventually I_{ss} becomes *true*; i.e., every $PredC_i$ in the $Clauses$ predicate becomes *true*. The one-to-one correspondence created by the mapping between each state predicate $PredC_i$ and each clause C_i implies that $PredC_i$ *holds iff at least one literal in* C_i *holds*. Therefore, all clauses are satisfied with the truth-value assignment based on the actions of p_{ss}.

Theorem 4.6. *Adding stabilization to low atomicity programs is NP-complete.*

Proof. The NP-hardness of adding stabilization follows from Lemmas 4.5 and 4.3. The NP membership of adding stabilization has already been established in [6]; hence the NP-completeness. □

Corollary 4.7. *Adding nonmasking fault tolerance to low atomicity programs under no fairness is NP-complete.*

Proof follows from Theorem 4.6 and the fact that Problem 3.3 is a special case of Problem 3.2.

4.3 Hardness under Weak Fairness

This section illustrates that, even under the assumption of weak fairness, addition of nonmasking fault tolerance in general, and self-stabilization in particular remain hard problems.

Theorem 4.8. *Adding stabilization under weak fairness is NP-complete.*

Proof. Consider the mapping from 3-SAT to Problem 3.3 presented in Section 4.2. We leverage the same mapping in order to create a mapping from an instance of 3-SAT to an instance of Problem 3.3 where $\mathcal{F} =$ weak. Let p_{ss} denote the instance of Problem 3.3 with the invariant $I_{ss} \equiv Clauses \wedge Iden \wedge (sat = 1)$ constructed corresponding to the 3-SAT formula. The instance of Problem 3.3 where $\mathcal{F} =$ weak includes exactly the same processes and variables in p_{ss}. Moreover, we compose p_{ss} with the token ring program introduced in Section 2. Since the state space of the TR program includes 27 states, the size of the state space of the instance of Problem 3.3 under weak fairness remains polynomial in the size of the 3-SAT formula. (The size of the state space of the instance of Problem 3.3 where $\mathcal{F} =$ weak is $27 \times |S_{p_{ss}}|$.) Let the invariant of the resulting program be equal to the conjunction of the invariants of the two programs; i.e., $I_w \equiv I_{ss} \wedge I_{TR}$. Thus, the resulting composed program will converge to I_w *iff* both p_{ss} and the TR program stabilize to their corresponding invariants.

$\Rightarrow$ *Proof*: We show that if the 3-SAT instance is satisfiable then the composition of p_{ss} and TR is self-stabilizing from I_w under weak fairness. If the 3-SAT formula is satisfiable then p_{ss} is strongly stabilizing from I_{ss}. Moreover, Dijkstra [1] has illustrated that the TR program is strongly stabilizing. Outside I_w, if I_{ss} has become true and I_{TR} is false, then the TR program will eventually recover to I_{TR}. If the TR program has recovered to its invariant, but p_{ss} has not yet recovered to I_{ss}, then there must be some action A of p_{ss} that is enabled (because the 3-SAT formula is satisfiable). At the same time, TR's computations in I_{TR} are infinite. That is, the action A is continuously enabled in a cycle formed in the state predicate $\neg I_{ss} \wedge I_{TR}$. Such a cycle is a non-progress cycle only under no fairness assumption; i.e., under a weakly fair scheduler, the composed program will eventually stabilize to I_w.

$\Leftarrow$ *Proof*: Let there be a program p_w composed of the variables of TR and p_{ss}, and actions that enable stabilization to I_w from any state under weak fairness. That is, I_{ss} and I_{TR} must both become true eventually. Our proof strategy is two-fold. First, we make the following observations to enable compositional reasoning about the two components of p_w:

Observation 4.9. Only processes of TR can make I_{TR} *true* and only processes of p_{ss} can contribute to reaching I_{ss} from any state.

Proof of Observation 4.9 is straightforward due to the read/write restrictions of processes and the independence of the two components in reading/writing the variables of each other. That is, even if the actions of the two components get interleaved, they will not impact the recovery of each component to its invariant under weak fairness. Second, since the TR component does not intervene the convergence of p_{ss} to I_{ss} (based on Observation 4.9), we can reason about p_{ss} separately. We prove that, even under weak fairness, the p_{ss} component of p_w should be strongly self-stabilizing from I_{ss}. This way, we can reuse the proof of Lemma 4.5 to demonstrate how the instance of 3-SAT is satisfied. To this end, we prove that neither P_3 nor P_0, P_1 and P_2 can have any cycles in $\delta_{p_{ss}}|\neg I_{ss}$.

- *Actions of P_3 alone cannot create a cycle in $\delta_{p_{ss}}|\neg I_{ss}$.* From the proof of Lemma 4.5, we know that P_0, P_1 and P_2 can only execute if $sat = 0$; otherwise, the closure of I_{ss} would be violated. The only way P_3 can have a cycle in $\delta_{p_{ss}}|\neg I_{ss}$ is to toggle the value of sat (because only P_3 has the permission to write sat). Since P_0, P_1 and P_2 can only execute if $sat = 0$, no action would be continuously enabled in this cycle. This would constitute a non-progress cycle under weak fairness, which is a contradiction with p_w having no non-progress cycles under weak fairness.
- *There is no cycle in $\delta_{p_{ss}}|\neg I_{ss}$ where multiple processes participate.* Since the actions of processes P_0, P_1 and P_2 in p_{ss} are independent from each other, it is impossible that a cycle exists in which P_0, P_1 and P_2 participate. Moreover, there is no cycle that is formed by the interleaving of the actions of P_0, P_1 and P_2 with P_3's actions since all processes of p_{ss} would be participating in such a cycle; i.e., none of the actions of p_{ss} would be continuously enabled. Such a cycle would constitute a non-progress cycle under weak fairness.
- *Processes P_i ($i \in \mathbb{N}_3$) of p_{ss} cannot form any cycles alone in $\delta_{p_{ss}}|\neg I_{ss}$.* By contradiction, consider a case where some P_i contains a cycle including the actions $x_i = a \wedge y_i = 0 \wedge sat = 0 \rightarrow y_i := 1$ and $x_i = a \wedge y_i = 1 \wedge sat = 0 \rightarrow y_i := 0$. These two actions capture an equivalence class of cycles in the state space of p_{ss}. Each cycle in this equivalence class includes two global states where $x_i = a \wedge y_i = 0 \wedge sat = 0$ holds in one and $x_i = a \wedge y_i = 1 \wedge sat = 0$ holds in the other. Consider another process P_j, where $j \in \mathbb{N}_3$ and $j \neq i$. Either P_j is not in any cycles, or P_j is also trapped in a cycle similar to P_i's. The former case means that, by weak fairness, P_j will get to a state where all its actions are disabled. In this case, the cycles of P_i become non-progress cycles under weak fairness. In the latter case, both P_i and P_j would be in a cycle in which no action is continuously enabled; hence a non-progress cycle under weak fairness. Now, we illustrate that P_3 cannot help P_i to exit its cycle either. Toggling the value of y_i would affect the truth-value of the predicates $PredC_m$ that depend on the state of P_i, where $m \in \mathbb{N}_k$. This in turn could change the truth value of the predicate $Iden \wedge Clause$. Since the actions of P_3 must include $Iden$ and $Clause$ in their guards[1], P_3 cannot be continuously enabled in the cycle of P_i. Thus, the cycle of P_i forms a non-progress cycle under weak fairness, which is a contradiction with p_w being self-stabilizing from I_w under weak fairness.

Since p_{ss} must be a strongly stabilizing program, the proof of Lemma 4.5 can be reused to demonstrate that the instance of 3-SAT is satisfied. □

Corollary 4.10. *Adding nonmasking fault tolerance under the assumption of weak fairness is NP-complete.*

Proof of Corollary 4.10 follows from Theorem 4.8 and the fact that Problem 3.3 is a special case of Problem 3.2 where $\mathcal{F} =$ weak.

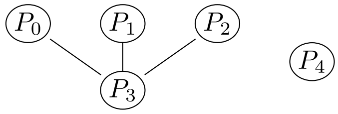
4.4 Hardness under Strong Fairness

In this section, we present a somewhat surprising result that adding nonmasking fault tolerance to low atomicity programs remains NP-hard even under strong fairness! This is surprising because adding self-stabilization under strong fairness (a.k.a. *weak stabilization* [11]) is known to be polynomial [11], [12], [13]. Our proof strategy is as follows. We first reuse the reduction presented in the proofs of Lemmas 4.3 and 4.5 to illustrate that adding nonmasking tolerance to low atomicity programs under no fairness is NP-hard. This may seem as a redundant result to Corollary 4.7, however, in the second step of our strategy, we reuse the mapping and reduction of this proof for showing the NP-hardness of adding nonmasking tolerance under strong fairness.

An alternative proof for the NP-hardness of adding nonmasking fault tolerance under no fairness (i.e., Corollary 4.7). First, we present a mapping from an arbitrary instance of 3-SAT to an instance of adding nonmasking fault tolerance (i.e., Problem 3.2). In Section 4.2.1, we augment the instance of Problem 3.3 where $\mathcal{F} =$ unfair with an additional process and two new types of faults. (Fig. 2 depicts the structure of the instance of Problem 3.2.) The idea behind this mapping is that finding a fault-span and a new invariant $I' \subseteq I$ for an intolerant program p with its invariant I is at least as hard as adding stabilization.

Fig. 4 illustrates the structure of our mapping for adding nonmasking fault tolerance. Processes P_0 to P_3 are taken from the system of Fig. 2. We add a new process P_4 that has a read-only binary variable $failed$ used to mark unrecoverable states. The invariant of the intolerant program p is $I \equiv I_{ss} \wedge (failed = 0)$ where $I_{ss} \equiv Iden \wedge Clauses \wedge (sat = 1)$ is the invariant of the system of p_{ss} in Fig. 2. Any state where $failed = 1$ is unrecoverable since $failed$ cannot be modified by any process. We consider two classes of faults f_{ss} and f_n denoted by $f \equiv f_{ss} \cup f_n$, where f_{ss} and f_n are defined as:

1. Otherwise, P_3 would include two actions $sat = 1 \rightarrow sat := 0$ and $sat = 0 \rightarrow sat := 1$ forming a cycle, whose impossibility we have already shown in the first item of our reasoning.



$P_0 \dots P_3$: Unchanged from Figure 2.

P_4 : $\{ failed \in \mathbb{N}_2$ (read-only)

Invariant: $I \equiv I_{ss} \wedge (failed = 0)$
Faults: $f \equiv f_{ss} \cup f_n$

Fig. 4. Instance of Problem 3.2 where $\mathcal{F}$ = unfair.

$$\begin{aligned} f_{ss}: \quad I \quad \rightarrow \quad & x_0 := select(\mathbb{N}_n);\ y_0 := select(\mathbb{N}_2); \\ & x_1 := select(\mathbb{N}_n);\ y_1 := select(\mathbb{N}_2); \\ & x_2 := select(\mathbb{N}_n);\ y_2 := select(\mathbb{N}_2); \\ & sat := 0 \end{aligned}$$

$$f_n: \quad \neg I_{ss} \wedge sat = 1 \quad \rightarrow \quad failed := 1$$

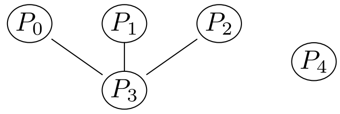
Faults ensure that a nonmasking f-tolerant program p' from I exists *iff* p_{ss} is stabilizing from I_{ss}. The fault f_{ss} may occur from states in $I' \subseteq I$ and perturb the program to any state where $sat = 0$ ($failed = 0$ is unchanged) and all x_i and y_i values are randomly chosen by the random function *select*. The fault-class f_n transitions to an unrecoverable state ($failed = 1$) when I_{ss} does not hold but $sat = 1$. In effect, P_3 is forced to assign $sat := 1$ only when $Iden \wedge Clauses$ holds. P_0 to P_2 must act to satisfy $Iden \wedge Clauses$ when $sat = 0$, preserving our mapping between program actions and a 3-SAT truth-assignment.

The size of the state space $|S_p|$ remains polynomial in the number of propositional variables n in the corresponding 3-SAT instance, specifically $|S_p| = 2^2(2n)^3$. It remains to show that a satisfying truth-assignment exists for the 3-SAT instance *iff* a nonmasking f-tolerant version of the instance in Fig. 4 exists from $I \equiv I_{ss} \wedge (failed = 0)$.

$\Rightarrow$ *Proof*: Given a satisfying valuation for a 3-SAT instance, we can create the corresponding stabilizing program p_{ss} with invariant I_{ss} for the system in Fig. 2 using the method in Lemma 4.3. Using all actions from p_{ss}, we can form a nonmasking f-tolerant program $p_{ft} = p_{ss}$ with invariant $I_{ft} \equiv I$ for this system.

For proof of p_{ft} being nonmasking f-tolerant from I, let us calculate its f-span. From a state in I, we can reach any state where $sat = 0$ and $failed = 0$ due to the occurrence of faults f_{ss}. Since only f_{ss} can occur from I, and $sat = 0$ holds after the occurrence of f_{ss}, f_n never gets enabled. Moreover, from the state predicate $\neg I_{ss} \wedge sat = 0$ computations of p will first satisfy $Iden \wedge Clauses$ and reach the invariant with a final action from P_3 which assigns $sat := 1$. At no point does P_3 assign $sat := 1$ when $Iden \wedge Clauses$ does not hold, leaving all source states of f_n out of the f-span. Thus, the f-span of p from I, denoted T, is equal to $(I_{ss} \vee sat = 0) \wedge failed = 0$, from where every computation eventually reaches I.

$\Leftarrow$ *Proof*: Let p' be a nonmasking f-tolerant program from an invariant $I' \subseteq I$ that meets the constraints of Problem 3.1 from a f-span T for the instance built in our mapping (see Fig. 4). The proof strategy is to show that a strongly stabilizing program p_{ss} for the corresponding system in Fig. 2 can be constructed from p', and then we shall reuse the proof of Lemma 4.5 to satisfy the 3-SAT instance. Observe that all states where $failed = 1$ must be excluded from T because recovery is impossible from $failed = 1$ due to write restrictions. Moreover, states where $\neg I_{ss} \wedge sat = 1$ holds cannot be in T either, otherwise f_n could assign $failed := 1$. Thus, the weakest and strongest predicates that can be considered as T are respectively equal to $(I \vee sat = 0) \wedge failed = 0$ (note I, not I') and $(I' \vee sat = 0) \wedge failed = 0$. From I', the occurrence of f_{ss} can perturb the state of the program to any state where $sat = 0 \wedge failed = 0$ holds. Thus, recovery to I' should be provided from $sat = 0$. In such states, either $(Iden \wedge Clauses)$ holds or not. If $(Iden \wedge Clauses)$ holds in states where $sat = 0$, then p' can recover to I' only with an action of P_3 that sets sat to 1. If $(Iden \wedge Clauses)$ does not hold when $sat = 0$, then P_3 must not set sat to 1 because then the state of p' will reach $\neg I_{ss} \wedge sat = 1$ from where fault f_n can occur and set $failed$ to 1, which is an unrecoverable state. The only processes that have read/write permission to make $(Iden \wedge Clauses)$ true are P_0, P_1 and P_2. Thus, p' must provide recovery from $\neg(Iden \wedge Clauses)$ to $(Iden \wedge Clauses)$ when $sat = 0$. We can use these actions, along with actions $\neg(Iden \wedge Clauses) \wedge sat = 1 \rightarrow sat := 0$ and $Iden \wedge Clauses \wedge sat = 0 \rightarrow sat := 1$ of P_3, to construct a program p_{ss} which is self-stabilizing for the corresponding instance given in Fig. 2. From this point, we use Lemma 4.5 on p_{ss} to find a truth-assignment which satisfies the 3-SAT instance.



$P_0 \dots P_3$: Unchanged from Figure 2.

P_4 : $\begin{cases} failed \in \mathbb{N}_2 & \text{(read-only)} \\ z_0, z_1, z_2 \in \mathbb{N}_3 & \text{(read \& write)} \end{cases}$

Invariant: $I \equiv I_{ss} \wedge (z_0 = z_1 = z_2 = 0) \wedge (failed = 0)$
Faults: $f \equiv f_{ss} \cup f_n \cup f_{s0} \cup f_{s1} \cup f_{s2}$

Fig. 5. Instance of Problem 3.2 where $\mathcal{F}$ = strong.

Theorem 4.11. *Adding nonmasking fault tolerance to low atomicity programs under strong fairness is NP-complete.*

Proof. Our proof strategy is to augment the mapping presented in the alternative proof of Corollary 4.7 and then show that the instance of 3-SAT is satisfiable *iff* nonmasking fault tolerance can be added to the instance of Problem 3.2 where $\mathcal{F}$ = strong. The proposed polynomial-time mapping is as follows. We construct an intolerant program as demonstrated in Fig. 5. Processes P_0 to P_3 are the same as those in the program of Fig. 4. We include three new variables z_0, z_1 and z_2 in process P_4 which can be read and written only by P_4. The domain of each z_i, where $i \in \mathbb{N}_3$, is equal to $\{0, 1, 2\}$. We also consider a new fault-class f_{si} for $i \in \mathbb{N}_3$. The invariant of the instance of Problem 3.2 is $I \equiv I_{ss} \wedge (z_0 = z_1 = z_2 = 0) \wedge (failed = 0)$.

The classes of faults include $f \equiv f_{ss} \cup f_n \cup f_{s0} \cup f_{s1} \cup f_{s2}$, where f_{ss} and f_n are taken from Fig. 4, and f_{si} is defined as follows ($i \in \mathbb{N}_3$):

$$f_{si}: \quad y_i = 0 \wedge sat = 0 \wedge z_i = 0 \quad \rightarrow \quad z_i := 1$$
$$f_{si}: \quad y_i = 1 \wedge sat = 0 \wedge z_i = 1 \quad \rightarrow \quad z_i := 2$$
$$f_{si}: \quad y_i = 0 \wedge sat = 0 \wedge z_i = 2 \quad \rightarrow \quad failed := 1$$

The new fault-class f_{si} ensures that the processes P_0, P_1 and P_2 of any f-tolerant program p' do not form nontrivial cycles in the state predicate $sat = 0$ if p' is nonmasking f-tolerant from $I' \subseteq I$ under strong fairness. Without f_{si}, it would be trivial to add fault tolerance under strong fairness by including the actions $y_i = 0 \wedge sat = 0 \rightarrow y_i := 1$ and $y_i = 1 \wedge sat = 0 \rightarrow y_i := 0$ in P_i for each specific value of x_i, where $i \in \mathbb{N}_3$, and the action $Iden \wedge Clauses \wedge sat = 0 \rightarrow sat := 1$ in P_3.

Observe that the size of the state space $|S_p|$ remains polynomial in the number of propositional variables n from the corresponding 3-SAT instance as $|S_p| = 2^2(6n)^3$. Now, we illustrate that a satisfying truth-value assignment exists for the 3-SAT instance *iff* a nonmasking f-tolerant version of the instance of Problem 3.2 exists where $\mathcal{F} =$ strong.

$\Rightarrow$ *Proof*: Given a satisfying valuation for the 3-SAT instance, we can create a nonmasking f-tolerant program p' with invariant $I' = I$ as specified in Fig. 5, where $f \equiv f_{ss} \cup f_n \cup f_{s0} \cup f_{s1} \cup f_{s2}$. From I, f_{ss} can perturb the program to states where $sat = 0 \wedge failed = 0$. Thus, states in $\neg(Iden \wedge Clauses) \wedge sat = 1$ are unreachable in the f-span of p' from I, thereby ensuring that f_n cannot take the program to the unrecoverable state $failed = 1$. Moreover, the state $y_i = 0 \wedge z_i = 2$ must be excluded from the f-span; otherwise, fault f_{si} could perturb the program state to $failed = 1$. Thus, the weakest predicate we can consider to be the f-span of p' from I is equal to $T \equiv (I_{ss} \vee sat = 0) \wedge (failed = 0) \wedge (y_0 = 1 \vee z_0 \neq 2) \wedge (y_1 = 1 \vee z_1 \neq 2) \wedge (y_2 = 1 \vee z_2 \neq 2)$.

We include the actions of P_0, P_1 and P_2 in p' based on the method outlined in the proof of Lemma 4.3. Thus, only one of the actions $x_i = a \wedge y_i = 0 \wedge sat = 0 \rightarrow y_i := 1$ and $x_i = a \wedge y_i = 1 \wedge sat = 0 \rightarrow y_i := 0$ is included in each process P_i, where $i \in \mathbb{N}_3$. Process P_3 includes the actions $(Iden \wedge Clauses) \wedge sat = 0 \rightarrow sat := 1$ and $\neg(Iden \wedge Clauses) \wedge sat = 1 \rightarrow sat := 0$. Finally, the process P_4 includes the actions $z_i \neq 0 \rightarrow z_i := 0$ for $i \in \mathbb{N}_3$.

We show that the program p' (with the aforementioned actions) is nonmasking f-tolerant from I under strong fairness. Once p' is perturbed to $T - I$, recovery to I is achieved as follows. The processes P_0, P_1 and P_2 ensure that $(Iden \wedge Clauses)$ is satisfied, and then P_3 sets sat to 1. Moreover, P_4 sets z_i to 0, thereby recovering to I. Using Fig. 6, we show that no computation prefix of $p' [] f$ from invariant I reaches the state $failed = 1$ even if faults f_{si} occur.

The two values in Fig. 6 respectively denote the values of y_i and z_i, where $i \in \mathbb{N}_3$, $sat = 0$ and x_i is fixed. These variables are only affected by processes P_i and P_4 and the fault-class f_{si}. Dashed arrows represent the two *possible* actions of P_i if P_i included both actions that change y_i (i.e., $y_i = 0 \wedge sat = 0 \rightarrow y_i := 1$ and $y_i = 1 \wedge sat = 0 \rightarrow y_i := 0$), of which exactly one is chosen in our construction of p' (for each unique x_i value). Since only one action is chosen, there exists no computation prefix of $p' [] f$ from invariant I to an unrecoverable state where $failed = 1$. Notice that without the fault-class f_{si}, the program that includes both actions that change y_i would have been nonmasking f-tolerant under strong fairness because the cycles formed in $p' | (T - I)$ are not livelocks under strong fairness. Moreover, from every state in $T - I$ there is an enabled action; i.e., deadlock freedom in $T - I$. Thus, p' is nonmasking f-tolerant from I under strong fairness. □

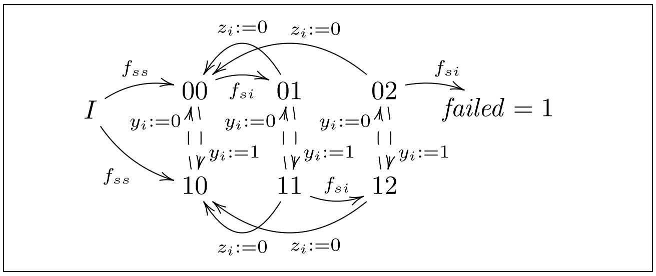


Fig. 6. Effects of f_{si}.

$\Leftarrow$ *Proof*: Given a program p' that is nonmasking f-tolerant under strong fairness and meets the constraints of Problem 3.2, we build a program p_{ft} with invariant $I_{ft} \equiv I_{ss} \wedge failed = 0$ that is nonmasking $f_{ss} \cup f_n$-tolerant from I_{ft} under no fairness (see Fig. 4). We note that, in the presence of faults f_{si}, P_4 must have actions to eventually assign 0 to z_i (required by invariant) from any nonzero value of z_i which is reached in the fault-span. Thus, in Fig. 6, P_4 transitions simply assign $z_i := 0$. Program p_{ft} includes the actions of P_0 to P_3 which do not form self-loops and recover from states where $sat = 0$. Actions of p_{ft} map to a satisfying truth-assignment for the instance of 3-SAT. Notice that, p_{ft} does not tolerate f_{si}.

The fault-span T_{ft} of p_{ft} is a subset of the fault-span T of p' since program and fault transitions of p_{ft} are a subset of those transitions of p'. It follows that T_{ft} does not contain unrecoverable states ($failed = 1$) nor does it include states $\neg I_{ss} \wedge sat = 1$ from where f_n could bring p_{ft} to an unrecoverable state. Thus, $T_{ft} \equiv (I_{ss} \vee sat = 0) \wedge failed = 0$ due to the definition of I_{ft}, fault class f_{ss}, and the excluded states which lead to $failed = 1$. Clearly a computation exists in p_{ft} from every state in T_{ft} to its invariant I_{ft} since $I' \subseteq I_{ft}$ (modulo z variables) and p' eventually reaches I' from all states in its fault-span T. The only states where $sat = 1$ holds in T_{ft} are also in I_{ft}. Moreover, we argue that actions of P_3 from states where $sat = 0$ (and are not self-loops) bring the system to a state where I_{ss} holds. Otherwise, some action would exist to set sat to 1 while preserving $\neg I_{ss}$. As a result, f_n could be enabled and could take the program state to $failed = 1$, which would be a contradiction with p' being nonmasking f-tolerant from some non-empty subset of I. Thus, P_3 actions only set sat to 1 when the resulting state is in I_{ss}.

Let us now show that if p' is to be nonmasking f-tolerant under strong fairness, then none of the processes P_i ($i \in \mathbb{N}_3$) can form nontrivial cycles in states $sat = 0$. When $sat = 0$, y_i and z_i take values shown in Fig. 6. We illustrate that, for a specific x_i value, each process P_i must have only one action that updates y_i to ensure no computation prefix of $p'[]f$ reaches $failed = 1$. By contradiction, assume P_i ($i \in \mathbb{N}_3$) has a non-trivial cycle for some fixed $x_i = a$ ($a \in \mathbb{N}_n$). Since the cycle exists in $p'|(sat = 0)$, therefore P_i must have actions $x_i = a \wedge y_i = 0 \wedge sat = 0 \rightarrow y_i := 1$ and $x_i = a \wedge y_i = 1 \wedge sat = 0 \rightarrow y_i := 0$. Now, we demonstrate the following computation prefix that reaches $failed = 1$.

1. Transitions of f_{ss} perturb p' to $x_i = a \wedge y_i = 0 \wedge sat = 0$, where $z_i = 0$ and $failed = 0$.
2. Then, transitions of f_{si} can occur, setting z_i to 1.
3. P_i sets y_i to 1.
4. Transitions of f_{si} occur again, setting z_i to 2.
5. P_i sets y_i to 0.
6. From this state, fault f_{si} can occur, setting $failed$ to 1 from where no recovery is possible.

Thus, P_i ($i \in \mathbb{N}_3$) cannot have cycles. Recall that when P_3 acts to change sat from 0 to 1, the resulting state must satisfy I_{ss}. As a result, the program p_{ft} constructed from the actions of processes $P_0 \ldots P_3$ when $sat = 0$ is nonmasking $(f_{ss} \cup f_n)$-tolerant from I_{ft}. The actions of p_{ft} can be mapped to a satisfying truth-assignment for the instance of 3-SAT. □

We now discuss the impact of our hardness results on failsafe and masking fault tolerance. Failsafe fault tolerance does not require recovery to invariant. Thus, the issue of fairness is irrelevant for failsafe fault tolerance. For masking fault tolerance, we observe that in the proof of NP-completeness of Problem 3.3 under no fairness in Section 4.2.2, one can consider the write restrictions of each process as part of a safety property where a process P_j is not allowed to write any x_j, where $0 \leq j \leq 2$. Thus, it follows that adding stabilization under no fairness would become an instance of the problem of adding masking fault tolerance. This way, we simply reuse the proof of NP-completeness of adding strong stabilization to prove the NP-completeness of adding masking fault tolerance under no fairness. (This result matches with Kulkarni and Arora's results in [6].) The hardness of adding masking fault tolerance under weak and strong fairness follow accordingly from the NP-completeness proofs of this section.

Corollary 4.12. *Adding masking fault tolerance is NP-complete under weak or strong fairness.*

5 Discussion

This section discusses algorithmic design of self-stabilization, complexity of algorithmic design and fairness assumptions. Existing methods for the algorithmic design of self-stabilization include constraint-based methods [26] and sound heuristics [13], [27]. Abujarad and Kulkarni [26] consider the program invariant as a conjunction of a set of local constraints, each representing the set of local legitimate states of a process. Then, they synthesize convergence actions for correcting the local constraints. Nonetheless, they do not explicitly address cases where local constraints have cyclic dependencies (e.g., maximal matching on a ring), and their case studies include only acyclic topologies. In our previous work [13], [27], we partition the state space to a hierarchy of state predicates based on the length of the shortest computation prefix from each state to some state in the invariant. Then, we systematically explore the space of all candidate recovery transitions that could contribute in recovery to the invariant without creating non-progress cycles.

Most hardness results [6], [9], [28] presented for the addition of fault tolerance lack the additional constraint of *recovery from any state*, which we have in the addition of stabilization. The proof of NP-hardness of adding failsafe fault tolerance presented in [9] is based on a reduction from 3-SAT, nonetheless, a failsafe fault-tolerant program does not need to recover to its invariant when faults occur. The problem of adding masking fault tolerance relies on finding a subset of the state space from where recovery is possible; no need to provide recovery from every state. As such, the hardness proof presented in [6] is based on a reduction in which such a subset of state space is identified along with corresponding convergence actions *iff* the instance of 3-SAT is satisfiable. This means that some states are allowed to be excluded from the fault-span; this is not an option in the case of adding self-stabilization. The essence of the proof in [28] also relies on the same principle where Bonakdarpour and Kulkarni illustrate the NP-hardness of designing progress from one state predicate to another in low atomicity programs. Most existing algorithmic methods [6], [13], [26], [27], [28], [29] investigate the problem of adding fault tolerance under no fairness assumption. To the best of our knowledge, this paper is the first to investigate the impact of fairness on the addition of fault tolerance.

6 Conclusions and Future Work

This paper illustrates that adding nonmasking fault tolerance to low atomicity programs is an NP-hard problem under no fairness, weak, and strong fairness. In the low atomicity model, program processes have read/write restrictions with respect to the variables of other processes. The presented proof of hardness is from 3-SAT to the problem of adding stabilization to non-stabilizing programs, which is a special case of adding nonmasking fault tolerance. We first presented a proof for the NP-hardness of adding stabilization under no fairness. Then we showed that, even under weak fairness adding stabilization remains an NP-hard problem, which implies the NP-hardness of adding nonmasking tolerance under weak fairness. While it is known that adding stabilization under strong fairness (a.k.a. weak stabilization) can be done in polynomial time (in the size of state space), we showed that adding nonmasking tolerance under strong fairness remains NP-hard in general. To extend this work, we will investigate special cases where the addition of stabilization in particular and nonmasking in general can be performed efficiently. That is, *for what programs, classes of faults and invariants can the addition of nonmasking fault tolerance be done efficiently*?

Acknowledgments

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