Test Help

A tour of some ideas

Warning: not certified as typo-free. If it looks like it must be a typo, it very well might be a typo. All typos are the responsibility of the Spook from Scary Movie.
In several days, an updated and vastly extended version of this worksheet will be available at `www.math.mtu.edu/~daolson`.

1. Is there a copy of $\mathbb{Z}_2$, somewhat disguised, hiding inside $\mathbb{Z}_{10}$?

   (a) Fill out the group table, using $a = 0$ and $b = 1$ in $\mathbb{Z}_2$.

+	a	b
a		
b		

   (b) Fill out the group table, using $a = 0$ and $b = 5$ in $\mathbb{Z}_{10}$.

+	a	b
a		
b		

   (c) Make the (hopefully) obvious conclusion. If we want to use the map $\phi : \mathbb{Z}_2 \to \mathbb{Z}_{10}$ to show how $\mathbb{Z}_2$ is undercover in $\mathbb{Z}_{10}$, we would let $\phi(0) = 0$. What should $\phi(1)$ be?

   (d) Because we want ϕ to preserve group actions, we must have $\phi(0) = \phi(0 + 0) = \phi(0) + \phi(0)$. In your first high school algebra class, you probably learned how to solve $x = x + x$ for x. Is this situation any different?

2. Repeat: Is there a copy of $\mathbb{Z}_3$, somewhat disguised, hiding inside $\mathbb{Z}_{12}$?

   (a) Fill out the group table, using $a = 0$ and $b = 1$ in $\mathbb{Z}_3$ (c must be...).

+	a	b	c
a			
b			
c			

   (b) Fill out the group table, using $a = 0$ and $b = 4$ in $\mathbb{Z}_{12}$ (c must be...).

+	a	b	c
a			
b			
c			

   (c) Make the (again, hopefully) obvious conclusion. If we want to use the map $\phi : \mathbb{Z}_3 \to \mathbb{Z}_{12}$ to show how $\mathbb{Z}_3$ is undercover in $\mathbb{Z}_{12}$, we would let $\phi(0) = 0$. What should $\phi(1)$ be?

(d) Because we want the map ϕ to preserve group actions, we know that $\phi(2) = \phi(1+1) = \phi(1) + \phi(1)$. Does this work?

(e) What if some twisted soul wanted to use $\varphi(1) = 8$. What must $\varphi(2)$ be, if φ preserves the group action?

(f) Try to fill out the group table, using $a = 0$ and $b = 8$ in $\mathbb{Z}_{12}$.

+	a	b	c
a			
b			
c			

3. Take another look at Problem 1 from the test. Saying that ϕ is an injective homomorphism is just saying that $\mathbb{Z}_2$ is hiding out, in disguise, in $\mathbb{Z}_{1000}$. Phrased that way, is the answer now obvious?

4. Saying that $\phi : \mathbb{Z}_2 \longrightarrow \mathbb{Z}_{1000}$ is a homomorphism just means that it preserves group actions: $\phi(a+b) = \phi(a) + \phi(b)$. What does it mean for it to be injective?

5. Find a homomorphism $\phi : \mathbb{Z}_2 \longrightarrow \mathbb{Z}_{1000}$ which is not injective. Hint: you don't have many choices.

6. Is there a function $\phi : \mathbb{Z}_2 \longrightarrow \mathbb{Z}_{1000}$ which is surjective? Give an example or (preferably) explain why it is impossible.

7. Suppose $\phi : \mathbb{Z}_5 \longrightarrow \mathbb{Z}_{10}$ is a homomorphism.

   (a) What must $\phi(0)$ be? Can you recreate the argument given earlier in this worksheet, without peeking?

   (b) Assume that $\phi(1) = 4$. Calculate $\phi(2) = \phi(1+1) = \phi(1) + \phi(1)$, and use a similar process to calculate $\phi(3)$ and $\phi(4)$.

   (c) Check whether $\phi(1+3) = \phi(1) + \phi(3)$.

   (d) Check whether $\phi(2+3) = \phi(2) + \phi(3)$.

   (e) Check whether $\phi(4+3) = \phi(4) + \phi(3)$.

   (f) Obviously 1 generates $\mathbb{Z}_5$. Does 4 generate $\mathbb{Z}_{10}$? Does 4 generate a disguised copy of $\mathbb{Z}_5$ hiding in $\mathbb{Z}_{10}$?

   (g) Does 6 generate the disguised copy of $\mathbb{Z}_5$ hiding in $\mathbb{Z}_{10}$? What does that answer tell you about what would have happened if $\phi(1) = 6$?

8. Gilligan says that a homomorphism $\phi : \mathbb{Z}_{15} \longrightarrow \mathbb{Z}_5$, where $\phi(1) = 1$, is like a triple–layer chocolate cake. Would you agree? Is it surjective? Is it injective?

9. Suppose that H is a normal subgroup of a group G. Let $h_1 \in H$ and $h_2 \in H$. What can you say about the product $h_1 h_2$ if $h_1 \neq h_2$? Would it make any difference if they were the same element?

10. Suppose that H is a normal subgroup of a group G. Suppose $h \in H$. Could you say the same thing about h^2? How about h^3? And h^{67}? Could you prove that $h^n \in H$ for any integer n? (Negative integers too!)

11. Let $g \in G$, a group, and $h \in H$, a normal subgroup. What can you say about ghg^{-1}? What about $g^{-1}hg$? As a misdirection question, can you say anything about hgh^{-1}?

12. Let $g \in G$, a abelian group. Suppose $h \in H$, a subgroup. Is the subgroup H abelian? Is it normal?

13. Let $g \in G$, a group, and $h_1 \in H$, a normal subgroup. Lunatic Larry says that elements in a normal subgroup "almost" commute, because $gh_1 = h_2g$, where $h_2 \in H$. Lunacy? Or freakishly correct?

14. Add $1/5$ and $1/3$. Would it make any difference if you added $3/15$ and $5/15$ instead?

15. Let k and l be integers. If you added $k * 1/5 + l * 1/3$, Could you always express the result as $m/15$, where m is some integer?

16. Can you put $3/23$, $34/42$, $38/n$ over a common denominator? No matter how many times you added or subtracted those numbers, would you ever need a larger common denominator?