1. **Comparison of Solutions to the Time-Dependent Eqn. of Motion**

Equations 4.1-15 and 4.1-40 are solutions to the time-dependent equation of motion for \( v_x(t,y) \) for a flow induced by the movement of a boundary at velocity of \( v_0 \). These equations are shown below.

\[
\phi(y,t) = 1 - \text{erf}\left(\frac{y}{\sqrt{4vt}}\right) = 1 - \text{erf}\left(\frac{\eta}{2\sqrt{\tau}}\right),
\]

where \( \phi \equiv \frac{v_x(y,t)}{v_o} \), \( \eta \equiv \frac{y}{b} \), \( \tau \equiv \frac{vt}{b^2} \), Eqn. 4.1-15

\[
\phi(\eta, \tau) = (1 - \eta) - \sum_{n=1}^{\infty} \frac{2}{n\pi} \exp(-n^2\pi^2\tau) \sin n\pi\eta,
\]

where \( \phi \equiv \frac{v_x(y,t)}{v_o} \), \( \eta \equiv \frac{y}{b} \), \( \tau \equiv \frac{vt}{b^2} \), Eqn. 4.1-40

Eqn. 4.1-15 is valid for a semi-infinite domain, \( 0 < y < \infty \), while eqn. 4.1-40 is valid for a finite domain, \( 0 < y < b \). You are to apply eqn. 4.1-15 to the case of a finite fluid depth, \( 0 < y < b \), at early times and compare to eqn. 4.1-40. Notice that eqn. 4.1-15 has been expressed in terms of the dimensionless variables used in eqn. 4.1-40. At early enough time, the penetration of momentum into the fluid above the moving plate is very limited and both eqn. will provide the same prediction.

- **a.** Determine a maximum value of \( \tau \) that is early enough such that eqn. 4.1-15 is in excellent agreement with eqn. 4.1-40. Recall from the lecture covering eqn. 4.1-15 the definition of \( \delta \). Use this definition to calculate \( \tau \) and compare predictions.

- **b.** Compare predictions when \( \tau \) is much larger than this limiting value of \( \tau \).

2. **Boundary Layer Theory: Integral Analysis and Approximate Solution**

Based on the derivations in lecture, evaluate the constants \( \alpha \) and \( \beta \). Assume that \( f(y/\delta(x)) \) takes for form of

\[
f \equiv \frac{v_x}{U} = \frac{3}{2} \frac{y}{\delta_{(x)}} - \frac{1}{2} \left( \frac{y}{\delta_{(x)}} \right)^3 = \frac{3}{2} \eta - \frac{1}{2} \eta^3
\]

Then, show that the boundary layer depth varies with \( x \) as

\[
\frac{\delta_{(x)}}{x} = 4.64 \text{ Re}^{-1/2}
\]
where the local Reynold's Number is defined as

$$Re_x = \frac{\rho U x}{\mu}$$

Furthermore, show that the force exerted by the fluid in the boundary layer on a flat plate of length L and width W is

$$F_x = 1.293 \sqrt{\rho \mu LW^2 U^3}$$

3. Stokes Law Derivation

Making the equations of motion dimensionless: Show that with the definitions provided in lecture of dimensionless r, θ, vr, vθ, and p, that the r- and q-components of the equations of motion contain the Reynold's Number preceding the inertial terms on the left hand side.

a. Continuity Equation: Insert the trial solutions for vr and vθ into the Continuity Equation and show that you get the following relationship between F(s) and G(s).

$$F(s) + \frac{s}{2} \frac{\partial F(s)}{\partial s} + G(s) = 0$$

b. As shown in class, the governing equation for F(s) is

$$s^4 \frac{d^4 F(s)}{ds^4} + 8s^3 \frac{d^3 F(s)}{ds^3} + 8s^2 \frac{d^2 F(s)}{ds^2} - 8s \frac{d F(s)}{ds} = 0$$

Show that the solution of this equation, after applying the boundary conditions, is

$$F(s) = 1 - \frac{2}{3s} + \frac{1}{2s^3}$$

and for G(s) it is

$$G(s) = -1 + \frac{3}{4s} + \frac{1}{4s^3}$$

c. Forces on the Falling Sphere: Show that the normal force acting on the falling sphere is

$$F^{(n)} = 4\pi R^3 \rho g + 2\pi R \mu U$$

In order to calculate $F^{(n)}$, it is necessary to know the fluid pressure at the surface of the falling sphere, which is (remember that for incompressible Newtonian fluids, $(\tau_{rr})_{r=R} = 0$)

$$\pi_{rr} \bigg|_{r=R} = (p + \tau_{rr})_{r=R} = \left(-p_{\infty} + \rho g \cos \theta + \frac{3}{2} \frac{\mu U}{R} \left( \frac{R}{r} \right)^2 \cos \theta \right)_{r=R}$$
You must first draw a diagram of the falling sphere and determine the component of $F^{(n)}$ in the direction opposing the falling of the sphere.

In addition, show that the tangential force acting on the sphere is

$$F^{(t)} = 4\pi R \mu U$$

Similarly, you must determine the component of $F^{(t)}$ in the direction opposing the falling of the sphere. Use Table B.1 to aid in the determination of $F^{(t)}$.

Due Mon. 13 Feb., 2006