**Stream Function:** Section 4.2. also Middleman text. p366

The Stream Function and solutions in terms of the Stream Function are very useful in solving 2-dimensional steady flows of Newtonian fluids. For 2-dim. flows, we have 3 dependent variables (2 velocity components + pressure) to solve for. By using the Stream Function, we are able to reduce this to just one dependent variable.

The Stream Function also has a geometric interpretation, as we will demonstrate next.

In Cartesian coordinates and for 2-dim. flow, we have 2 velocity components which can be expressed in terms of the Stream Function,

\[
\nu_x = \frac{\partial \Psi}{\partial y}, \quad \nu_y = \frac{\partial \Psi}{\partial x}.
\]

*Definition of \( \Psi \)*

where \( \Psi \) is stream function. \( \Psi = \Psi(x, y) \) varies in \( x \) and \( y \) directions for 2-dim. flow. A differential change in \( \Psi \) can be related to \( dx \) and \( dy \),

\[
d\Psi = \frac{\partial \Psi}{\partial x} \, dx + \frac{\partial \Psi}{\partial y} \, dy.
\]

\[
\int = \nu_y \, dx - \nu_x \, dy \quad \text{(using definitions of \( \Psi \) from above)}
\]

Along a line of constant \( \Psi \), \( d\Psi = 0 \)

\[
\frac{dy}{dx}\bigg|_{\Psi} = \frac{\nu_y}{\nu_x}
\]
In other words, the path that would be taken by a fluid element through space \( \left[\frac{dy}{dx}\right]_\psi \) at a constant value of \( \psi \) (\( d\psi = 0 \)) would be aligned with the fluid velocity \( (\frac{v_y}{v_x}) \) components!

Thus, the flow path within the fluid are along paths of constant \( \psi \), thus the Stream Function is valuable for flow visualization.

Another useful property of \( \psi \) is its relationship to the mass flow rate.

\[
m = \int_{\psi_1}^{\psi_2} \rho \nu \, dn
\]

where

\[
m = \text{mass} \div \text{time} \div \text{unit width} \div \text{in } z\text{-direction}
\]

\( \rho = \text{fluid density} \)  \( \frac{\text{mass}}{\text{volume}} \)

\( \nu = \text{velocity vector (magnitude only)} \)  \( \sqrt{v_x^2 + v_y^2} \)

\( dn = \text{a differential distance } d \text{ to a streamline (const. } \psi \text{)} \)

To complete this analysis, we need to relate \( d\psi \) to \( dn \) and to \( \nu \).

We note that

\[
\frac{dx}{dn} = \frac{v_y}{\nu} \\
\frac{dy}{dn} = \frac{v_x}{\nu}
\]
\[
\frac{d\psi}{dn} = \left( \frac{\partial \psi}{\partial x} \right) \frac{dx}{dn} + \left( \frac{\partial \psi}{\partial y} \right) \frac{dy}{dn}
\]

\[
= + \nu_y \frac{dx}{dn} + \nu_x \frac{dy}{dn}
\]

\[
= + \nu_y \left( \frac{\nu_y}{\nu} \right) + \nu_x \left( \frac{\nu_x}{\nu} \right)
\]

\[
= \frac{\nu_y^2 + \nu_x^2}{\nu} = \frac{\nu^2}{\nu} \quad \Rightarrow \quad \nu = \frac{d\psi}{dn}
\]

\[
\dot{m} = \int \rho \nu d\psi = \int \rho \frac{d\psi}{\nu} d\psi = \int \rho d\psi = \rho \left( \psi_2 - \psi_1 \right)
\]

Thus, the mass flow rate between streamlines is proportional to the difference between them. Thus, if \( \psi \) is displayed at equal \( \Delta \psi \) between them, regions of the greatest mass flow rate will have streamlines closest together.

**Example: Laminar Flow Between Parallel Plates:**

The general 2-dim. flow case is when the plates are not exactly parallel.

\[
\begin{align*}
B \uparrow & \quad \gamma \uparrow \rightarrow x \\
\psi_2 & = \nu_x (x, y) \\
\psi_1 & = \nu_y (x, y)
\end{align*}
\]

The Continuity Eqn and Navier-Stokes Eqns. are.
\[ \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \]  
\[ \text{3 eqns.} \quad \text{3 unknowns:} \quad u_x, u_y, p \]

Continuity.

\[ p \left( u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} \right) = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} \right) \]  
\[ \text{x-component} \]

\[ p \left( u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} \right) = -\frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} \right) \]  
\[ \text{y-component} \]

In terms of \( \psi \), the Continuity Eqn. becomes:

\[ -\frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial y \partial x} = 0 \]

which is satisfied for any \( \psi(x,y) \)!

Thus we lose 1 eqn., but we have only 2 unknowns, \( \psi, p \).

We eliminate \( p \) by:

\[ \text{subst. definition of } \psi \text{ into } x- \text{ and } y- \text{ component eqns.} \]

\[ \frac{\partial}{\partial y} \]  
\[ \text{x-comp. eqn.} \]

\[ \frac{\partial}{\partial x} \]  
\[ \text{y-comp. eqn.} \quad \text{from x-comp. eqn.} \quad \frac{\partial^2 \psi}{\partial y \partial x} \text{ terms cancel.} \]

\[ \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x^2} = \nabla \cdot \nabla \psi \]

\[ \nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \]

\[ \nabla^4 \psi = \frac{\partial^4 \psi}{\partial x^4} + 2 \frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4} \]

see Table 4. for \( \psi \) eqns in other coord. systems.