Heat Conduction in a Cooling Fin. 10.7

Assumptions
- \( T = f(z) \) only (fin is very thin, so \( T \neq f(x) \))
- \( k = \) constant
- heat is lost from the shell in the fin, \( h(T-T_a) \) where \( h \) is a constant.

Energy Balance:
For conduction only, \( v = 0 \)

\[ q_z = \frac{dT}{dz} = -k \frac{dT}{dz} \]

rate of energy in at \( z \) \( \quad \quad \quad \) \( \frac{(2BW)q_z|_z}{dz} \)
" " " " " out " " \( z+\Delta z \) \( \frac{(2BW)q_z|_{z+\Delta z}}{dz} \)
rate of heat loss from shell \( \quad \quad \quad \) \( (2W\Delta z)h(T-T_a) \)
\[ 2BW q_z/z - 2BW q_z/z + \Delta z - 2W \Delta z h (T - T_a) = 0 \]

by \[ 2BW \Delta z \]

\[ \lim_{\Delta z \to 0} \]

\[ - \frac{dq_z}{dz} = \frac{h}{B} (T - T_a) \]

Fourier's Law, \[ q_z = -k \frac{dT}{dz} \]

\[ \frac{d^2T}{dz^2} = \frac{h}{Bk} (T - T_a) \]

BC1 \[ z = 0 \quad T = T_w \]

BC2 \[ z = L \quad dT/dz = 0 \]

Let \[ \Theta = \frac{T - T_a}{T_w - T_a} \]

\[ \zeta = \frac{z}{L} \]

\[ N^2 = \frac{hL^2}{kB} = \frac{hL}{k} \frac{1}{B} \]

\[ Bi = \text{Biot Number} \]

\[ \frac{d^2 \Theta}{d\zeta^2} = N^2 \Theta \]

BC1 \[ \zeta = 0 \quad \Theta = 1 \]

BC2 \[ \zeta = 1 \quad d\Theta/d\zeta = 0 \]

From Appendix C, pg 852
\[ \Theta = C_1 \exp(N \beta) + C_2 \exp(-N \beta) \]

**BC1** \( I = C_1 (1) + C_2 (1) \Rightarrow C_2 = 1 - C_1 \)

\[ \frac{d\Theta}{d\beta} = \frac{C_1}{N} \exp(N \beta) - \frac{C_2}{N} \exp(-N \beta) \]

**BC2** \( O = \frac{C_1}{N} \exp(N) - \frac{C_2}{N} \exp(-N) \)

\[ O = C_1 \exp(N) - (1-C_1) \exp(-N) \]

\[ O = C_1 (\exp(N) + \exp(-N)) - \exp(N) \]

\[ C_1 = \frac{\exp(N)}{\exp(N) + \exp(-N)} \]

\[ C_2 = 1 - C_1 = 1 - \frac{\exp(N)}{\exp(N) + \exp(-N)} = \frac{\exp(-N)}{\exp(N) + \exp(-N)} \]

\[ \Xi = \frac{\exp(N) \exp(N \beta)}{\exp(N) + \exp(-N)} + \frac{\exp(N) \exp(-N \beta)}{\exp(N) + \exp(-N)} \]

\[ = \frac{\exp(-N(1-\beta)) + \exp(-N(1-\beta))}{\exp(N) + \exp(-N)} = \frac{\cosh(N(1-\beta))}{\cosh N} \]
Free Convection Heat Transfer; 10.9.

Assumptions:
- Steady-state for \( N_2(y) \) and \( T(y) \).
- \( N_2 = f(y) \) only.
- \( T = f(y) \) only.
- \( \mu, k \) constant.
- \( \rho = \bar{\rho} + \frac{\rho}{\rho} (\bar{T} - \bar{T}) \)

where \( \bar{T} = \frac{1}{2} (T_2 + T_1) \)
\( \bar{\rho} = \frac{1}{V} \left( \frac{\partial V}{\partial T} \right)_p \) coefficient of volume expansion
\[ e = \frac{1}{V_p} \left( \frac{\partial (V_p)}{\partial T} \right)_p = -\frac{1}{\rho} \frac{\partial (\rho)}{\partial T} \]

Shell Balance on Energy:

rate of energy in at \( y \), \( \Delta z \) \( \rho \Delta V \rho (\Delta V \rho) \)
" " " " out " \( \Delta V \rho (\Delta V \rho) \)
" " " " in at \( z = 0 \), \( \Delta y \rho (\Delta y \rho) \)
" " " " out at \( z = L \), \( \Delta y \rho (\Delta y \rho) \)
rate of work on shell by gravity, \( \Delta V \rho (\Delta V \rho) \)
\[ (\Delta W_{ey}|_y) - (\Delta W_{ey}|_{y+\Delta y}) + (\Delta W_{ez}|_z) - (\Delta W_{ez}|_{z+\Delta z}) + \frac{\partial W}{\partial z} \Delta z = 0 \]

\[ \frac{\partial W}{\partial y} \to 0 \]

\[ - \frac{\partial e_y}{\partial y} - \frac{\partial e_z}{\partial z} + \rho g z \Delta z = 0 \]

\[ e_y = \delta_y \cdot e = \left( \frac{1}{2} \rho v^2 + \rho \hat{H} \right) u^y + [\tau \cdot v] y + g_y \]

\[ [\tau \cdot v] y = \tau_{yx} u^x + \tau_{yy} u^y + \tau_{yz} u^z = -\mu \left( \frac{\partial u_z}{\partial y} \right) \Delta z \]

\[ q_y = -k \frac{\partial T}{\partial y} \]

\[ = -\mu \left( \frac{\partial u_z}{\partial y} \right) u_z - k \frac{\partial T}{\partial y} \]

\[ -\frac{\partial e_y}{\partial y} = \mu \left( \frac{\partial u_z}{\partial y} \right) \frac{\partial u_z}{\partial y} + \mu v_z \frac{\partial^2 u_z}{\partial y^2} + k \frac{\partial^2 T}{\partial y^2} \]

\[ e_z = \delta_z \cdot e = \left( \frac{1}{2} \rho v^2 + \rho \hat{H} \right) u^z + [\tau \cdot v] z + g_z \]

\[ v^2 = u_x^2 + u_y^2 + u_z^2 = u^2 \]

\[ \hat{H} = \hat{H}^0 + \hat{C}_p (T-T^0) + \frac{1}{\rho} (P-P^0) \]

\[ [\tau \cdot v] z = \tau_{xz} u^x + \tau_{yz} u^y + \tau_{zz} u^z = -2\mu \left( \frac{\partial u_z}{\partial z} \right) u_z + 0 \]

\[ q_z = -k \frac{\partial T}{\partial z} \to 0 \]

\[ T = f(y) \text{ only} \]

\[ = \frac{1}{2} \rho v_z^2 \cdot u_z + \rho \hat{H}^0 u_z + \rho \hat{C}_p (T-T^0) u_z + (P-P^0) u_z - 2\mu \left( \frac{\partial u_z}{\partial z} \right) u_z \]

\[ -\frac{\partial e_z}{\partial z} = 0 - 0 - 0 - 0 - \nu_z \frac{\partial P}{\partial z} - 0 - 0 \]