Chapter 4. \( v \) in terms of more than 1 Independent Variable

Time-Dependent Flows:

Flow Near a Boundary Set in Motion.
(Method of Similarity Solutions or Method of Combination of Variables).

\( t < 0 \), fluid at rest.
\( \rho = \mu = \text{constant} \)
\( t = 0 \), wall set in motion

\( u_y = u_z = 0 \), \( p \neq p(x) \), \( q_x = 0 \)
\( t > 0 \), unsteady flow

Eqn. of Continuity, Table B.4
\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u_x) + \frac{\partial}{\partial y} (\rho u_y) + \frac{\partial}{\partial z} (\rho u_z) = 0
\]

Eqn. of Motion, Table B.6
\[
\rho \frac{\partial u_x}{\partial t} = -\frac{\partial P}{\partial x} + \mu \frac{\partial^2 u_x}{\partial y^2} \quad \text{or} \quad \frac{\partial u_x}{\partial t} = -\frac{\partial P}{\partial x} + \nu \frac{\partial^2 u_x}{\partial y^2}, \quad \nu = \frac{\mu}{\rho}
\]
Initial Condition (IC): \( t \leq 0 \), \( u_x = 0 \) for all \( y \).

BC 1: \( y = 0 \), \( u_x = v_0 \) for \( t > 0 \).

BC 2: \( y = \infty \), \( u_x = 0 \) for \( t > 0 \).

Make Eqn. of Motion Dimensionless,

define \( \phi = u_x/v_0 \) so

\[
\frac{du_x}{dt} = v_0 \frac{\partial \phi}{\partial t} = \frac{d^2u_x}{dy^2} = v_0 \frac{\partial^2 \phi}{\partial y^2}.
\]

\[
\frac{d\phi}{dt} = v \frac{d^2\phi}{dy^2} \quad \Rightarrow \quad \frac{d\phi}{dt} = v \frac{d^2\phi}{dy^2}.
\]

\( \phi = \phi(y, t; v) \), but is dimensionless, \( \therefore \) it must be dependent on a dimensionless quantity made up of \( y, t, \) and \( v \).

\[
\eta = \frac{y}{\sqrt{4vt}} \quad \Rightarrow \quad \phi = \phi(\eta)
\]

We need to convert Eqn. of Motion and IC / BC to dimensionless form.

\[
\frac{d\phi}{dt} = \frac{d\phi}{dn} \frac{dn}{dt} = \frac{d\phi}{dn} \frac{d\phi}{dt} \left( \frac{y}{2\sqrt{v}t^{3/2}} \right) = \frac{d\phi}{dn} \frac{y}{4\sqrt{v}t^{3/2}} \frac{1}{t^{3/2}}
\]

\[
= \frac{d\phi}{dn} \left( -\frac{1}{2} \right) \frac{\eta}{t}
\]
\[
\frac{d\phi}{dy} = \frac{\partial\phi}{\partial\eta} \frac{d\eta}{dy} = \frac{\partial\phi}{\partial\eta} \frac{1}{\sqrt{4\nu t}}
\]
\[
\frac{d^2\phi}{dy^2} = \frac{d}{dy} \left( \frac{d\phi}{dy} \right) = \frac{d}{\partial\eta} \frac{d\eta}{dy} \left( \frac{\partial\phi}{\partial\eta} \frac{1}{\sqrt{4\nu t}} \right) = \frac{\partial^2\phi}{\partial\eta^2} \frac{1}{4\nu t}
\]

Subst. into Eqn. of Motion

\[
\frac{d\phi}{dt} = v \frac{d^2\phi}{dy^2} \quad \Rightarrow \quad \frac{d\phi}{\partial\eta} \left( \frac{\partial\phi}{\partial\eta} \right) \frac{1}{4\nu t} = v \frac{1}{4\nu t} \frac{\partial^2\phi}{\partial\eta^2}
\]

\[-\frac{d\phi}{\partial\eta} = \frac{2vt}{4\nu t} \frac{\partial^2\phi}{\partial\eta^2} \quad \Rightarrow \quad \frac{d^2\phi}{\partial\eta^2} + 2\eta \frac{d\phi}{\partial\eta} = 0
\]

Through this combination of variables substitution, we have converted the Eqn. of Motion from a partial differential eqn. to an ordinary differential eqn. We need 2 "boundary" conditions to solve this eqn.

\[\text{IC} \implies \text{BC} \quad \rightarrow \quad \eta = \frac{y}{\sqrt{4\nu t}} = \frac{\infty}{\sqrt{4\nu t}} = \frac{y}{\sqrt{4\nu t}} \quad \Rightarrow \quad \phi = \frac{v_x}{v_0} = \frac{0}{v_0} = 0
\]

\[\text{BC} \quad \rightarrow \quad \eta = \frac{0}{\sqrt{4\nu t}} = 0
\]

\[\phi = \frac{v_x}{v_0} = \frac{v_0}{v_0} = 1
\]

\[\sqrt{27}/06\]