First-Order IVP

Solve: \( \frac{dy}{dx} = f(x, y) \)

Subject to: \( y(x_0) = y_0 \)

Example: \( y = cx^2 \) is a one-parameter family of solutions of the ODE \( xy' - 2y = 0 \) on the interval \( (-\infty, \infty) \).

Find the solution subject to the condition \( y(2) = 1 \).
Example: $y = \frac{1}{x^2 + c}$ is a one-parameter family of solutions of the ODE $y' + 2xy^2 = 0$.

Find the solution curve passing through the point $(0, -1)$. 
Note: If we consider \( y = \frac{1}{x^2 - 1} \) as a:

- **function:** the domain of \( y \) is the set of real numbers for which \( y(x) \) is defined. The domain of \( y \) is all real numbers except \(-1\) and \(1\).

- **solution of the ODE \( y' + 2xy^2 = 0 \):** the interval \( I \) of definition of \( y \) could be taken to be any interval over which \( y(x) \) is defined and differentiable. The largest intervals on which \( y \) is a solution of the ODE are \(-\infty < x < -1, -1 < x < 1, \) and \(1 < x < \infty\).

- **solution of the IVP \( y' + 2xy^2 = 0, \ y(0) = -1 \):** the interval \( I \) of definition of \( y \) could be taken to be any interval over which \( y(x) \) is defined, differentiable, and contains the initial point \( x = 0 \). The largest interval on which \( y \) is a solution of the IVP is \(-1 < x < 1\).
Second-Order IVP

Solve: \( \frac{d^2y}{dx^2} = f(x, y, y') \)

Subject to: \( y(x_0) = y_0 \)
\( y'(x_0) = y_1 \)

Example: \( y = c_1 e^x + c_2 \) is a two-parameter family of solutions of the ODE \( y'' = y' \) on \( (-\infty, \infty) \).

Find a solution to the IVP:

Solve: \( y'' = y' \)

Subject to: \( y(0) = 1 \)
\( y'(0) = 2 \)
In general, for an $n^{th}$-order ODE, particular solutions arise from the requirement that for some specified value of $x$ in an interval $I$, the function $y$ and its first $n - 1$ derivatives take on prescribed values.

\[ \frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}) \text{ on } I \]

Subject to:
\[
\begin{align*}
y(x_0) &= y_0 \\
y'(x_0) &= y_1 \\
&\vdots \\
y^{(n-1)}(x_0) &= y_{n-1}
\end{align*}
\]

where $x_0 \in I$ and $y_0, y_1, \ldots, y_{n-1}$ are arbitrarily specified real constants.
Existence and Uniqueness

In studying IVPs, fundamental questions arise concerning the existence and uniqueness of solutions.

Consider the first-order IVP:

\[
\begin{cases}
\frac{dy}{dx} = f(x, y) \\
y(x_0) = y_0
\end{cases}
\]

Existence questions:

- Under what conditions does the IVP have a solution? That is: Does the ODE have a solution? If so, do any of the solution curves pass through the point \((x_0, y_0)\)?
- If a solution exists, what is the interval of definition?

Uniqueness questions:

- Under what conditions does the IVP possess a unique solution?
- If there is a unique solution, what is the interval of uniqueness?
Theorem 1.1: Existence of a Unique Solution for First-Order IVP

Consider the first-order IVP

\[ \begin{cases} \frac{dy}{dx} = f(x, y) \\ y(x_0) = y_0 \end{cases} \]

Let \( R \) be a rectangular region in the \( xy \)-plane defined by \( a \leq x \leq b, c \leq y \leq d \) that contains the point \((x_0, y_0)\) in its interior.

If \( f \) and \( \frac{\partial f}{\partial y} \) are continuous on \( R \), then there exists an interval \( I_0 = \{x : x_0 - h < x < x_0 + h\}, h > 0 \), contained in \( a \leq x \leq b \), on which there is a unique solution \( y(x) \) to this IVP.
Interval of Existence / Uniqueness:

Let $y(x)$ be a solution to the first-order IVP. There are three sets of $x$ values related to this function, which may or may not be the same:

1. The domain of $y(x)$.
2. The interval $I$ over which the solution $y(x)$ is defined or exists (interval of definition).
3. The interval $I_0$ over which the solution $y(x)$ exists and is unique (interval of existence and uniqueness).
Comments on Theorem 1.1:

1. The conditions on \( f \) and \( \partial f / \partial y \) are **sufficient conditions**, not necessary conditions. That is:

   If \( f \) and \( \partial f / \partial y \) are continuous on \( R \), then there is a unique solution in some interval \( I_0 \) around \( x_0 \) whenever \((x_0, y_0)\) is a point interior to \( R \).

   If \( f \) and/or \( \partial f / \partial y \) are not continuous in \( R \), then no conclusions can be made: The IVP may have a unique solution, or may have multiple solutions, or may have no solutions.

2. If \((x_0, y_0)\) is a point in the interior of \( R \), then continuity of \( f \) on \( R \) is sufficient to guarantee the existence of a solution to the first-order IVP on some interval \( I \) of definition.

3. This theorem does not tell us how large \( I \) or \( I_0 \) are. For example, \( I \) does not need to be as wide as \( R \), and \( I_0 \) need not equal \( I \). Since \( I_0 \) could be very small, we say that the solution \( y(x) \) to the IVP is unique in a local sense.

4. Even if an IVP has a solution, we may not be able to construct it analytically. Not all ODEs have analytical solutions. Perhaps all we can do is approximate it.

5. There is an extension of this theorem to \( n^{\text{th}} \)-order IVPs.
Example: Consider the ODE \( xy' - 2y = 0 \)