Exam I
CBE614
October 5, 2005

Please be neat.
Please write on only one side of each piece of paper in your solution.

Navier-Stokes Equation: \[ \rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g} \]

Continuity Equation: \[ \frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}) \]

Newtonian Incompressible Constitutive Equation: \[ \mathbf{e} = -\mu \left[ \nabla \mathbf{v} + (\nabla \mathbf{v})^T \right] \]

Shear flow: \[ \mathbf{v} = \begin{pmatrix} \zeta(t)x_2 \\ 0 \\ 0 \end{pmatrix}_{123} \]

Elongational flow: \[ \mathbf{v} = \begin{pmatrix} \frac{1}{2} \varepsilon(t)x_1 \\ -\frac{1}{2} \varepsilon(t)x_2 \\ \varepsilon(t)x_3 \end{pmatrix}_{123} \]

1. (20 points) What is \( \mathbf{A} \cdot \mathbf{B}^T \) written in Einstein notation?

2. (20 points) For the scalar function \( f \) written below, what is \( \nabla \cdot \nabla f \)?

\[ f(x_1, x_2, x_3) = 4x_1^2 + 2x_2 + x_3 \]

3. (10 points) In shear flow, which components of the stress tensor are equal to zero for all fluids (Newtonian and non-Newtonian)?
4. (20 points) What does the Newtonian incompressible constitutive equation predict for \( \overline{\eta}'(t) \), the material function for start-up of steady uniaxial elongational flow? The kinematics of the flow and the definition of the material function are shown below. Please show how you obtain your answer. Please sketch your answer.

\[
v = \begin{cases} 
-\frac{1}{2} \varepsilon(t)x_1 \\
\frac{1}{2} \varepsilon(t)x_2 \\
\varepsilon(t)x_3 
\end{cases}_{123}
\]

\[
\varepsilon(t) = \begin{cases} 
0 & t < 0 \\
\varepsilon_0 & t \geq 0 
\end{cases}
\]

\[
\overline{\eta}' = -\frac{\tau_{33} - \tau_{11}}{\varepsilon_0}
\]

5. (30 points) A steady flow of an incompressible, Newtonian fluid is created between two very wide, parallel plates as shown below. The pressure at position \( x=0 \) is \( P_0 \) and the pressure at position \( x=L \) is \( P_L \). The top plate moves with a steady velocity \( V \). What is the steady state velocity profile? You may neglect gravity. Please show your work.
6. (5 point bonus problem) For the tensor given below, the first invariant \( I_1 = \text{trace}(\mathbf{T}) = 0 \). What is the first invariant equal to in the cylindrical coordinate system?

\[
\mathbf{T} = \begin{pmatrix}
-4 & 0 & 0 \\
0 & -4 & 0 \\
0 & 0 & 8
\end{pmatrix}_{123}
\]