We can improve this fit by adjusting the Maxwell model to allow multiple relaxation modes.

\[ \mathbf{G}(t) = \sum_{i=1}^{N} \frac{\eta_i}{\lambda_i} e^{-\lambda_i t} \mathbf{g}(t) \]

\[ \mathbf{G}(t) = \sum_{i=1}^{N} \int \frac{\eta_i}{\lambda_i} e^{-\lambda_i t} \mathbf{g}(t') \, dt' \]

**Generalized Maxwell Model**

2N parameters (can fit anything)

---

**Predictions of the Generalized Maxwell Model**

<table>
<thead>
<tr>
<th>Steady shear</th>
<th>[ \eta = \sum_{i=1}^{N} \eta_i ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \Psi_1 = \Psi_2 = 0 ]</td>
<td></td>
</tr>
</tbody>
</table>

Fails to predict shear normal stresses

<table>
<thead>
<tr>
<th>Step shear strain</th>
<th>[ G(t) = \sum_{i=1}^{N} \frac{\eta_i}{\lambda_i} e^{-\lambda_i t} ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ G_{\alpha_1} = G_{\alpha_2} = 0 ]</td>
<td></td>
</tr>
</tbody>
</table>

This function can fit any observed data; note that the GMM does not predict shear normal stresses.

---

**The Linear-Viscoelastic Models**

**Differential Maxwell (one mode):**

\[ \dot{\tau} + \frac{\eta}{G} \frac{\partial \tau}{\partial t} = -\eta \dot{\gamma} \]

**Integral Maxwell (one mode):**

\[ \tau = \int \frac{\eta}{\lambda} e^{-\lambda t} \mathbf{g}(t') \, dt' \]

**Generalized Maxwell model (N modes):**

\[ \tau = \int \sum_{i=1}^{N} \frac{\eta_i}{\lambda_i} e^{-\lambda_i t} \mathbf{g}(t') \, dt' \]

Since the term in brackets is just the predicted relaxation modulus \( G(t) \), we can write an even more general linear viscoelastic model by leaving this function unspecified.

---

**The Linear-Viscoelastic Models**

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**Generalized Linear-Viscoelastic Model:**

\[ \tau = \int G(t - t') \dot{\gamma}(t') \, dt' \]
Small-Amplitude Oscillatory Shear Material Functions

**Kinematics:**

\[ \mathbf{\ddot{\gamma}}(t) = \bar{\gamma}_0 \cos \omega t \]

\[ \bar{\gamma}_0 = \frac{\gamma_0}{\omega} \]

**Material Functions:**

\[ G'(\omega) = \frac{\gamma_0}{\omega} \cos \delta \]

\[ G''(\omega) = \frac{\gamma_0}{\omega} \sin \delta \]

\( \delta \) is the phase difference between stress and strain.

Storage modulus

Loss modulus

Predictions of (single-mode) Maxwell Model in SAOS

\[ G'(\omega) = \frac{\eta_s \omega^2}{1 + (\lambda_s \omega)^2} \]

\[ G''(\omega) = \frac{\eta_s \omega}{1 + (\lambda_s \omega)^2} \]

Predictions of (multi-mode) Maxwell Model in SAOS

\[ G'(\omega) = \sum_{i=1}^{N} \frac{\gamma_i \omega^2}{1 + (\lambda_i \omega)^2} \]

\[ G''(\omega) = \sum_{i=1}^{N} \frac{\gamma_i \omega}{1 + (\lambda_i \omega)^2} \]

GLVE

\[ G'(\omega) = a \int G'(\gamma) \cos \omega \gamma d\gamma \]

\[ G''(\omega) = a \int G'(\gamma) \sin \omega \gamma d\gamma \]

GMM

\[ G'(\omega) = \frac{\eta_s}{\omega} \int \frac{G'(\gamma)}{1 + (\lambda_s \omega)^2} d\gamma \]

Limitations of the GLVE Models

- Predicts constant shear viscosity
- Only valid in regime where strain is additive (small-strain, low rates)
- All stresses are proportional to the deformation rate tensor; thus shear normal stresses cannot be predicted
- Cannot describe flows with a superposed rigid rotation (as we will now discuss; see Morrison p296)
Steady shear viscosity and first and second normal stress coefficients

**BOGER FLUIDS**

Example: Drag flow of a Generalized Linear-Viscoelastic fluid between infinite parallel plates
- steady state
- incompressible fluid
- infinitely wide, long

**Shear flow in a rotating frame of reference**

Shear flow in a rotating frame of reference

**Summary:** Generalized Linear-Viscoelastic Constitutive Equations

**PRO:**
- A first constitutive equation with memory
- Can match SAOS, step-strain data very well
- Captures start-up/cessation effects
- Simple to calculate with
- Can be used to calculate the LVE spectrum

**CON:**
- Fails to predict shear normal stresses
- Fails to predict shear-thinning/thickening
- Only valid at small strains, small rates
- Not frame-invariant