MICROSCOPIC MOMENTUM BALANCES WITH THE NAVIER-STOKES EQN

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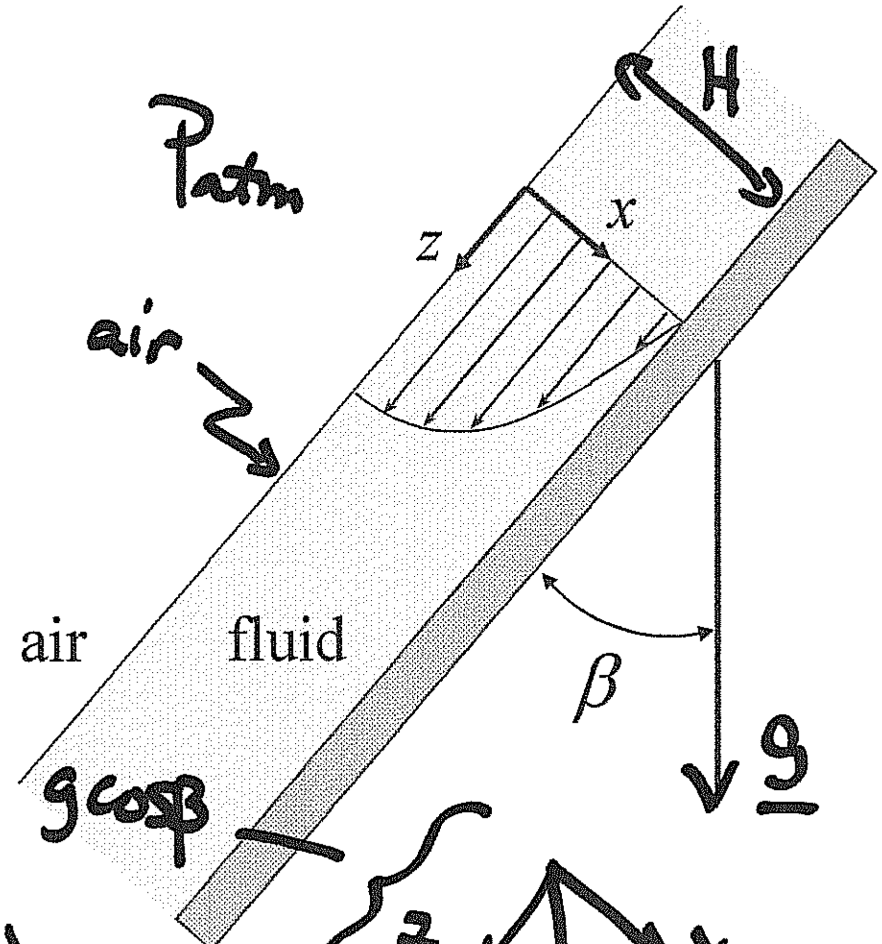
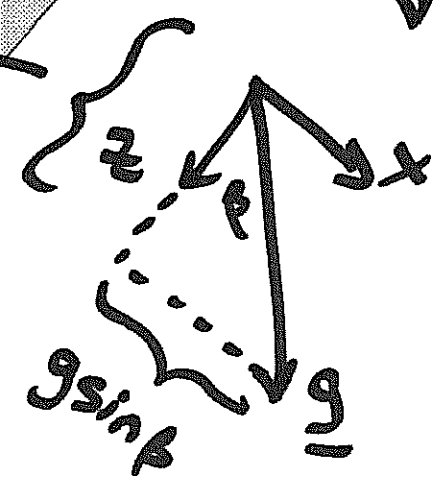
Problem: Calculate the steady state velocity profile for the flow of an incompressible Newtonian fluid down an inclined plane. You may assume the plate is wide and that end effects may be neglected. What is the average velocity per unit width?





$$\underline{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}_{xyz} = \begin{pmatrix} 0 \\ 0 \\ v_z \end{pmatrix}_{xyz}$$

$$\underline{g} = \begin{pmatrix} g\sin\beta \\ 0 \\ g\cos\beta \end{pmatrix}$$

Navier - Stokes (Gibbs)

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\right) = -\nabla P + \mu\nabla^2\underline{v} + \rho\underline{g}$$

Continuity Equation (Gibbs)

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\underline{v}) = 0$$

The Equation of Continuity and the Equation of Motion in Cartesian, cylindrical, and spherical coordinates

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continuity EQN

Continuity Equation, Cartesian coordinates

$$\frac{\partial \rho}{\partial t} + \left(v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} \right) + \rho \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) = 0$$

Continuity Equation, cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial (\rho r v_r)}{\partial r} + \frac{1}{r}\frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{\partial (\rho v_z)}{\partial z} = 0$$

Continuity Equation, spherical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial (\rho r^2 v_r)}{\partial r} + \frac{1}{r \sin\theta}\frac{\partial (\rho v_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial (\rho v_\phi)}{\partial \phi} = 0$$

Equation of Motion for an incompressible fluid, 3 components in Cartesian coordinates

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial P}{\partial x} - \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) + \rho g_x$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial P}{\partial y} - \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) + \rho g_y$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z} - \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right) + \rho g_z$$

Equation of Motion for an incompressible fluid, 3 components in cylindrical coordinates

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial P}{\partial r} - \left(\frac{1}{r}\frac{\partial (r\tau_{rr})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta r}}{\partial \theta} - \frac{\tau_{\theta\theta}}{r} + \frac{\partial \tau_{zr}}{\partial z}\right) + \rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta v_r}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial P}{\partial \theta} - \left(\frac{1}{r^2}\frac{\partial (r^2\tau_{r\theta})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{z\theta}}{\partial z} + \frac{\tau_{\theta r} - \tau_{r\theta}}{r}\right) + \rho g_\theta$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z} - \left(\frac{1}{r}\frac{\partial (r\tau_{rz})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z}\right) + \rho g_z$$

Equation of Motion for an incompressible fluid, 3 components in spherical coordinates

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right)$$

$$= -\frac{\partial P}{\partial r} - \left(\frac{1}{r^2}\frac{\partial (r^2\tau_{rr})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (\tau_{\theta r}\sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tau_{\phi r}}{\partial \phi} - \frac{\tau_{\theta\theta} + \tau_{\phi\phi}}{r}\right) + \rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2\cot\theta}{r}\right)$$

$$= -\frac{1}{r}\frac{\partial P}{\partial \theta} - \left(\frac{1}{r^3}\frac{\partial (r^3\tau_{r\theta})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (\tau_{\theta\theta}\sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tau_{\phi\theta}}{\partial \phi} + \frac{\tau_{\theta r} - \tau_{r\theta}}{r} - \frac{\tau_{\phi\phi}\cot\theta}{r}\right) + \rho g_\theta$$

$$\rho\left(\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{v_\phi v_\theta\cot\theta}{r}\right)$$

$$= -\frac{1}{r\sin\theta}\frac{\partial P}{\partial \phi} - \left(\frac{1}{r^3}\frac{\partial (r^3\tau_{r\phi})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (\tau_{\theta\phi}\sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tau_{\phi\phi}}{\partial \phi} + \frac{\tau_{\phi r} - \tau_{r\phi}}{r} + \frac{\tau_{\phi\theta}\cot\theta}{r}\right) + \rho g_\phi$$

MASS BAL (Continuity Eqn)

steady state $\rightarrow$ $\frac{\partial}{\partial t} \rightarrow 0$

incomp. fluid $\rightarrow$ ρ = constant

$v_x = v_y = 0$

$$\Rightarrow \boxed{\frac{\partial v_z}{\partial z} = 0}$$

due to mass balance

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation) 3 components in Cartesian coordinates

$$
\begin{aligned}
\rho\left(\cancel{\frac{\partial v_x}{\partial t}}+v_x\cancel{\frac{\partial v_x}{\partial x}}+v_y\cancel{\frac{\partial v_x}{\partial y}}+v_z\cancel{\frac{\partial v_x}{\partial z}}\right) &= -\frac{\partial P}{\partial x}+\mu\left(\cancel{\frac{\partial^2 v_x}{\partial x^2}}+\cancel{\frac{\partial^2 v_x}{\partial y^2}}+\cancel{\frac{\partial^2 v_x}{\partial z^2}}\right)+\rho g_x \\
\rho\left(\cancel{\frac{\partial v_y}{\partial t}}+\cancel{v_x}\frac{\partial v_y}{\partial x}+v_y\cancel{\frac{\partial v_y}{\partial y}}+v_z\cancel{\frac{\partial v_y}{\partial z}}\right) &= -\frac{\partial P}{\partial y}+\mu\left(\cancel{\frac{\partial^2 v_y}{\partial x^2}}+\cancel{\frac{\partial^2 v_y}{\partial y^2}}+\cancel{\frac{\partial^2 v_y}{\partial z^2}}\right)+\rho g_y \\
\rho\left(\cancel{\frac{\partial v_z}{\partial t}}+\cancel{v_x}\frac{\partial v_z}{\partial x}+\cancel{v_y}\frac{\partial v_z}{\partial y}+v_z\cancel{\frac{\partial v_z}{\partial z}}\right) &= -\frac{\partial P}{\partial z}+\mu\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\cancel{\frac{\partial^2 v_z}{\partial z^2}}\right)+\rho g_z
\end{aligned}
$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in cylindrical coordinates

$$
\begin{aligned}
\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right) &= -\frac{\partial P}{\partial r}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rv_r)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}\right)+\rho g_r \\
\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right) &= -\frac{1}{r}\frac{\partial P}{\partial \theta}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rv_\theta)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right)+\rho g_\theta \\
\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right) &= -\frac{\partial P}{\partial z}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right)+\rho g_z
\end{aligned}
$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in spherical coordinates

$$
\begin{aligned}
&\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi}-\frac{v_\theta^2+v_\phi^2}{r}\right) \\
&\quad = -\frac{\partial P}{\partial r}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right)\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v_r}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial \phi^2}\right. \\
&\qquad \left.-\frac{2}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(v_\theta\sin\theta\right)-\frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right)+\rho g_r \\
&\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}+\frac{v_r v_\theta}{r}-\frac{v_\phi^2\cot\theta}{r}\right) \\
&\quad = -\frac{1}{r}\frac{\partial P}{\partial \theta}+\mu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(v_\theta\sin\theta\right)\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial \phi^2}\right. \\
&\qquad \left.+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}-\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right)+\rho g_\theta \\
&\rho\left(\frac{\partial v_\phi}{\partial t}+v_r\frac{\partial v_\phi}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}+\frac{v_r v_\phi}{r}+\frac{v_\phi v_\theta\cot\theta}{r}\right) \\
&\quad = -\frac{1}{r\sin\theta}\frac{\partial P}{\partial \phi}+\mu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(v_\phi\sin\theta\right)\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial \phi^2}\right. \\
&\qquad \left.+\frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial \phi}+\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right)+\rho g_\phi
\end{aligned}
$$

NAVIER STOKES

Note: the r-component of the Navier-Stokes equation in spherical coordinates may be simplified by adding $0 = \frac{2}{r}\nabla\cdot\underline{v}$ to the component shown above. This term is zero due to the continuity equation (mass conservation). See Bird et. al.

References:

1. R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena*, 2^{nd} edition, Wiley: NY, 2002.
2. R. B. Bird, R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric Fluids: Volume 1 Fluid Mechanics*, Wiley: NY, 1987.

x-comp: $0 = -\frac{\partial p}{\partial x} + \rho g_x \leftarrow g\sin\beta$ Navier-Stokes

y-comp: $0 = -\frac{\partial p}{\partial y} + \cancel{\rho g_y}$

z-comp: $0 = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2}\right) + \rho g_z \leftarrow g\cos\beta$

steady state $\rightarrow \frac{\partial}{\partial t} = 0$

mass bal $\rightarrow \frac{\partial v_z}{\partial z} = 0$

$v_x = v_y = 0$

$\underline{g} = \begin{pmatrix} g\sin\beta \\ 0 \\ g\cos\beta \end{pmatrix}$

x-comp NS: $0 = -\frac{\partial P}{\partial x} + \rho g \sin\beta$

$$\boxed{\frac{\partial P}{\partial x} = \rho g \sin\beta}$$

y-comp NS: $\boxed{0 = \frac{\partial P}{\partial y}}$

z-comp NS: $0 = -\cancel{\frac{\partial P}{\partial z}} + \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \cancel{\frac{\partial^2 v_z}{\partial y^2}}\right) + \rho g \cos\beta$

Additional assumptions:
(observations)

$P \neq f(z)$

$v_z \neq f(y)$ (wide

z - comp

$$\boxed{0 = \mu \frac{\partial^2 v_z}{\partial x^2} + \rho g \cos\beta}$$

Since:

$\frac{\partial v_z}{\partial z} = 0$ (mass bal)

$\frac{\partial v_z}{\partial y} = 0$ (wide flow) $\Rightarrow$ $\frac{\partial}{\partial x}$ goes to $\frac{d}{dx}$

$$0 = \mu \frac{d^2 v_z}{dx^2} + \rho g \cos\beta$$

$$\frac{d^2 v_z}{dx^2} = \frac{-\rho g \cos\beta}{\mu} \equiv a$$

Now, integrate:

$$\frac{d^2 V_z}{dx^2} = a$$

$$\frac{dV_z}{dx} = ax + C_1$$

$$\boxed{V_z = \frac{ax^2}{2} + C_1 x + C_2}$$ ← SOLN; now Boundary conditions

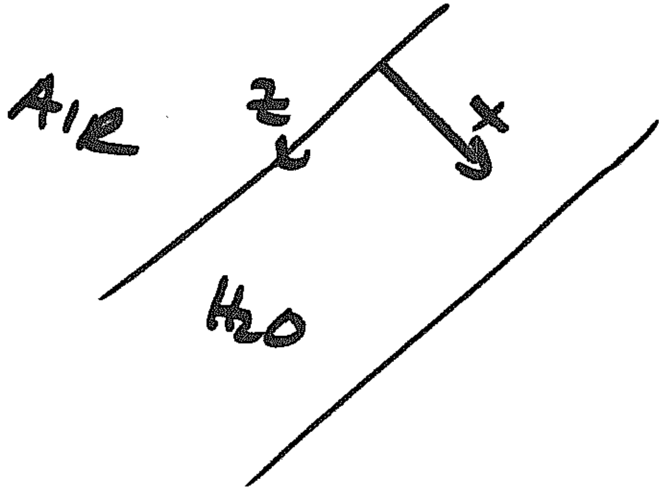
Boundary Conditions:

BC1: $V_z = 0$ $x = H$ no slip

$$\boxed{0 = \frac{aH^2}{2} + C_1 H + C_2}$$ ← 1 eqn 2 unknowns

BC2: $\frac{\partial V_z}{dx} = 0$ $x = 0$ $\Rightarrow$ $\boxed{C_1 = 0}$

see next pg



BC 2: stress matches at free surface $\Rightarrow$ $\frac{dV_z}{dx} = 0$ there.

AIR

$$\tau_{xz}^{AIR} = -\underbrace{\mu_{air}}_{\to 0} \frac{dV_z}{dx} = 0$$

$$\|$$

WATER

$$0 = \tau_{xz}^{WATER} = -\mu_{H_2O} \frac{dV_z}{dx}$$

$$\Rightarrow \boxed{\frac{dV_z}{dx} = 0 \text{ at } x = 0}$$

Since $c_1 = 0$, substituting above yields,

$$V_z = \frac{ax^2}{2} + \cancel{c_1 x} + c_2$$

$$V_z(x) = \frac{ax^2}{2} + c_2$$

From BC1: $$0 = \frac{aH^2}{2} + \cancel{c_1 H} + c_2$$

$$\Rightarrow c_2 = -\frac{aH^2}{2}$$

$$\Rightarrow V_z(x) = \frac{ax^2}{2} - \frac{aH^2}{2} = \frac{a}{2}\left(x^2 - H^2\right)$$

$$\boxed{V_z(x) = \frac{\rho g \cos\beta}{2\mu}\left(H^2 - x^2\right)}$$

//