CM3110
Transport I
Part II: Heat Transfer

MichiganTech

One-Dimensional Heat Transfer - Unsteady

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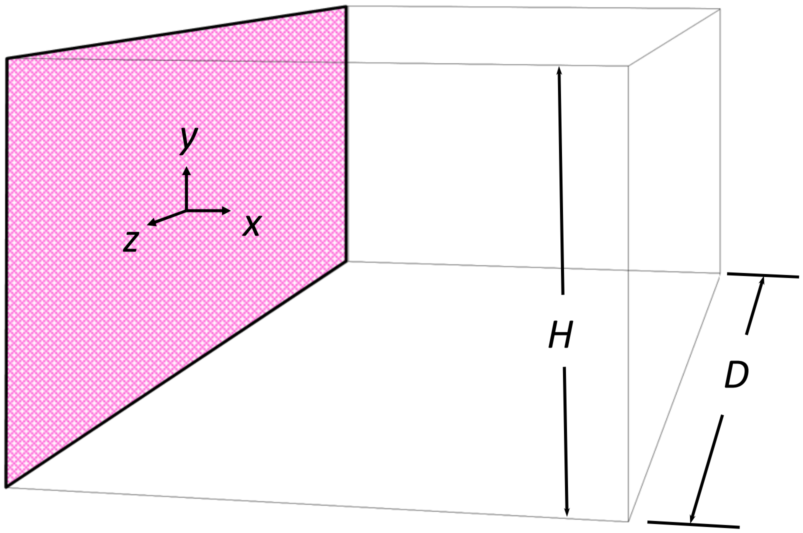
1

Example 1: **Unsteady Heat Conduction in a Semi-infinite solid**

A very long, very wide, very tall slab is initially at a temperature T_o. At time t = 0, the left face of the slab is exposed to an environment at temperature T_1. What is the time-dependent temperature profile in the slab? The slab is a homogeneous material of thermal conductivity, k, density, ρ, and heat capacity, C_p.

Unsteady State Heat Transfer

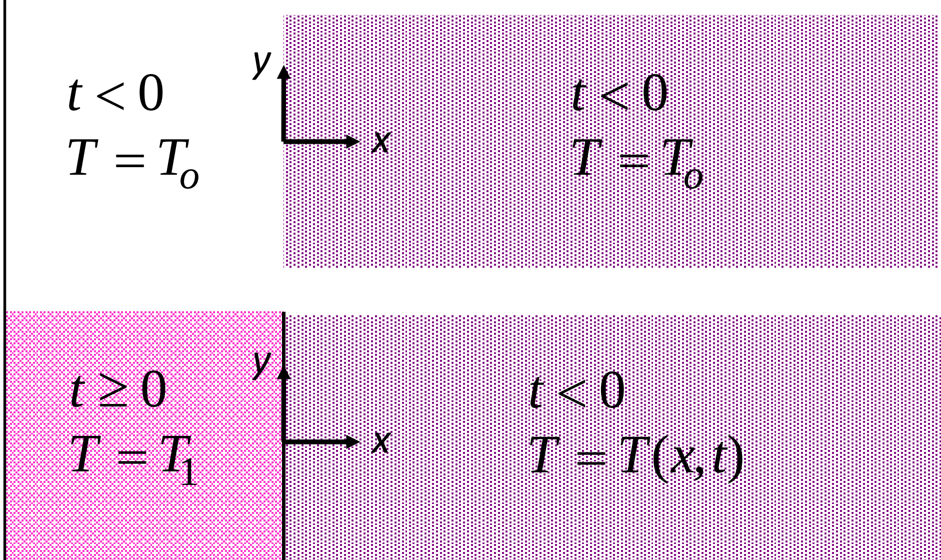
Example: Unsteady Heat Conduction in a Semi-infinite solid



H, D, very large

Initial Condition:

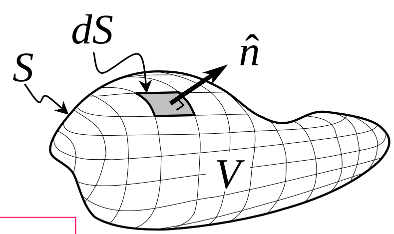
$t < 0$
$T = T_o$



General Energy Transport Equation

(microscopic energy balance)

As for the derivation of the microscopic momentum balance, the microscopic energy balance is derived on an arbitrary volume, *V*, enclosed by a surface, *S*.



Gibbs notation:

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + \underline{v} \cdot \nabla T \right) = k \nabla^2 T + S$$

see handout for component notation

General Energy Transport Equation

(microscopic energy balance)

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + \underline{v} \cdot \nabla T \right) = k \nabla^2 T + S$$

- $\frac{\partial T}{\partial t}$: rate of change
- $\underline{v} \cdot \nabla T$: convection
- $k \nabla^2 T$: conduction (all directions)
- S: source (energy generated per unit volume per time)

velocity must satisfy equation of motion, equation of continuity

see handout for component notation

Equation of energy for Newtonian fluids of constant density, ρ, and thermal conductivity, k, with source term (source could be viscous dissipation, electrical energy, chemical energy, etc., with units of energy/(volume time)).

CM310 Fall 1999 Faith Morrison

Source: R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Processes*, Wiley, NY, 1960, page 319.

Note: this handout is on the web:
www.chem.mtu.edu/~fmorriso/cm310/energy2013.pdf

Gibbs notation (vector notation)

$$\left(\frac{\partial T}{\partial t} + \underline{v} \cdot \nabla T\right) = \frac{k}{\rho \hat{C}_p} \nabla^2 T + \frac{S}{\rho \hat{C}_p}$$

Cartesian (xyz) coordinates:

$$\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} = \frac{k}{\rho \hat{C}_p}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{S}{\rho \hat{C}_p}$$

Cylindrical (rθz) coordinates:

$$\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} = \frac{k}{\rho \hat{C}_p}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{S}{\rho \hat{C}_p}$$

Spherical (rθϕ) coordinates:

$$\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial T}{\partial \phi} = \frac{k}{\rho \hat{C}_p}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2}$$

Example 1: **Unsteady Heat Conduction in a Semi-infinite solid**

A very long, very wide, very tall slab is initially at a temperature T_o. At time $t = 0$, the left face of the slab is exposed to an environment at temperature T_1. What is the time-dependent temperature profile in the slab? The slab is a homogeneous material of thermal conductivity, k, density, ρ, and heat capacity, C_p.

Newton's law of cooling BC's:

$$q_x = hA(T_{bulk} - T_{surface})$$

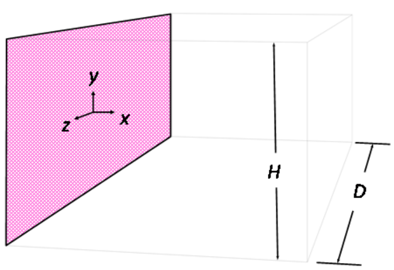
Microscopic Energy Equation in Cartesian Coordinates

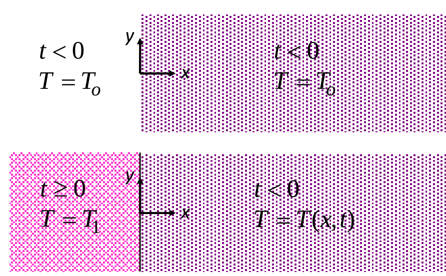
$$\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} = \frac{k}{\rho \hat{C}_p}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{S}{\rho \hat{C}_p}$$

$$\alpha \equiv \frac{k}{\rho \hat{C}_p} =$$ thermal diffusivity

what are the boundary conditions? initial conditions?

Example 7: Unsteady Heat Conduction in a Semi-infinite solid





You try.

10

Unsteady State Heat Conduction in a Semi-Infinite Slab

$$\frac{\partial T}{\partial t} = \frac{k}{\rho \hat{C}_p}\left(\frac{\partial^2 T}{\partial x^2}\right) = \alpha \left(\frac{\partial^2 T}{\partial x^2}\right)$$

Initial condition: $t = 0, T = T_o \ \forall \ x$

Boundary conditions:

$$x = 0, \quad q_x = hA(T - T_1) \quad \forall \ t > 0$$

$$x = \infty, \quad T = T_o \quad \forall \ t$$

Unsteady State Heat Conduction in a Semi-Infinite Slab

The solution is obtained by combination of variables.

$$\frac{\partial T}{\partial t} = \frac{k}{\rho \hat{C}_p}\left(\frac{\partial^2 T}{\partial x^2}\right) = \alpha \left(\frac{\partial^2 T}{\partial x^2}\right)$$

Initial condition: $t = 0, T = T_o \ \forall \ x$

Boundary conditions:

$$x = 0, \quad q_x = hA(T - T_1) \quad \forall \ t > 0$$

$$x = \infty, \quad T = T_o \quad \forall \ t$$

$$\beta \equiv \frac{h\sqrt{\alpha t}}{k} \qquad \zeta \equiv \frac{x}{2\sqrt{\alpha t}}$$

Unsteady State Heat Conduction in a Semi-Infinite Slab

$$\left(\frac{T-T_o}{T_1-T_o}\right)=erfc(\zeta)-e^{\beta(2\zeta+\beta)}erfc(\zeta+\beta)$$

Geankoplis 4th ed., eqn 5.3-7, page 363

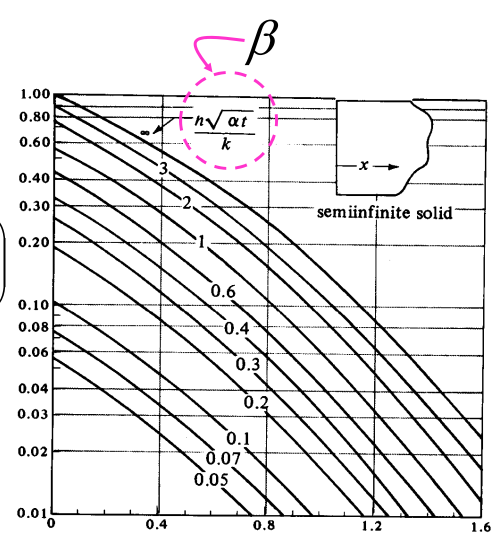
$$\beta\equiv\frac{h\sqrt{\alpha\, t}}{k}\qquad \zeta\equiv\frac{x}{2\sqrt{\alpha\, t}}$$

complementary error function $\quad erfc(x)=1-erf(x)$

error function $\quad erf(x)\equiv\frac{2}{\sqrt{\pi}}\int_0^x e^{-(x')^2}dx'$

Unsteady State Heat Conduction in a Semi-Infinite Slab

$$\left(\frac{T-T_o}{T_1-T_o}\right)$$



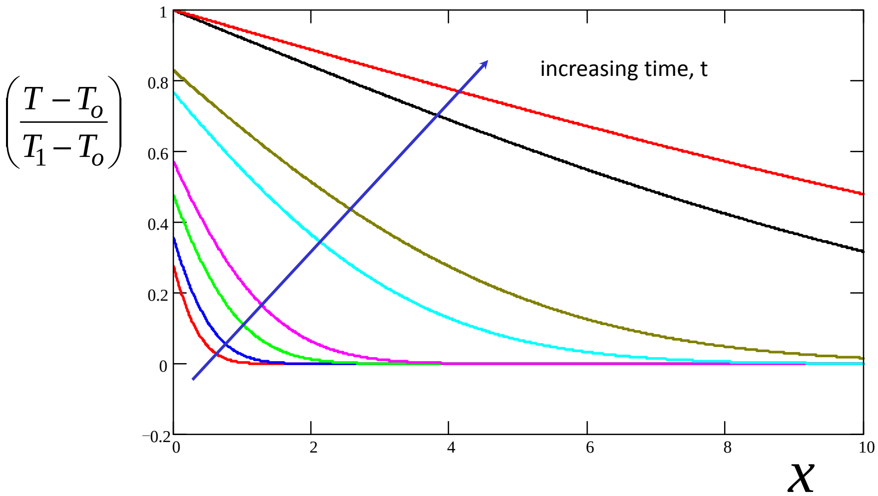
$$\zeta=\frac{x}{2\sqrt{\alpha\, t}}$$

Geankoplis 4th ed., Figure 5.3-3, page 364

Unsteady State Heat Conduction in a Semi-Infinite Slab

h = *k* = α = 1

t = 0.1, 0.2, 0.5, 1, 5, 10, 50, 100



evaluated on Mathcad Plus

How could we use this solution?

Example: Will my pipes freeze?

The temperature has been 35°F for a while now, sufficient to chill the ground to this temperature for many tens of feet below the surface. Suddenly the temperature drops to -20°F. How long will it take for freezing temperatures (32°F) to reach my pipes, which are 8 *ft* under ground? Use the following physical properties:

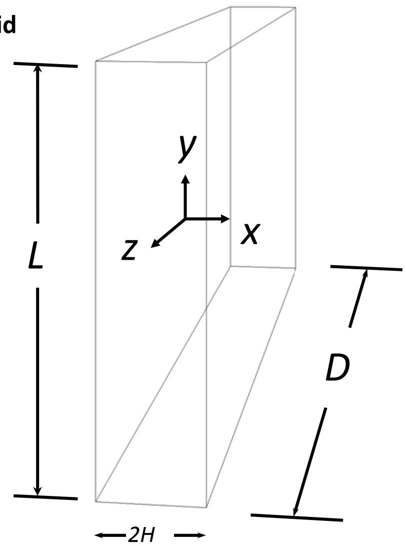
$$h = 2.0 \frac{BTU}{h\ ft^2\ {}^oF}$$

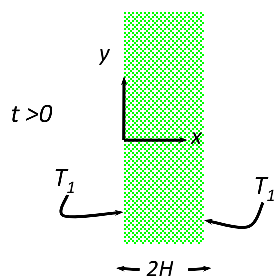
$$\alpha_{soil} = 0.018 \frac{ft^2}{h}$$

$$k_{soil} = 0.5 \frac{BTU}{h\ ft\ {}^oF}$$

Example 8: **Unsteady Heat Conduction in a Finite-sized solid**

- The slab is tall and wide, but of thickness 2*H*
- Initially at T_o
- at time $t = 0$ the temperature of the sides is changed to T_1





Unsteady State Heat Transfer

Use same microscopic energy balance eqn as before.

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + \underline{v} \cdot \nabla T \right) = k \nabla^2 T + S$$

- $\frac{\partial T}{\partial t}$: rate of change
- $\underline{v} \cdot \nabla T$: convection
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see handout for component notation

Microscopic Energy Equation in Cartesian Coordinates

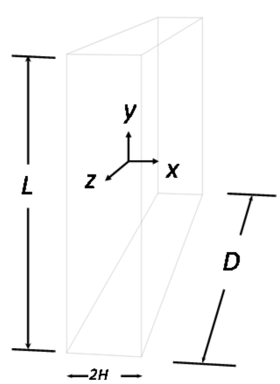
$$\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} = \frac{k}{\rho \hat{C}_p}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{S}{\rho \hat{C}_p}$$

$$\alpha \equiv \frac{k}{\rho \hat{C}_p} =$$ thermal diffusivity

what are the boundary conditions? initial conditions?

***Example 8:* Unsteady Heat Conduction in a Finite-sized solid**

You try.



20

Unsteady State Heat Conduction in a Finite Slab

$$\frac{\partial T}{\partial t} = \frac{k}{\rho \hat{C}_p}\left(\frac{\partial^2 T}{\partial x^2}\right) = \alpha\left(\frac{\partial^2 T}{\partial x^2}\right)$$

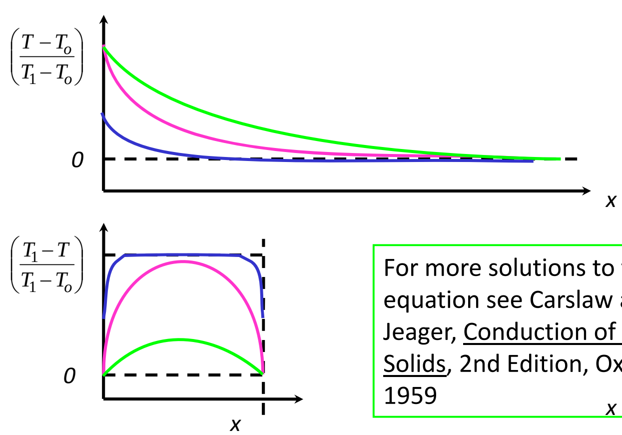
Initial condition: $t = 0, T = T_o \; \forall \; x$

Boundary conditions:

$$\left.\begin{array}{ll} x = 0, & T = T_1 \\ x = 2H, & T = T_1 \end{array}\right\} \; \forall \; t > 0$$

Q: *How can two completely different situations give the same governing equation?*

A: The boundary conditions make all the difference



For more solutions to this equation see Carslaw and Jeager, Conduction of Heat in Solids, 2nd Edition, Oxford, 1959

Unsteady State Heat Conduction in a Finite Slab

$$\frac{\partial T}{\partial t} = \frac{k}{\rho \hat{C}_p}\left(\frac{\partial^2 T}{\partial x^2}\right) = \alpha \left(\frac{\partial^2 T}{\partial x^2}\right)$$

The solution is obtained by separation of variables.

Initial condition: $t = 0, T = T_o \; \forall \; x$

Boundary conditions:

$$\left.\begin{array}{ll} x = 0, & T = T_1 \\ x = 2H, & T = T_1 \end{array}\right\} \quad \forall \; t > 0$$

Unsteady State Heat Conduction in a Finite Slab: solution by separation of variables

Let $Y \equiv \left(\frac{T_1 - T}{T_1 - T_o}\right)$

$$\frac{\partial Y}{\partial t} = \alpha \left(\frac{\partial^2 Y}{\partial x^2}\right)$$

Guess: $Y = X(x)\Theta(t)$

Initial condition:

$$t = 0, T = T_o \; \forall \; x \Rightarrow Y = 1$$

Boundary conditions:

$$\left.\begin{array}{ll} x = 0, & T = T_1 \Rightarrow Y = 0 \\ x = 2H, & T = T_1 \Rightarrow Y = 0 \end{array}\right\} \quad \forall \; t > 0$$

Unsteady State Heat Conduction in a Finite Slab: soln by separation of variables

$$Y = X(x)\Theta(t) \qquad \frac{\partial Y}{\partial t} = \alpha\left(\frac{\partial^2 Y}{\partial x^2}\right)$$

$$\frac{\partial Y}{\partial t} = \frac{\partial}{\partial t}\left(X(x)\Theta(t)\right) = X(x)\frac{d\Theta(t)}{dt}$$

$$\frac{\partial Y}{\partial x} = \frac{\partial}{\partial x}\left(X(x)\Theta(t)\right) = \frac{dX(x)}{dx}\Theta(t)$$

$$\frac{\partial^2 Y}{\partial x^2} = \frac{d^2X(x)}{dx^2}\Theta(t)$$

Unsteady State Heat Conduction in a Finite Slab: soln by separation of variables

$$\frac{\partial Y}{\partial t} = \alpha\left(\frac{\partial^2 Y}{\partial x^2}\right)$$

Substituting:

$$X(x)\frac{d\Theta(t)}{dt} = \alpha\frac{d^2X(x)}{dx^2}\Theta(t)$$

$$\frac{1}{\Theta(t)}\frac{d\Theta(t)}{dt} = \alpha\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} \Rightarrow = \lambda$$

constant

function of time only

function of position (x) only

Unsteady State Heat Conduction in a Finite Slab: soln by separation of variables

Separates into two ordinary differential equations:

$$\frac{1}{\Theta(t)}\frac{d\Theta(t)}{dt}=\lambda$$

$$\alpha\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}=\lambda$$

Solve.

Apply BCs.

Apply ICs.

Temperature Profile for Unsteady State Heat Conduction in a Finite Slab

$$\left(\frac{T_1-T}{T_1-T_o}\right)=\frac{4}{\pi}\left\{e^{\frac{-\pi^2\alpha t}{4H^2}}\sin\frac{\pi x}{2H}+\frac{1}{3}e^{\frac{-3^2\pi^2\alpha t}{4H^2}}\sin\frac{3\pi x}{2H}+\frac{1}{5}e^{\frac{-5^2\pi^2\alpha t}{4H^2}}\sin\frac{5\pi x}{2H}+\cdots\right\}$$

Geankoplis 4th ed., eqn 5.3-6, p363

Microscopic Energy Balance – is the correct physics for many problems!

Tricky step:

solving for *T* field; this can be mathematically difficult

- partial differential equation in up to three variables
- boundaries may be complex
- multiple materials, multiple phases present
- may not be separable from mass and momentum balances

Strategy: solve using numerical methods

(e.g. ***Comsol***)

****** Or ******

Develop correlations on complex systems by using ***Dimensional Analysis***

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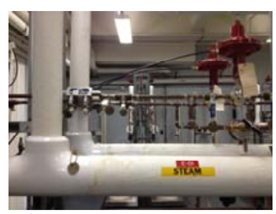
Part II: Heat Transfer



More Complex Heat Transfer – Dimensional Analysis

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30