**Example 1:** Unsteady Heat Conduction in a Finite-sized Solid

- The slab is tall and wide, but of thickness $2H$
- Initially at $T_0$
- At time $t = 0$ the temperature of the sides is changed to $T_f$

Use same microscopic energy balance eqn as before.

$$\frac{\partial T}{\partial t} + \nabla \cdot (\rho \cdot C_p \cdot \mathbf{v}) = k \nabla^2 T + S$$

- **Convection**
- **Source** (energy generated per unit volume per time)
- **Rate of change**
- **Conduction** (all directions)

See handout for component notation.
Microscopic Energy Equation in Cartesian Coordinates

\[ \frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} = \frac{k}{\rho C_p} \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{S}{\rho C_p} \]

\[ \alpha \equiv \frac{k}{\rho C_p} \quad \text{thermal diffusivity} \]

what are the boundary conditions? initial conditions?

Unsteady State Heat Conduction in a Finite Slab

\[ \frac{\partial T}{\partial t} = \frac{k}{\rho C_p} \left( \frac{\partial^2 T}{\partial x^2} \right) = \alpha \left( \frac{\partial^2 T}{\partial x^2} \right) \]

Initial condition: \( t = 0, T = T_o \ \forall \ x \)

Boundary conditions:
\[
\begin{align*}
  x &= 0, & T &= T_i \\
  x &= 2H, & T &= T_i \\
\end{align*}
\]
\( \forall \ t > 0 \)
Q: How can two completely different situations give the same governing equation?
A: The boundary conditions make all the difference


Unsteady State Heat Conduction in a Finite Slab: solution by separation of variables

Let \( Y \equiv \frac{T_i - T}{T_i - T_o} \)

\[
\frac{\partial Y}{\partial t} = \alpha \left( \frac{\partial^2 Y}{\partial x^2} \right)
\]

Guess: \( Y = X(x)\Theta(t) \)

Initial condition:
\[ t = 0, \ T = T_o \ \forall \ x \ \Rightarrow \ Y = 1 \]

Boundary conditions:
\[
\begin{align*}
  x &= 0, \quad T = T_i \Rightarrow Y = 0 \quad \forall \ t > 0 \\
  x &= 2H, \quad T = T_i \Rightarrow Y = 0 \quad \forall \ t > 0
\end{align*}
\]
Unsteady State Heat Conduction in a Finite Slab: soln by separation of variables

\[ Y = X(x)\Theta(t) \]

\[ \frac{\partial Y}{\partial t} = \alpha \left( \frac{\partial^2 Y}{\partial x^2} \right) \]

\[ \frac{\partial Y}{\partial t} = \frac{\partial}{\partial t} (X(x)\Theta(t)) = X(x) \frac{d\Theta(t)}{dt} \]

\[ \frac{\partial Y}{\partial x} = \frac{\partial}{\partial x} (X(x)\Theta(t)) = \frac{dX(x)}{dx} \Theta(t) \]

\[ \frac{\partial^2 Y}{\partial x^2} = \frac{d^2 X(x)}{dx^2} \Theta(t) \]

Substituting:

\[ X(x) \frac{d\Theta(t)}{dt} = \alpha \frac{d^2 X(x)}{dx^2} \Theta(t) \]

\[ \frac{1}{\Theta(t)} \frac{d\Theta(t)}{dt} = \alpha \frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} \]

\[ \Rightarrow = \lambda \]

\[ \lambda \]

constant

function of time only

function of position (x) only

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Unsteady State Heat Conduction in a Finite Slab: soln by separation of variables

Separates into two ordinary differential equations:

\[
\frac{1}{\Theta(t)} \frac{d\Theta(t)}{dt} = \lambda \\
\alpha \frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} = \lambda
\]

Solve.
Apply BCs.
Apply ICs.

Temperature Profile for Unsteady State Heat Conduction in a Finite Slab

\[
\left( \frac{T_1 - T}{T_1 - T_o} \right) = 4 \pi \left\{ \frac{\pi^2 \alpha t}{4H^2} \sin \frac{\pi x}{2H} + \frac{1}{3} \frac{3\pi^2 \alpha t}{4H^2} \sin \frac{3\pi x}{2H} \right. \\
\left. + \frac{1}{5} \frac{5\pi^2 \alpha t}{4H^2} \sin \frac{5\pi x}{2H} + \cdots \right\}
\]

Geankoplis 4th ed., eqn 5.3-6, p363
Unsteady Macroscopic Energy Balances

Example 3: If a piece of steel at $T = T_0$ is dropped into a large reservoir of fluid at $T_\infty$, what is the temperature of the steel as a function of time?

$k = \text{large}$, which means that there is no resistance to heat transfer in the steel. Therefore, we are NOT calculating a temperature profile. 

Use Macroscopic Energy Balance

$$T = T(t)$$

balance over time interval $\Delta t$

amount of energy that enters with the flow between $t$ and $t+\Delta t$

amount of energy that exits with the flow between $t$ and $t+\Delta t$

$W_{s,\text{on}} \Delta t$

$m_{\text{in}} \left( \dot{H} + \frac{V^2}{2} + gz \right)_{\text{in}} \Delta t$

$m_{\text{out}} \left( \dot{H} + \frac{V^2}{2} + gz \right)_{\text{out}} \Delta t$

see Felder and Rousseau or Himmelblau

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Unsteady Macroscopic Energy Balances

\[ \text{accumulation} = \text{input} - \text{output} \]

\[
\frac{d(U_{\text{sys}} + E_{p,\text{sys}} + E_{k,\text{sys}})}{dt} = -\Delta H - \Delta E_p - \Delta E_k + Q_{in} + W_{on,sys}
\]

For negligible changes in \( E_p \) and \( E_k \), and no phase change, and single-input, single-output system,

\[
\frac{dU_{\text{sys}}}{dt} = M_{\text{sys}} C_v \frac{dT_{\text{sys}}}{dt}
\]

\[ \Delta H = m C_p (T_{\text{out}} - T_{\text{in}}) \]

Our case, no shafts

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Unsteady Macroscopic Energy Balances

\[
M_{\text{sys}} C_v \frac{dT_{\text{sys}}}{dt} = m C_p (T_{\text{in}} - T_{\text{sys}}) + Q_{in} + W_{on,sys}
\]

Our case, no flow

Our case, no shafts

• For negligible changes in \( E_p \) and \( E_k \)
• and no phase change
• and single-input, single-output system
• assume \( T_{\text{out}} = T_{\text{sys}} \)

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Unsteady Macroscopic Energy Balance
Applied to cooling steel part:

\[ M_{sys} C_v \frac{dT_{sys}}{dt} = Q_{in} \]

The temperature changes in the slug are due to the heat loss.

The heat loss depends on the heat-transfer coefficient from the slug to the environment.

\[ Q_{in} = hA(T - T_\infty) \]

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