Mini Exam 2
CM3110
08 Oct 2008

1. \( R_{\text{min}} = 4000 = \frac{\rho V D}{\mu} \)

\[ \langle V \rangle = \frac{4000 \mu}{\rho D} \]

\[ = \left( \frac{62.4 \text{ lbm}}{\text{ft}^3} \right) \left( 0.364 \text{ in} \right) \left( \frac{\text{ft}}{12 \text{ in}} \right) \]

\[ \langle V \rangle = 1.42005 \text{ ft/s} \]

\[ Q = \langle V \rangle A = \langle V \rangle \pi R^2 \]

\[ = \left( \frac{1.42005 \text{ ft}}{s} \right) \pi \left( \frac{0.364 \text{ in}}{2} \right)^2 \left( \frac{\text{ft}}{12 \text{ in}} \right)^2 \]

\[ = 1.024208 \frac{\text{ft}^3}{s} \times 10^{-3} \]
\[ Q = \left( \frac{0.26208 \, \text{ft}^2}{5} \right) \left( \frac{7.485 \, \text{gal}}{1 \, \text{ft}^3} \right) \left( \frac{60 \text{sec}}{\text{min}} \right) \]

\[ Q = 0.461 \, \text{gpm} \]

\[ Q \approx 0.5 \, \text{gallons/min} \]

2. Poiseuille flow in a slit.

- Steady
- Flow in \( x \)-dir
- \( p \) = constant

(See later)

\[ \frac{\text{D}V_x}{\text{D}x} = 0 \]

**Continuity**
The Equation of Continuity and the Equation of Motion in Cartesian, cylindrical, and spherical coordinates

**Continuity Equation, Cartesian coordinates**

\[
\frac{\partial \rho}{\partial t} + \left( \frac{\partial \rho}{\partial x} v_x + \frac{\partial \rho}{\partial y} v_y + \frac{\partial \rho}{\partial z} v_z \right) + \rho \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) = 0
\]

**Continuity Equation, cylindrical coordinates**

\[
\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial (r \rho v_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{\partial (\rho v_z)}{\partial z} = 0
\]

**Continuity Equation, spherical coordinates**

\[
\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial (r^2 \rho v_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{1}{r \sin \theta \sin \phi} \frac{\partial (\rho v_\phi)}{\partial \phi} = 0
\]

**Equation of Motion for an incompressible fluid, 3 components in Cartesian coordinates**

\[
\rho \left( \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial P}{\partial x} - \left( \frac{1}{r} \frac{\partial (r \tau_{rr})}{\partial r} + \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} \right) + \rho g_x
\]

\[
\rho \left( \frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = -\frac{\partial P}{\partial y} - \left( \frac{1}{r} \frac{\partial (r \tau_{r\theta})}{\partial r} + \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{r\phi}}{\partial \phi} \right) + \rho g_y
\]

\[
\rho \left( \frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial P}{\partial z} - \left( \frac{1}{r} \frac{\partial (r \tau_{rz})}{\partial r} + \frac{\partial \tau_{r\phi}}{\partial \phi} + \frac{\partial \tau_{\phi z}}{\partial \phi} \right) + \rho g_z
\]

**Equation of Motion for an incompressible fluid, 3 components in cylindrical coordinates**

\[
\rho \left( \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_r^2 + v_\theta^2}{r} \right) = -\frac{\partial P}{\partial r} - \left( \frac{1}{r} \frac{\partial (r \tau_{rr})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{r\phi}}{\partial \phi} \right) + \rho g_r
\]

\[
\rho \left( \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} \right) = -\frac{1}{r \sin \theta} \frac{\partial P}{\partial \theta} - \left( \frac{1}{r^2} \frac{\partial (r^2 \tau_{r\theta})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{\theta \phi}}{\partial \phi} + \frac{\tau_{r\theta} - \tau_{r\phi} \sin \theta}{r} \right) + \rho g_\theta
\]

\[
\rho \left( \frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} \right) = -\frac{1}{r \sin \theta \sin \phi} \frac{\partial P}{\partial \phi} - \left( \frac{1}{r^2} \frac{\partial (r^2 \tau_{r\phi})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial \tau_{r\phi}}{\partial \theta} + \frac{\partial \tau_{\theta \phi}}{\partial \phi} + \frac{\tau_{r\phi} - \tau_{r\theta} \sin \theta}{r} \right) + \rho g_\phi
\]
Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in Cartesian coordinates

\[ \rho \left( \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial P}{\partial x} + \mu \left( \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) + \rho g_x \]

\[ \rho \left( \frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = -\frac{\partial P}{\partial y} + \mu \left( \frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right) + \rho g_y \]

\[ \rho \left( \frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial P}{\partial z} + \mu \left( \frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho g_z \]

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in cylindrical coordinates

\[ \rho \left( \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_r v_\theta}{r} + \frac{v_\phi}{r} \frac{\partial v_r}{\partial \phi} \frac{\partial^2 v_r}{\partial \theta^2} \right) = -\frac{\partial P}{\partial r} + \mu \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_r}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \phi^2} \right) + \rho g_r \]

\[ \rho \left( \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + v_r \frac{\partial v_\theta}{\partial \phi} \frac{\partial^2 v_\theta}{\partial \phi \partial \theta} \right) = -\frac{\partial P}{\partial \theta} + \mu \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_\theta}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \phi^2} \right) - \frac{2}{r^2} \frac{\partial}{\partial \theta} \left( v_\theta \sin \theta \right) - \frac{2}{r^2} \frac{\partial}{\partial \phi} \left( \frac{\partial v_\phi}{\partial \phi} \right) + \rho g_\theta \]

\[ \rho \left( \frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\phi}{r} \frac{\partial v_\phi}{\partial \phi} + v_r \frac{\partial v_\phi}{\partial \theta} \frac{\partial^2 v_\phi}{\partial \theta \partial \phi} \right) = -\frac{\partial P}{r \sin \theta} + \mu \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_\phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_\phi}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^2 v_\phi}{\partial \phi^2} + \frac{2}{r^2} \frac{\partial^2 v_\phi}{\partial \theta \partial \phi} + \frac{2}{r^2} \frac{\partial^2 v_\phi}{\partial \phi^2} \right) + \rho g_\phi \]

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in spherical coordinates

\[ \rho \left( \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_r v_\theta}{r} + \frac{v_\phi}{r} \frac{\partial v_r}{\partial \phi} \frac{\partial^2 v_r}{\partial \theta^2} \right) = -\frac{\partial P}{\partial r} + \mu \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_r}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \phi^2} \right) \]

\[ \rho \left( \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + v_r \frac{\partial v_\theta}{\partial \phi} \frac{\partial^2 v_\theta}{\partial \phi \partial \theta} \right) = -\frac{\partial P}{r \sin \theta} + \mu \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_\theta}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \phi^2} \right) + \frac{2}{r} \frac{\partial}{\partial \theta} \left( v_\theta \sin \theta \right) + \frac{2}{r^2} \frac{\partial}{\partial \phi} \left( \frac{\partial v_\phi}{\partial \phi} \right) + \rho g_\theta \]

\[ \rho \left( \frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\phi}{r} \frac{\partial v_\phi}{\partial \phi} + v_r \frac{\partial v_\phi}{\partial \theta} \frac{\partial^2 v_\phi}{\partial \theta \partial \phi} \right) = -\frac{\partial P}{r^2 \sin \theta} + \mu \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_\phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_\phi}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^2 v_\phi}{\partial \phi^2} \right) + \frac{2}{r^2} \frac{\partial}{\partial \theta} \left( v_\phi \sin \theta \right) + \frac{2}{r^2} \frac{\partial}{\partial \phi} \left( \frac{\partial v_\phi}{\partial \phi} \right) + \rho g_\phi \]

Note: the r-component of the Navier-Stokes equation in spherical coordinates may be simplified by adding \(0 = \frac{\partial}{\partial r} \nabla \cdot g\) to the component shown above. This term is zero due to the continuity equation (mass conservation). See Bird et al.

References:
Navier-Stokes

x-component: \[ \frac{\partial P}{\partial x} = \mu \frac{\partial^2 U_x}{\partial y^2} \]

y-component: \[ \frac{\partial P}{\partial y} = -pg \]

z-component: \[ \frac{\partial P}{\partial z} = 0 \]

velocity B.C.: \[ y = 0 \quad \frac{\partial U_x}{\partial y} = 0 \]
\[ y = H \quad U_x = 0 \]

or
\[ y = H \quad U_x = 0 \]
\[ y = -H \quad U_x = 0 \]

pressure:
\[ x = 0 \quad P = P_0 \]
\[ x = L \quad P = P_L \]