Temperature Charts for Induction and Constant-Temperature Heating

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Charts are presented for determining complete temperature histories in spheres, cylinders, and plates. It is shown that for values of the dimensionless time ratio X greater than 0.2 the heating equations reduce to such a simple form that for each shape two charts which give temperatures at any position within the heated or cooled bodies can be plotted. It is also shown that the usual simple heating and cooling charts can also be used for the determination of temperatures and heating times in bodies heated by a constant rate of heat generation at the surface (induction heating). Finally, a two-dimensional chart is given for finding heating times in short cylinders, thereby eliminating the trial-and-error solution that is necessary when heating times are found from the present one-dimensional charts.

Nomenclature

The following nomenclature is used in the paper:

- exp = base of natural logarithms, for example $\exp(a) = e^a$
- F = deg F
- h = coefficient of heat transfer between surroundings and surface of heated body, Btu/sq ft hr F
- I^* = dimensionless heating-up temperature ratio for induction heating $(t - t_b)/(h/q)$
- I = dimensionless induction-temperature ratio expressed in relaxation (cooling) form, $1 - (t - t_b)/(h/q) = 1 - I^*$
- L = half-thickness of a semi-infinite plate or a short cylinder heated from opposite sides, ft
- m = relative boundary resistance (conductivity)/$(L)(h)$
- n = relative position ratio (distance in feet from center of plate, cylinder, or sphere)/(L or r)
- q = heat flux per unit area applied to the surface of a plate, cylinder, or sphere in induction heating, Btu/sq ft hr
- r = radius of cylinder or sphere, ft
- t = temperature, deg F
- t' = furnace temperature, deg F (temperature of hot ambient)
- Y = dimensionless temperature ratio in relaxation form, $(t' - t)/(t' - t_b)$
- Y^* = dimensionless temperature ratio in heating-up form, $(t - t_b)/(t' - t_b) = 1 - Y$
- X = dimensionless time (diffusivity × time)/(half-thickness squared) for a semi-infinite plate heated from opposite sides, a long cylinder, or a sphere
- τ = time, hr

Subscripts:

- n refers to dimensionless position n
- 1 refers to surface of sphere, cylinder, or plate
- 0 refers to center of sphere, cylinder, or plate
- p refers to plate
- c refers to cylinder
- b refers to initial base temperature of surroundings

Introduction

The solution of many heat-flow problems requires the use of charts from which temperature distribution in solids of simple shape such as plates, cylinders, and spheres, can be found. A survey of the existing charts shows that no set of them fills all the requirements of accuracy and readability over the entire temperature range, of sufficient curves to permit easy interpolation, and of coverage of extreme conditions of time, thickness, thermal properties, and boundary conductance.

The original charts of Gurney and Lurie (1)[2] give temperature distributions in the interior of the body, but in doing so are limited to relatively few values of boundary resistance and permit good readability only in the larger time ratios. Charts of the Groeber type (2) are plotted so as to give more accurate reading for short times, but each chart is limited to temperature ratios for only one position in the body. As a result, a large number of such charts is needed to give complete spatial coverage and in practical applications the type has been restricted to surface and center temperatures. They have the added disadvantage of poor readability for even moderately small temperature ratios; and, in common with the Gurney-Lurie charts, do not cover the extreme range of conditions met in industrial heat transfer.

Schack (3) took the charts of Groeber and by graphical interpolations was able to add many more values for the dimensionless time parameter X. Newman (4) added to the existing data charts of the Groeber type for cylinders, and in addition ingeniously derived simple mathematical relations for solving two- and three-dimensional heat-flow problems. Hottel (5) greatly extended the range of charts of the Gurney-Lurie type for surface, center, and "space-mean" temperatures for semi-infinite slabs. Bachmann (6) provided additional charts for slabs, cylinders, and spheres, but in his charts used dimensionless parameters different from those used in the charts previously described.

Considered as a group, existing charts are particularly poor for those problems which involve large boundary resistances, such as occur in low-temperature work, convection heating, or the heating of thin pieces in the high-temperature range. They also are not applicable to many problems involving extremely short heating or cooling periods such as occur in quenching and hardening.

All available charts can be applied only to heating by exposure to a constant ambient temperature. An entirely different, but equally important heating problem is presented by induction heating, in which heat is generated at a constant rate at or near the surface. If the frequency is sufficiently high, it is permissible to consider this heat generation to take place entirely at the surface. The widespread application of high-frequency induction heating has created a need for heating charts similar to those which are available for constant-ambient-temperature

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Note: Statements and opinions advanced in papers are to be understood as individual expressions of their authors and not those of the Society.

[2] Numbers in parentheses refer to the Bibliography at the end of the paper.

heating. A solution of the equations governing heat flow in bodies subjected to a constant rate of heat generation brings out the important fact that when expressed in dimensionless units these equations are identical in from to those which govern heat flow in the same bodies but subjected to constant ambient temperature. As a result, the charts for constant-temperature heating also can be used for all heating problems characterized by constant surface heat generation.

It is the purpose of the present paper: (*a*) To present short-time charts for simple bodies subjected to very short heating cycles; (*b*) to extend the range of relative boundary-resistance factors in charts of the Gurney-Lurie type so as to include all values from zero to infinity; (*c*) to present temperature charts that, for relatively long time ratios, will give temperature distributions throughout the body rather than only at the surface and center; (*d*) to show that the radiation-convection type of heating charts can be used for those cases of induction heating which can be considered to have purely surface-heating effects.

All of the experimental work connected with the temperature charts was done on the "heat and mass flow analyzer" at Columbia University. The analyzer and the method upon which it is based have been described repeatedly (7) and therefore do not need to be described here. All of the short-time charts, with the exception of those for $n = 1$ for plates and spheres, were obtained by means of the analyzer. The long-time charts were obtained for the greater part by calculations; some check tests were run on the analyzer.

PART 1 CONSTANT-TEMPERATURE (C-T) HEATING

Charts for Short-Time Heating Cycles

The purpose of short-time heating charts is to give accurate temperature readings for very small dimensionless time ratios. Charts of this nature are shown in Figs. 1 to 6, inclusive. There is little difference between these charts and the other heating-cooling charts familiar to the engineer. As is customary, a dimensionless time ratio X is plotted as abscissa and a dimensionless temperature ratio Y as ordinate. Each curve holds for a different relative boundary-resistance factor m. Each set of curves is labeled with a suitable dimensionless position ratio n.

Consider, for example, Fig. 1, which applies to semi-infinite slabs, and which has a position ratio n of 1. For a plate heated on both sides n is by definition (distance from mid-plane of plate)/(half thickness of plate). Since n is equal to 1, the temperature chart therefore applies to the surface of a slab. Fig. 2 shows curves for the center ($n = 0$), and for the plane midway between surface and center ($n = 0.5$). It should be noted that a different temperature ratio is plotted here than is customarily found in the standard heating-cooling charts mentioned. In those charts, and in Figs. 7, 8, 9, and 13 of this paper, the cooling ratio which is customarily designated by Y is used; in this paper the heating ratio, which is $1 - Y$, is used for Figs. 1 to 6, and is designated by Y^*. Y^* and Y are connected by the relation $Y^* = 1 - Y$.

Two scales are used for each ordinate. This has been done

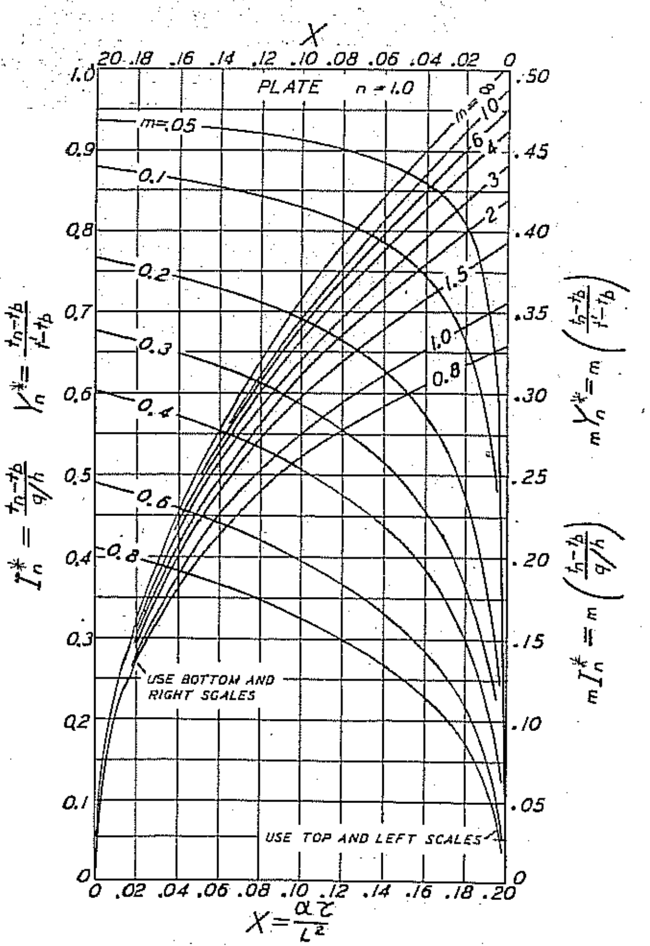

Fig. 1 Chart for Determining Temperature History at Surface of Semi-Infinite Plate

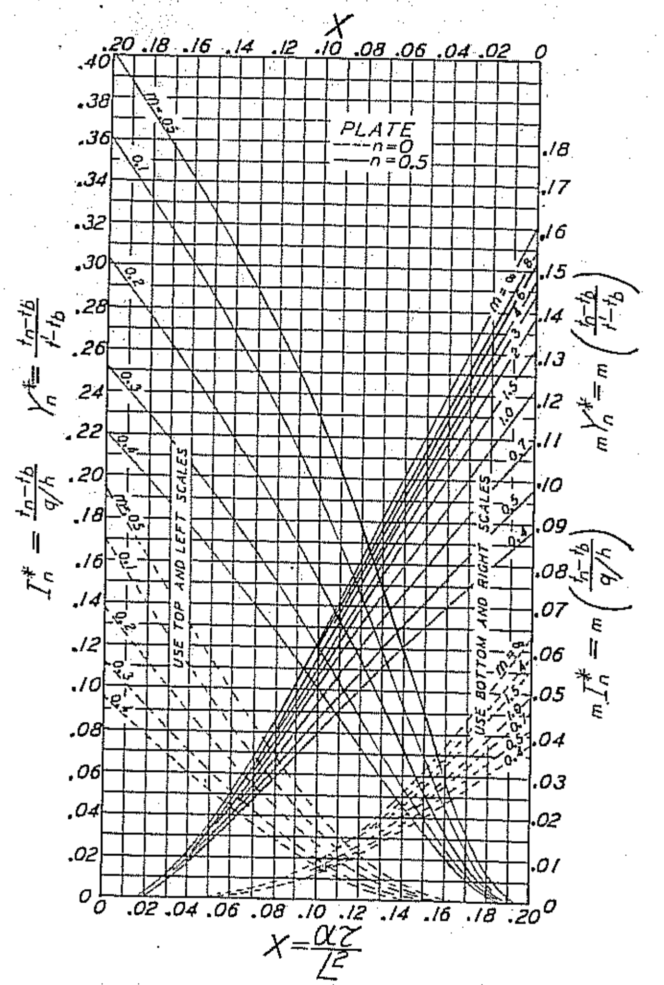

Fig. 2 Chart for Determining Temperature History at Center ($n = 0$) and Mid-Plane ($n = 0.5$) of Semi-Infinite Plate

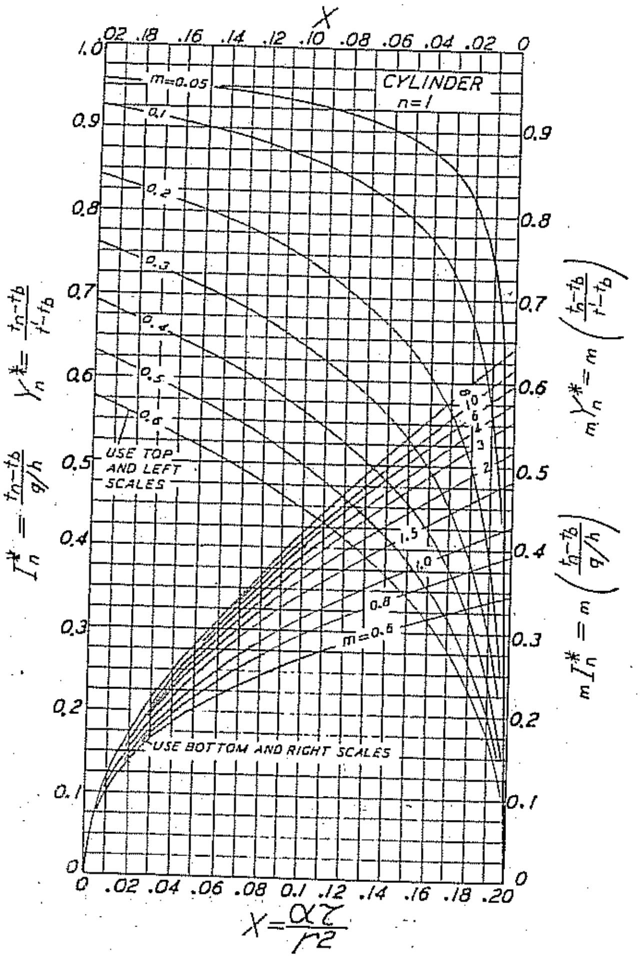

FIG. 3 CHART FOR DETERMINING TEMPERATURE HISTORY AT SURFACE OF INFINITELY LONG CYLINDER

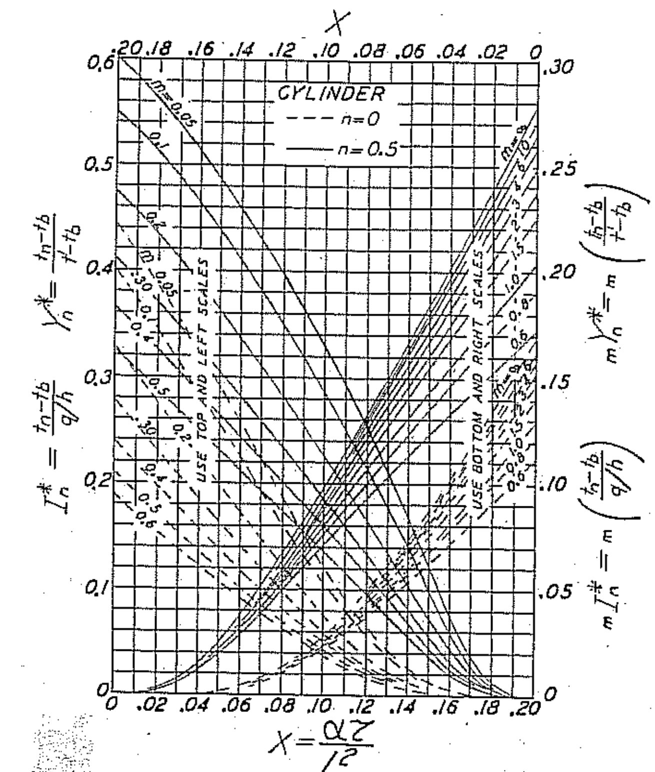

FIG. 4 CHART FOR DETERMINING TEMPERATURE HISTORY AT CENTER (n = 0) AND HALF-RADIUS (n = 0.5) OF INFINITELY LONG CYLINDER

because the charts can be used not only for constant (ambient) temperature (c-t) heating but also for induction heating. In c-t heating, a large relative boundary resistance m means very low surface temperatures for the short time of exposure to heat. In induction heating, on the other hand, a large relative boundary resistance means that little heat can escape from the surface; therefore high temperatures result from high m values, in contrast to the low temperatures in c-t heating. In order to have a readable scale, the product mY^* has been plotted for large m values. The I^* factor applies to induction heating and will be explained in Part 2. The use of the charts will be illustrated in the following example:

Example 1. A 9-in-thick steel slab initially at 70 F is suddenly put into a 2400 F constant-temperature furnace and heated from one side only. If the conductivity is 25 Btu/ft, hr, F, specific weight is 460 lb per cu ft, specific heat is 0.120 Btu/lb, F, and the average surface heat-transfer coefficient between flame and stock is assumed to be 94 Btu/sq ft, hr, F, what will be the temperatures at the surfaces and at the center after 12 min of heating?

$$X = (25)(12/60)/(0.12 \times 460)(9/12)^2 = 0.1607$$
$$m = (25)/(94)(9/12) = 0.354$$

From Fig. 1: At the exposed surface; $Y^*_1 = 0.621 = (t_1 - 70)/(2400 - 70)$. From this the exposed surface temperature t_1 is found to be 1515 F.

From Fig. 2: At the unexposed surface; $Y^*_0 = 0.091 = (t_0 - 70)/(2400 - 70)$. From this the unexposed temperature t_0 is found to be 282 F.

From Fig. 2: At the center $n = (4.5)/(9)$; $Y^*_n = 0.191 = (t_n - 70)/(2400 - 70)$. From this the center temperature is found to be 515 F.

If the slab had cooled from 2400 F instead of heating from 70 F, all other conditions remaining unchanged, the temperatures would be as follows:

At the exposed surface: $0.621 = (2400 - t_1)/(2400 - 70)$. From this the exposed surface temperature is found to be 954 F.

At the unexposed surface: $0.091 = (2400 - t_0)/(2400 - 70)$. From this the unexposed surface temperature is found to be 2188 F.

At the center: $0.191 = (2400 - t_n)/(2400 - 70)$. From this the center temperature is 1954 F.

CHARTS FOR LONG-TIME HEATING CYCLES

When X is greater than 0.2 and m is less than 100, Figs. 7 to 12, inclusive, can be used. Figs. 7, 8, and 9 give temperatures at the mid-plane of a plate, the axis of a cylinder, and the center of a sphere, respectively. Figs. 10, 11, and 12 give a multiplication factor by which the temperature at any other position n can be found.

Suppose, for example, the temperature ratio at $n = 0.4$ is desired when the temperature ratio Y_0 at $n = 0$ has been found, from Fig. 7, 8, or 9. With the same m that was used in finding Y_0 go to Fig. 10, 11, or 12, enter the abscissa with this m go vertically to the curve for $n = 0.4$, and read the ordinate Y_n/Y_0. Then form the product $Y_0(Y_n/Y_0) = Y_n$.

The heating of a semi-infinite slab will be used to illustrate the reasoning upon which the figures are based. Consider the following equation, which governs the heating-up of semi-infinite plates:

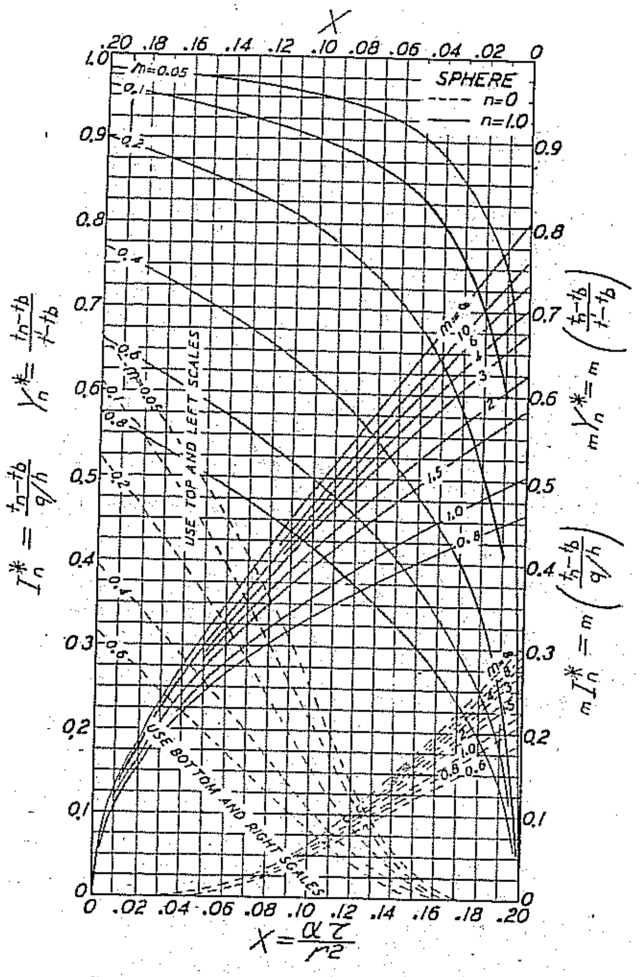

Fig. 5 Chart for Determining Temperature History at Center ($n = 0$) and Surface ($n = 1$) of Sphere

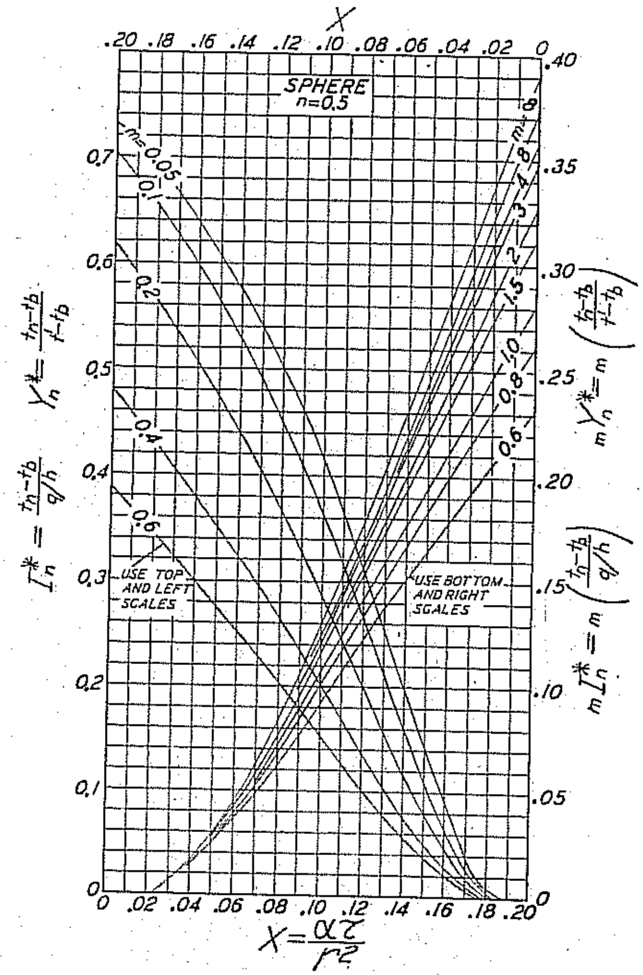

Fig. 6 Chart for Determining Temperature History at Half-Radius of Sphere

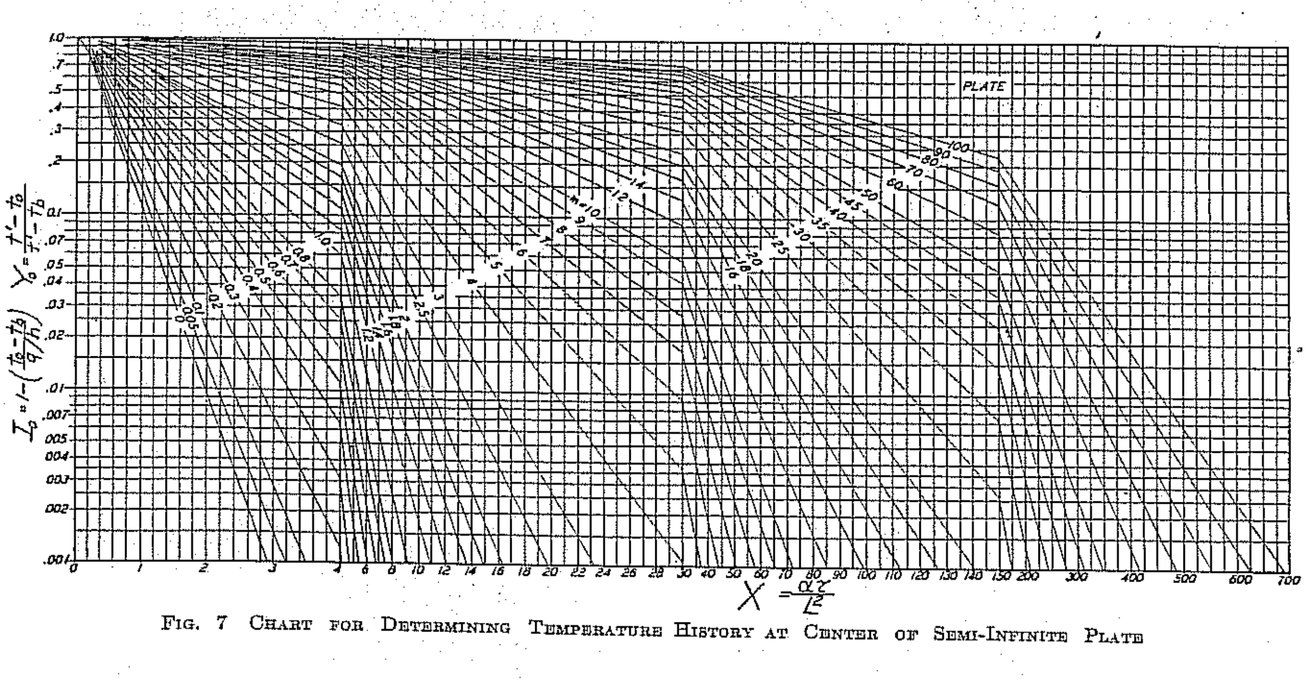

Fig. 7 Chart for Determining Temperature History at Center of Semi-Infinite Plate

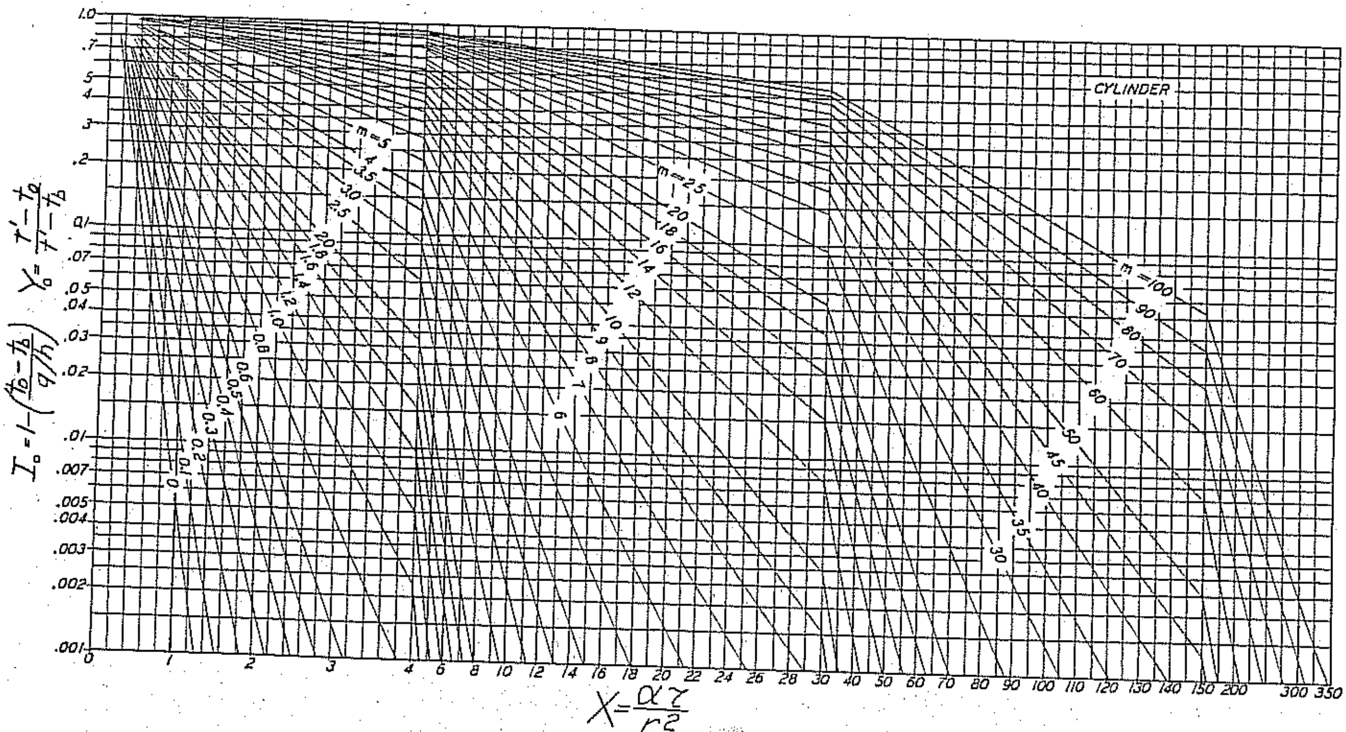

FIG. 8 CHART FOR DETERMINING TEMPERATURE HISTORY AT CENTER OF INFINITELY LONG CYLINDER

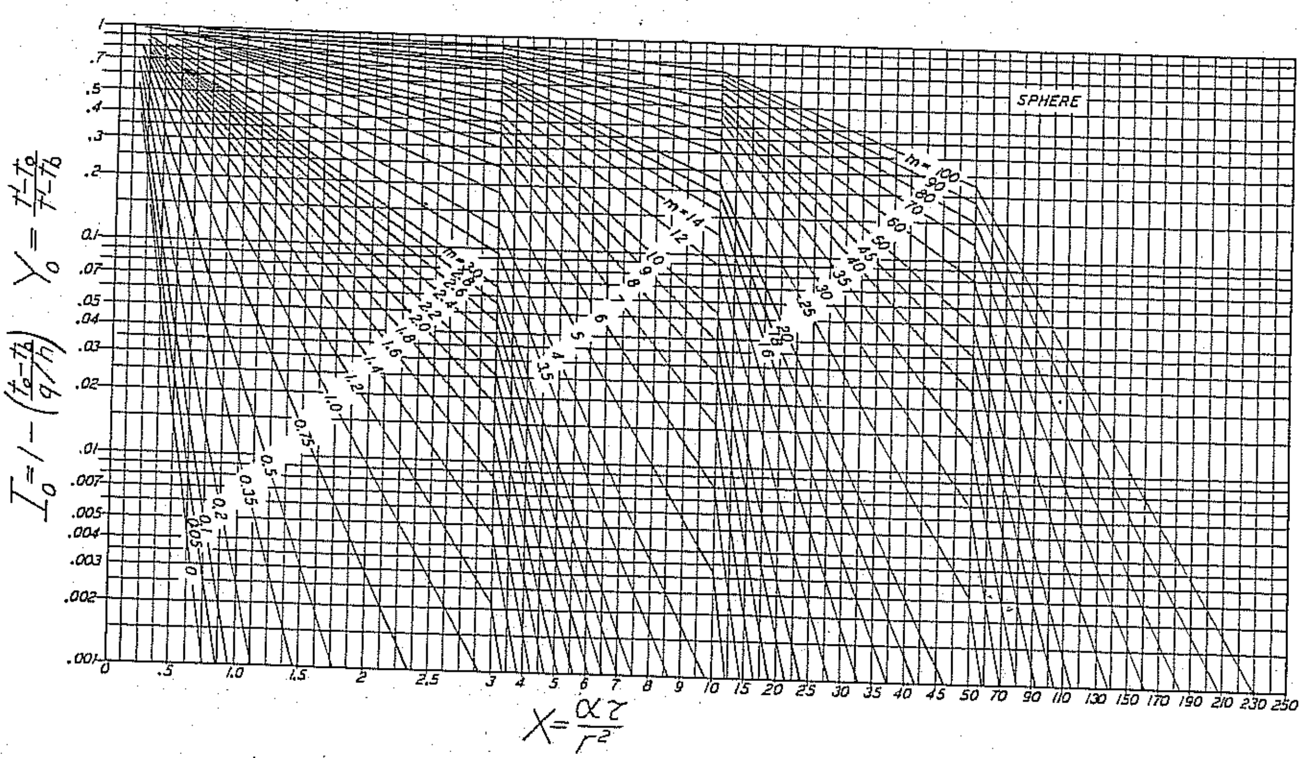

FIG. 9 CHART FOR DETERMINING TEMPERATURE HISTORY AT CENTER OF SPHERE

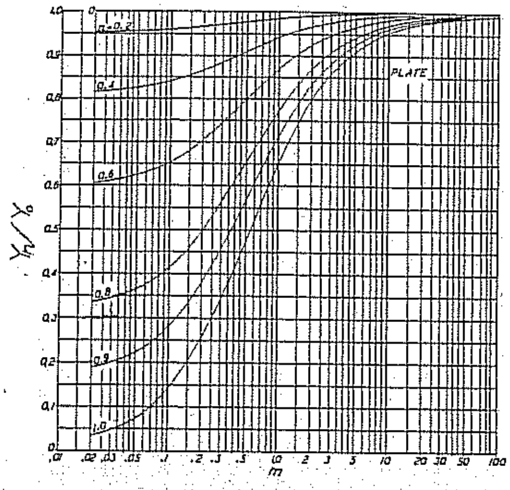

FIG. 10 POSITION CORRECTION FACTORS FOR DIMENSIONLESS TEMPERATURE RATIOS FOR SEMI-INFINITE PLATE

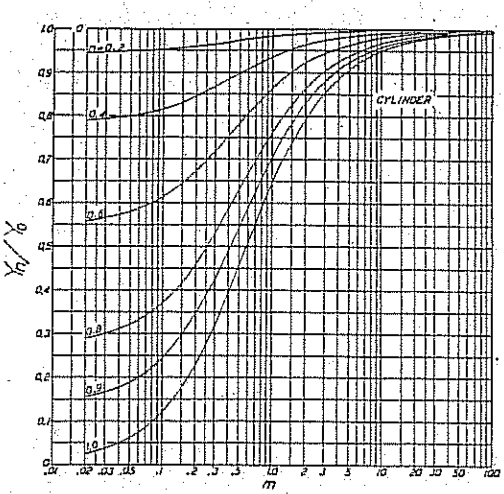

FIG. 11 POSITION CORRECTION FACTORS FOR DIMENSIONLESS TEMPERATURE RATIOS FOR INFINITELY LONG CYLINDER

$$Y_n = \sum_{k=1}^{\infty} \frac{2(\sin w_k)(\cos w_k n)\exp(-w_k^2 X)}{w_k + (\sin w_k)(\cos w_k)} \quad \ldots\ldots [1]$$

This equation converges rapidly and for values of X larger than 0.2 all terms after the first can be neglected. Equation [1] then reduces to

$$Y_n = \frac{2(\sin w)(\cos wn)\exp(-w^2X)}{w + (\sin w)(\cos w)} \quad \ldots\ldots [2]$$

Since at the center n is zero, the equation for the temperature at the center can be written

$$Y_0 = \frac{2(\sin w)\exp(-w^2X)}{w + (\sin w)(\cos w)} = a_w \exp(-w^2X) \quad \ldots\ldots [3]$$

Dividing Equation [2] by Equation [3] gives $Y_n/Y_0 = (\cos w)$, where w is the first root of the characteristic number equation $mw = \cot w$. It is clearly evident that two sets of charts, one giving the temperature ratios at the center, and the other the ratio $Y_n/Y_0 = (\cos w)$ will serve to give all temperature ratios from $n = 0$ to $n = 1$. For the plate, Y_0 is plotted in Fig. 7, and

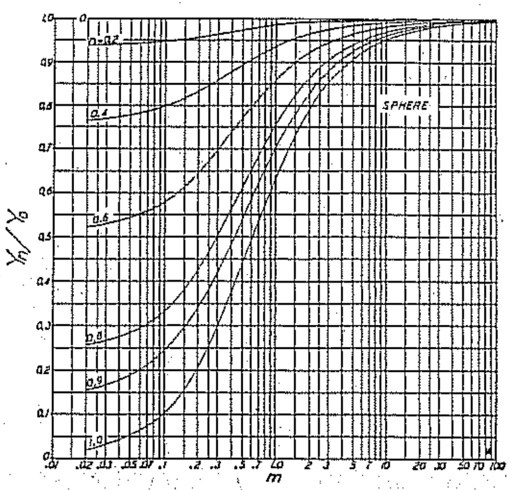

FIG. 12 POSITION CORRECTION FACTORS FOR DIMENSIONLESS TEMPERATURE RATIOS FOR SPHERE

the position factor in Fig. 10. The use of the charts will be illustrated by means of a simple example.

Example 2. A steel rod 1 in. diam and having the same properties as the steel plate of Example 1, is initially at a uniform temperature of 70 F. It is suddenly put into a 1600 F furnace and kept there until the center reaches a temperature of 1400 F. What is the temperature distribution at distances of 0.1, 0.2, 0.3, 0.4, and 0.5 in. from the center? What is the heating time when the center has reached 1400 F? Assume h to be 36.9 Btu/sq ft hr F.

$$m = (25)/(36.9)(1/24) = 16.3$$

$$y_0 = (1600 - 1400)/(1600 - 70) = 0.1307$$

At	n is	From Fig. 11 Y_n/Y_0 is	Therefore, $Y_n = (Y_n/Y_0)(0.1307) = \frac{(1600 - t_n)}{(1600 - 70)}$	From which t_n is
0.1 in.	0.2	0.998	0.1304	1400.3 F
0.2 in.	0.4	0.995	0.1300	1401.2 F
0.3 in.	0.6	0.990	0.1294	1402.0 F
0.4 in.	0.8	0.980	0.1282	1403.8 F
0.45 in.	0.9	0.975	0.1274	1405.0 F
0.5 in.	1.0	0.970	0.1267	1406.0 F

From Fig. 8, for $m = 16.3$ and $Y_0 = 0.1307$, X is 16.7. From this the heating time is 3.84 min.

In the example just given it was assumed that the center temperature was known and that the other temperatures and the heating time were the unknowns. Problems in which the heating time is known and the temperatures are the unknowns can be solved just as readily by finding Y_0 from the given heating time and relative boundary resistance and forming the $(Y_n/Y_0)Y_0$ products. Cylinders and spheres are treated in a similar manner.

RELATIVE BOUNDARY RESISTANCES GREATER THAN 100

There are many problems in which m values much greater than 100 are met. These can be solved by using Fig. 13, it being understood that X is greater than 0.2. To illustrate the principles underlying this graph, it will be convenient to consider Equation [3], although it should be noted that the analysis is equally true for cylinders and spheres. It can readily be shown that as m approaches infinity, the first root of $mw = \cot w$ approaches zero. Therefore, for small values of w, $2(\sin w)/(w - \sin w \cos w)$ approaches $2w/(w + w)$, since for small angles $\sin w$ approaches w and $\cos w$ approaches 1. Consequently, Equation

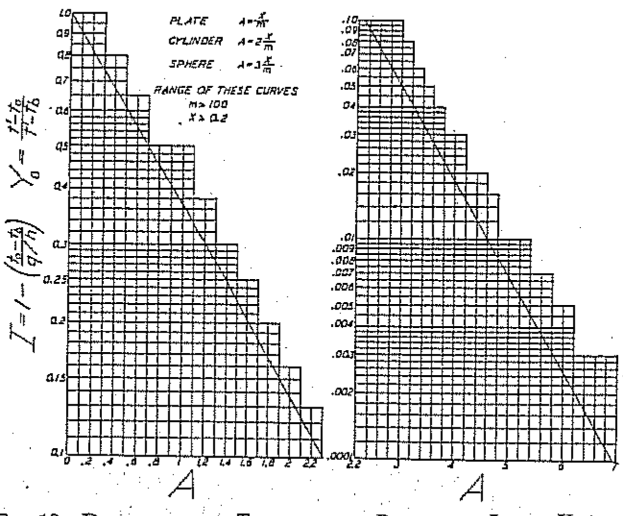

FIG. 13 DIMENSIONLESS TEMPERATURE RATIOS FOR LARGE VALUES OF RELATIVE BOUNDARY RESISTANCE ($m > 100$)

[3] reduces to $Y_0 = \exp(-w^2X)$. If, in addition, $mw = \cot w$ is written as $(1/m) = (w) \tan w$, and $\tan w$ is expressed in its series form, it can readily be shown that for the small values of w under consideration it is sufficiently accurate to consider only the first term in the expansion. As a result $(1/m) = (w) \tan w$ reduces to $1/m = w^2$. Therefore, Y_0 can be written as $Y_0 = \exp(-X/m)$. By similar arguments it can be shown that for cylinders $Y_0 = \exp(-2X/m)$ and for spheres $Y_0 = \exp(-3X/m)$. Or, using a general form, $Y_0 = \exp(-A)$, where for plates $A = X/m$, for cylinders $2X/m$, and for spheres $3X/m$. This is the form that has been plotted in Fig. 13. As a matter of practical interest, it is pointed out that for values of A less than 0.01, Y_0 reduces to $1 - A$. To illustrate the use of the chart consider the following example:

Example 3. A $^1/_{32}$-in. partition in a steel ventilation duct is at 70 F. Hot air at a constant temperature of 200 F is suddenly blown through the duct. Assuming the average surface heat-transfer coefficient to be 4 Btu/sq ft, hr, F, find the time it will take for the center ($^1/_{64}$ in. from either surface) of the partition to reach 195 F.

$$Y_0 = (200 - 195)/(200 - 70) = 0.0385$$
$$m = 25 \times (64 \times 12)/4 = 4800$$

From Fig. 13, for $Y_0 = 0.0385$, $A = X/m$ is 3.26. From this X is $3.26 \times 4800 = 15{,}640$. If we assume that the thermal properties of the steel are the same as those given in the preceding examples, the heating time is $15{,}640/(0.452)(64 \times 12)^2 = 0.0587$ hr = 210 sec.

PART 2 INDUCTION HEATING

High-frequency induction heating, using frequencies in the order of millions of cycles per second, is fundamentally different in its effects, both mathematically and physically, from low-frequency heating. Where high-frequency heating is characterized by the generation of heat in a very thin layer near the surface of the workpiece, low-frequency heating is associated with a space-varying generation of heat. For this latter case the relation between strength of generation and depth of penetration depends, for a given material, directly upon the frequency. A mathematical solution is possible for problems of this type, provided that the relation between frequency and depth can be formulated. For those frequencies in which the heating effect is concentrated in such a thin shell that, for all practical purposes, the heating can be considered to be surface heating, a simple solution is possible (see Appendix).

In high-frequency heating of metals there is a pronounced skin-heating effect, the thickness of the skin being, among other things, a function of the frequency of the current in the induction coils. For commonly used steels, Curtis (8) quotes the following figures as representative of the relation between depth of penetration and frequency: For 2000 cycles per sec, a skin depth of $^1/_8$ in., and for 200,000 cycles per sec a depth of 0.020 in. Sherman (9) gives the equivalent depth of uniform penetration as less than 0.001 in. at 5,000,000 cycles. Such high frequencies have come into widespread use in thin-case hardening, in which the quenching of the surface layer is accomplished either entirely or in large part by the cold core just beneath the surface layer of heated metal.

The actual conditions of high-frequency induction heating can be closely simulated by applying a constant heat flux to the surface of the workpiece. If a constant heat flux is applied, for example, to a long cylindrical rod, part of the flux will penetrate into the rod by conduction and the remainder will be lost to the surroundings by convection and radiation. As the temperature of the rod increases, a larger and larger part of the heat input will be lost until the surface temperature finally assumes a value at which all the heat flux is lost to the surroundings. Thus if q is the flux in Btu/sq ft, hr, and h is the surface heat-transfer coefficient at steady-state conditions in Btu/sq ft, hr, F, the rod will, after an infinite time, assume a final uniform temperature of q/h deg F above the original temperature of the body and surroundings.

It is shown in the Appendix that, if a dimensionless temperature ratio I^* (for induction heating) is defined as $(t_n - t_b)/(q/h)$, where t_n is the temperature at position n in the workpiece, and t_b is the original uniform temperature, then the equations which govern the flow of heat in a body subject to c-t heating are identical with the equations governing the flow of heat in the same body subject to induction heating. It then follows that the same curves which are used for c-t heating can also be used for high-frequency induction heating. It should be pointed out that in Figs. 1 to 6, inclusive, the relation $I^*_n = (t_n - t_b)/(q/h)$ is plotted, whereas in Figs. 7 to 9, inclusive, the relation $I_0 = 1 - (t_0 - t_b)/(q/h)$ is plotted. For the latter case, temperatures at any position n can be found from Figs. 10 to 12, by forming the product $I_n = I_0(I_n/I_0)$ just as is done for c-t heating. The procedure will be illustrated by the following examples:

Example 4—Short-Time Induction Heating. The 1-in. bar of Example 2 is to be heated by induction to a surface temperature of 1600 F in 1.5 sec. (a) What energy input, in kilowatts per square inch, must be applied at the surface? (b) What will be the final center temperature of the bar?

Assuming that the average surface temperature is (2/3) (1600) = 1070 F, that the temperature of the surroundings is 70 F, and that heat is lost by radiation only, h will be 9.9 Btu/sq ft, hr, F; therefore

$$m = 25 \times 24/9.9 = 60.1$$
$$X = 0.425 \times 1.5/(1/24)^2(3600) = 0.1084$$

From Fig. 3, $mI^*_1 = m(t_1 - 70)/(q/9.9) = 0.444$

$$q = (1600 - 70)(60.1 \times 9.9)/(0.444)$$
$$= 2{,}050{,}000 \text{ Btu/sq ft, hr, F}$$
$$= (2{,}050{,}000)/(3413 \times 144)$$
$$= 4.17 \text{ kw per sq in.}$$

To find the temperature at the center:

From Fig. 4, $mI^*_0 = m(t_0 - 70)/(2{,}050{,}000/9.9) = 0.036$, from which t_0 is 1312 F.

Example 5—Long-Time Induction Heating. If, in the foregoing example, the heating time is changed to 200 sec: (*a*) What must be the energy input in kilowatts per square inch? (*b*) What will be the center temperature? Assume all properties are unchanged.

$$X = 0.452(200/3600)/(1/24)^2 = 14.5$$

From Fig. 11

$$I_1/I_0 = 0.991$$

From Fig. 8

$$I_0 = 1 - (t_0 - t_b)/(q/h) = 0.65$$

Therefore

$$I_1 = 0.991 \times 0.65 = 0.644 = 1 - (1600 - 70)/(2/9.9)$$

From this q is found to be 42,600 Btu/sq ft, hr = 0.0865 kw per sq in. Then

$$I_0 = 1 - (t_0 - 70)/(42{,}600/9.9) = 0.65$$

and, therefore

$$t_0 = 1580 \text{ F}$$

It is emphasized that the methods given for induction heating are precisely accurate only when all of the energy is generated at the surface of the workpiece. Therefore the application of the curves to actual conditions must be tempered with judgment. For very high frequencies, it is reasonable to assume that negligible error is involved; for lower frequencies, which involve greatest errors, the methods can be usefully applied to determine general trends or limiting values.

PART 3 TWO-DIMENSIONAL HEAT FLOW

The fact that the infinite series represented by Equation [1] reduces to the simple form of Equation [3] can be used to considerable advantage in determining heating times and temperatures within bodies subjected to two-dimensional heat flow. Consider the equation for the heating of an infinite cylinder (the subscript c denotes cylinder)

$$(Y_n)_c = \sum_{k=1}^{\infty} \frac{2J_1(u_k)J_0(u_k n)\exp(-u^2_k X_c)}{u_k[J^2_0(u_k) + J^2_1(u_k)]} \quad \ldots\ldots [4]$$

where J_0 and J_1 are the Bessel functions of order 0 and 1.

Now, it is just as true for cylinders as it is for plates that, for X greater than 0.2, all terms after the first become negligible. Therefore for n equal to zero

$$(Y_0)_c = \frac{2J_1(u)\exp(-u^2X_c)}{u[J^2_0(u) + J^2_1(u)]} = a_u \exp(-u^2X_c) \ldots\ldots [5]$$

where u is found from the characteristic number equation $mu = J_0(u)/J_1(u)$. It is clearly evident that at any position n, $(Y_n) = J_0(un)(Y_0)_c$. $(Y_0)_c$ is plotted in Fig. 8, and $J_0(un)$ in Fig. 11. It is possible from these two figures to find the temperature at any point in an infinitely long cylinder by using the methods for the slab illustrated in Part 1, Example 2.

The extension of this simple principle to finite cylinders follows readily. Newman (4) showed that, for finite cylinders, the temperature ratio at any point (n_p, n_c) in the cylinder is equal to the product $(Y_n)_p(Y_n)_c$. Therefore for all X greater than 0.2

$$Y_0 = [a_w \exp(-w^2X_p)]\cdot[a_u \exp(-u^2X_c)] = (Y_0)_p(Y_0)_c \ldots [6]$$

Since $X_p = \left(\frac{r}{L}\right)^2 \cdot X_c$ this can be written

$$Y_0 = a_w a_u \exp\left\{-X_c\left[u^2 + \left(\frac{wr}{L}\right)^2\right]\right\} \ldots\ldots\ldots [7]$$

Equation [7] is a function of the variables m_p, m_c, $\frac{r}{L}$ and X_c; it is plotted in Fig. 14. Fig. 14 enables one to find the heating time required to reach any desired temperature within a short cylinder. This is not possible from the one-dimensional charts except by resorting to a trial-and-error solution.

The temperature at any position $n_p n_c$ can be found by means of the following relation

$$Y_{(n_p,n_c)} = (Y_0)_p(\cos wn)_p(Y_0)_c(J_0 un)_c \ldots\ldots\ldots [8]$$

$\text{Cos}(wn)_p$ and $J_0(un)_c$ are plotted in Figs. 10 and 11, in their equivalent temperature form Y_n/Y_i. Note that

$$\cos(wn)_p = (Y_n/Y_0)_p;\ J_0(un)_c = (Y_n/Y_0)_c$$

The use of Figs. 10, 11, and 14, in finding heating times and temperatures will be illustrated in the following example:

Example 6—Short Cylinder. A steel cylinder, conductivity = 25 Btu/ft, hr, F, diffusivity 0.452 sq ft/hr, 6 in. long and 4 in. diam, is put into a 2000 F furnace. Assuming heat is transmitted over the entire surface and h is 75 Btu/sq ft, hr, F, find: (*a*) How long it will take the center to reach 1900 F; and (*b*) what the temperatures will be at points which are 1 in. from the axis and $^1/_2$ in. from either face.

Since both faces are heated

$$\frac{r}{L} = \frac{2}{3} \text{ and } (r/L)^2 = 0.444$$

$$m_p = 25/(3/12)75 = 1.333$$

$$m_c = 25/(2/12)75 = 2$$

$$Y = \frac{2000 - 1900}{2000 - 70} = 0.0519$$

To find the heating time, enter the bottom right quadrant of Fig. 14, at $(r/L)^2 = 0.444$, move horizontally along the broken line to $m_p = 1.333$, and then go vertically to $m_c = 2$. From this point draw a horizontal whose termination is found by going to the top right quadrant at $m_c = 2$, following the broken line to $m_p = 1.333$, and thence horizontally to $Y_0 = 0.0519$. The intersection of a perpendicular from this point to the horizontal located in the preceding step gives $X_c = 2.7$. Before finding the heating time it first must be determined whether X_p is less than 0.2, since the graph is not valid for either X_c or X_p less than 0.2.

$$X_p = (r/L)^2(X_c) = 0.444 \times 2.7 = 1.2$$

Therefore the method is applicable to this problem.

The temperatures are found from the following:

At the edge $n_p = n_c = 1$. From Fig. 10, for $n_p = 1$ and $m_p = 1.333$, $(Y_n/Y_0)_p = 0.718$.

From Fig. 11, for $m_c = 2$ and $n_c = 1$, $(Y_n/Y_0)_c = 0.79$. Since at the edge $(Y_n)_p(Y_n)_c = (Y_n/Y_0)_p(Y_n/Y_0)_c \times (Y_0)_p(Y_0)_c$ and $(Y_0)_p(Y_0)_c = 0.0519$, it follows that $(Y_n)_p(Y_n)_c = \frac{2000 - t}{2000 - 70}$ = (0.718)(0.79)(0.0519), from which the temperature at the edge is found to be 1943 F. At 1 in. from the axis, $n_c = {}^1/_2 = 0.5$. From Fig. 11, $(Y_n/Y_0)_c = 0.946$. At $^1/_2$ in. from the face $n_p = 2.5/3 = 0.833$. From Fig. 10 $(Y_n/Y_0)_p = 0.8$; therefore

$$\frac{2000 - t}{2000 - 70} = (0.946)(0.8)(0.0519) = 0.0392$$

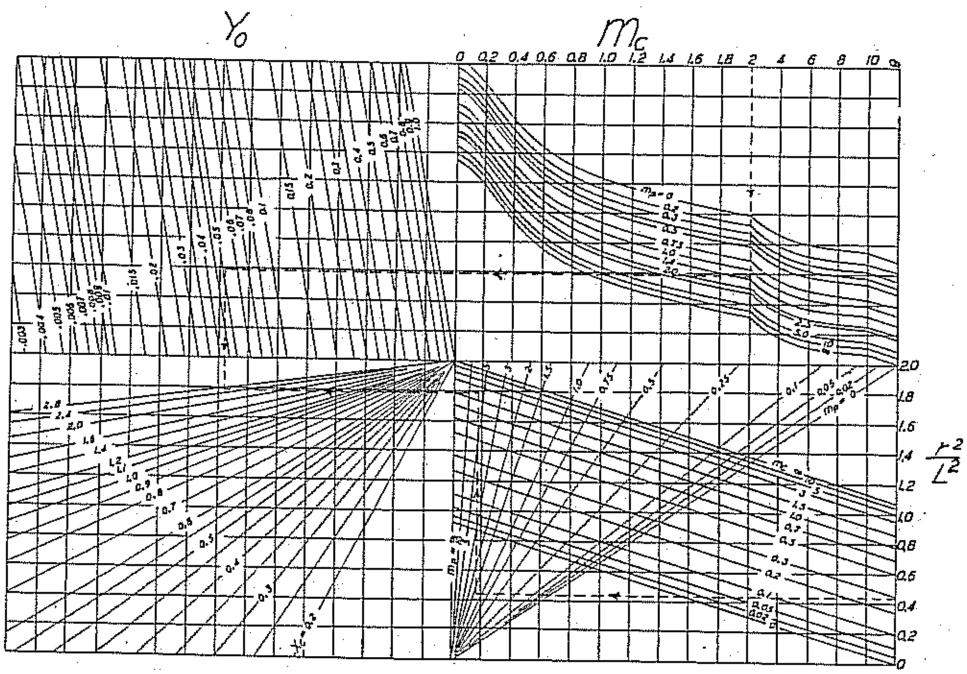

FIG. 14 CHART FOR DETERMINING TEMPERATURE HISTORY AT CENTER OF SHORT CYLINDER

Solving for t gives 1424 F for the temperature at a point $^1/_2$ in. from the axis and $^1/_2$ in. from either face.

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Appendix

DERIVATION OF EQUATIONS FOR HIGH-FREQUENCY INDUCTION HEATING

Additional nomenclature for the Appendix is as follows:

C = volumetric specific heat
α = diffusivity, (conductivity)/(volumetric specific heat)
R = thermal resistivity
R_f = boundary resistance (film resistance)
L^{-1} = inverse Laplace transform
n = as a subscript indicates partial differentiation with respect to relative position n
x = position in heated body; as a subscript denotes partial differentiation with respect to x
xx = as a subscript, second partial differentiation with respect to x

Other units are given in the nomenclature at the beginning of the paper.

The partial differential equation which governs heat flow in a semi-infinite solid of half-thickness L can be written

$$t_\tau = \alpha(t_{xx}) \quad [9]$$

The subscript τ indicates partial differentiation with respect to time, and the subscript xx the second partial differentiation with respect to space. The diffusivity α can be written as $1/RC$, where R is the thermal resistivity and C the volumetric thermal capacity. At the surface of the plate, the total flux q will be equal to the flux lost to the surroundings (through the boundary resistance) and the flux conducted into the plate, that is

$$q = \frac{t_x}{R} + \frac{t(L, \tau)}{R_f} \quad [10]$$

R_f being the film resistance and $t(L, \tau)$ the time-variable surface temperature distant L units from the center. It is convenient to introduce a set of dimensionless parameters. Introducing the following units

$$X = (\alpha\tau)/L^2$$

$$Y(n, X) = \frac{t(n, X) - t_b}{(q/h)}$$

$$n = x/L$$

Equation [9] becomes

$$Y_X(n, X) = Y_{nn}(n, X) \ldots\ldots\ldots\ldots [11]$$

Equation [10] becomes

$$Y_n(1, X) = \frac{1 - Y(1, X)}{m}, \ (X > 0) \ldots\ldots [12]$$

In addition, for the center, we can write

$$Y_n(0, X) = 0, \ (X > 0) \ldots\ldots\ldots\ldots [13]$$

Taking the Laplace transform of Equation [11] and the boundary conditions Equations [12] and [13]

$$sy(n, s) = y_{nn}(n, s), \ (0 > n > 1) \ldots\ldots [14]$$

$$y_n(1, s) = \frac{(1/s) - y(1, s)}{m} \ldots\ldots\ldots [15]$$

$$y_n(0, s) = 0 \ldots\ldots\ldots\ldots [16]$$

The solution of Equation [14] is

$$y(n, s) = A \sinh [n(s)^{\frac{1}{2}}] + B \cosh [n(s)^{\frac{1}{2}}]$$

Therefore

$$y_n(n, s) = (s)^{\frac{1}{2}} A \cosh [n(s)^{\frac{1}{2}}] + (s)^{\frac{1}{2}} B \sinh [n(s)^{\frac{1}{2}}]$$

To satisfy Equation [16], A must be zero. Applying Equation [15]

$$y_n(1, s) = (s)^{\frac{1}{2}} B \sinh (s)^{\frac{1}{2}} = \frac{(1/s) - B \cosh (s)^{\frac{1}{2}}}{m}$$

from which

$$B = \frac{1}{s} \cdot \frac{1}{\cosh (s)^{\frac{1}{2}} - m(s)^{\frac{1}{2}} \sinh (s)^{\frac{1}{2}}}$$

Therefore

$$y(n, s) = \frac{\cosh n(s)^{\frac{1}{2}}}{s\,[\cosh(s)^{\frac{1}{2}} - m(s)^{\frac{1}{2}} \sinh (s)^{\frac{1}{2}}]} \ldots\ldots [17]$$

Taking the inverse transform

$$Y(n, X) = \frac{1}{2\pi i} \int_{Br} \frac{\cosh n(s)^{\frac{1}{2}} \cdot \exp (sX)(dX)}{s[\cosh(s)^{\frac{1}{2}} - m(s)^{\frac{1}{2}} \sinh (s)^{\frac{1}{2}}]}$$

The residue at the simple pole $s = 0$ is readily found to be 1. The poles of $\cosh(s)^{\frac{1}{2}} - m(s)^{\frac{1}{2}}$ occur at $s = -w^2$. The residue is found by taking the inverse transform

$$Y(n, X) = L^{-1} \left\{ \frac{p(s)}{q(s)} \right\} = \sum_{1}^{\infty} \frac{p(s_k)}{q'(s_k)} \ (\text{for } s_k = -w^2_k)$$

This inversion gives

$$\frac{t(n, X) - t_b}{q/h} = Y(n, X)$$

$$= 1 - 2 \sum_{1}^{\infty} \frac{(\sin w_k)(\cos nw_k) \exp(-w_k{}^2X)}{w_k + (\sin w_k)(\cos w_k)} \ldots [18]$$

where w_k is found from the characteristic number equation $mw = \cot w$.

Equation [18] is identically equal to the equation for a slab subjected to a constant temperature. The equations for cylinders and spheres were derived by the same method and found to be identically equal to those for heating in a constant-temperature source. It would not have been necessary to carry out the complete evaluation of the problem because it is obvious from Equations [11], [12], and [13], that the solution must lead to the same equation as for simple heating-up. The Laplace transform has been used to bring out the method by which the short-time curves for the surface of plates and spheres were developed.

Where small values of X are involved, calculation of the temperature ratios Y_n becomes quite laborious because of the number of terms that must be used. An accurate approximation can be had for values of X less than 0.2 by making use of the fact that, as X approaches zero, s approaches infinity; for example, Equation [17] can be written

$$y(n, s) = \frac{\exp [n(s)^{\frac{1}{2}}] + \exp[-n(s)^{\frac{1}{2}}]}{s\{ [\exp(s^{\frac{1}{2}}) - \exp[-(s^{\frac{1}{2}})] - m(s^{\frac{1}{2}}) [\exp(s^{\frac{1}{2}}) - \exp (-s^{\frac{1}{2}})] \}} \ldots [19]$$

Since s is very large, because of the small value of X, $\exp -n(s)^{\frac{1}{2}}$ approaches zero, and

$$y(n, s) \cong \frac{\exp [-(1 - n)(s)^{\frac{1}{2}}]}{s[1 - m(s)^{\frac{1}{2}}]} \ldots\ldots [20]$$

At the surface $n = 1$, therefore

$$y(n, s) \cong \frac{1}{s[1 - m(s)^{\frac{1}{2}}]} \ldots\ldots [21]$$

Taking the inverse transform

$$y(1, X) = 1 - \exp(X/m^2)\, erfc \left(\frac{\sqrt{X}}{m}\right) \ldots\ldots [22]$$

Since the complementary error function is a tabulated function, Equation [22] is convenient for calculation. At the center

$$y(0, X) \cong 2 \left\{ erfc \left(\frac{1}{2\sqrt{X}}\right) - \exp(1/m + X/m^2)\, erfc \left(\frac{1}{2\sqrt{X}} + \frac{\sqrt{X}}{m}\right) \right\} \ldots\ldots [23]$$

These equations can be used where greater accuracy is desired than is possible from the charts. Similar formulas can be derived for other values of n, and the method can be applied with equal facility to cylinders and spheres.