Recursive Equation Solving with Excel

Faith A. Morrison
William R. Kenan Visiting Professor for Distinguished Teaching
Department of Chemical and Biological Engineering
Princeton University

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This document briefly explains using some recursive tools for problem solving with MS Excel 2010.

The Problem

In engineering problems we sometimes encounter equations that we wish to solve with trial-and-error techniques. For example, when using data correlations such as the friction-factor/Reynolds number Prandtl correlation of fluid mechanics, we cannot algebraically solve for friction factor $f$, but we can obtain a numerical solution recursively by guessing $f$, substituting the guessed value on the right-hand side, and subsequently solving for a new value of $f$:

$$\frac{1}{\sqrt{f}} = 4.0 \log_{10} Re \sqrt{f} - 0.4$$

(1)

This solution method is called successive substitution. Another equation that requires numerical solution is the characteristic equation $f(\beta) = 0$ for the eigenvalues $\beta_i$ in heat transfer in a rod. The dimensionless temperature distribution of this problem (Carlsaw and Jaeger, 1946) is given by:

$$\Theta \left( \frac{r}{a}, \tau \right) = \sum_{i=1}^{\infty} e^{-\beta_i^2 \tau} \frac{2Bi J_0 \left( \frac{r}{a} \beta_i \right)}{\left[ \beta_i^2 + Bi^2 \right] J_0 (\beta_i)}$$

(2)

with characteristic equation:

$$f(\beta) = \beta J_1 (\beta) - Bi J_0 (\beta) = 0$$

(3)

where $\Theta = \frac{T-T_{\text{bulk}}}{T_{\text{initial}}-T_{\text{bulk}}}$, $\tau = \frac{a t}{\rho} C_p$, $\alpha = \frac{k}{\rho} C_p$, $\rho$ is density, $C_p$ is heat capacity, $k$ is thermal conductivity, $Bi = \frac{ha}{k}$ is the Biot number, $h$ is heat transfer coefficient, $J_n$ is the Bessel function of the first kind of order $n$, $r$ is radial position in a cylindrical coordinate system, and $a$ is the cylinder radius. The characteristic equation $f(\beta) = 0$ has roots $\beta_i$ for which we must solve before we can evaluate the summation in the expression for $\Theta$; we cannot solve for $\beta_i$ algebraically, but we can do so numerically.

Prandtl Correlation

To evaluate the Prandtl Correlation using Excel we set up the equation as follows.

1. Turn on recursive calculating: Office Button, Excel Options, Formulas, Calculation Options, Workbook Calculation, Enable iterative calculation.
2. (Optional) Set “Enter mode” so that cursor does not move when the Enter key is pressed: Office Button, Excel Options, Advanced, Editing Options, de-select button for “After pressing Enter, move selection down.”
3. Enter the cells and formulas as shown in Figure 1. We have used Excel cell-labeling for clarity.
4. Close the loop of the recursion by going to the cell containing the first guess (B2 in this example) and enter the address of the new value of friction factor(B4). Excel will iterate until a final answer is obtained (Figure 2). If Excel is in “Manual recalculation” mode, press the F9 key to recalculate.
Eigenvalue Evaluation

To evaluate Equation 3 for the roots $\beta_i$, it is best that we first plot $f(\beta)$; $f(\beta)$ is shown for various Biot numbers in Figure 3. The Bessel functions were evaluated with the Excel function BesselJ(x, n), where x is the argument of the Bessel function and n is the order of the Bessel function.

The eigenvalues are the roots of the characteristic equation, and thus we seek the values of $\beta$ at which $f(\beta)$ crosses the x-axis. Choosing $Bi = 1$ for this illustration, we replot the data, zooming in on the first root (Figure 4). To solve for the value of $\beta$ associated with this first root, we proceed as follows:

1. Begin with a guess for $\beta = x_1$ that is not too far away from the root.
2. If the value of $f(x_1)$ is not equal to zero, we find a next guess $x_2$ by following the slope of the line down to where it intersects the x-axis.
3. If the value of $f(x_2)$ is not equal to zero, we repeat step 2 until $f(x_i) = 0$.

To estimate the slope needed in step 2, we could rigorously calculate the slope from the function, but it is often just as effective to estimate the slope from a two-point calculation. Thus, for two points $x$ and
$x + x_{\text{step}}$, the function evaluates to and $y|_x$ and $y|_{x + x_{\text{step}}}$, and the slope and intercept of this secant line are

\[
\text{slope} = \left( \frac{y|_{x + x_{\text{step}}} - y|_x}{x_{\text{step}}} \right) \\
\text{intercept} = y|_x - (x)(\text{slope})
\]

The new value of $x_2$ in our algorithm is thus

\[
x_{\text{new}} = x_2 = -\left( \frac{\text{intercept}}{\text{slope}} \right)
\]

We carried out this algorithm for the first root of the characteristic equation for $Bi = 1.0$ (Figure 5). To start the iteration, we create a circular reference between the cell labeled $x_{\text{guess}}$ and that labeled $x_{\text{new}}$. We must first have turned on recursive calculations as explained in the first example. The root found is $x_{\text{final}} = 1.255784 = \beta_1$. Note that if our first guess is not close enough to the root, we may find instead the second or even higher roots (try an initial guess of 0.2). This difficulty can be addressed with more sophisticated root-seeking algorithms.
Figure 4: The root seeking secant algorithm described in the text. The first guess is $x_1$; the slope of the function at $x_1$ is estimated by a secant line at that point and the next guess for $x$ is where the secant line intersects the x-axis. The algorithm then repeats.

Figure 5: The secant algorithm is implemented in Excel, taking advantage of the recursive calculation option. At left, we enter the formulas described in the text. At the middle, we create a circular reference by making the contents of cell B2 equal to “=B8” or “=x_new” (the cell B8 is labeled x_new). At the right, we see the result of the iterative calculation: $f(x)=0$ and $x\_guess=x\_new=1.255784$.

Reference