Recursive Equation Solving with Excel

Faith A. Morrison

William R. Kenan Visiting Professor for Distinguished Teaching
Department of Chemical and Biological Engineering
Princeton University

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This document briefly explains using some recursive tools for problem solving with MS Excel 2010.

The Problem

In engineering problems we sometimes encounter equations that we wish to solve with trial-and-error techniques. For example, when using data correlations such as the friction-factor/Reynolds number Prandtl correlation of fluid mechanics, we cannot algebraically solve for friction factor f, but we can obtain a numerical solution recursively by guessing f, substituting the guessed value on the right-hand side, and subsequently solving for a new value of f:

$$\frac{1}{\sqrt{f}} = 4.0 \log_{10} Re \sqrt{f} - 0.4 \tag{1}$$

This solution method is called successive substitution. Another equation that requires numerical solution is the characteristic equation $f(\beta) = 0$ for the eigenvalues β_i in heat transfer in a rod. The dimensionless temperature distribution of this problem (Carlsaw and Jaeger, 1946) is given by:

$$\Theta\left(\frac{r}{a}, \tau\right) = \sum_{i=1}^{\infty} e^{-\beta_i^2 \tau} \frac{2 Bi J_0\left(\frac{r}{a}\beta_i\right)}{\{\beta_i^2 + Bi^2\} J_0(\beta_i)} \tag{2}$$

with characteristic equation:

$$f(\beta) = \beta J_1(\beta) - Bi\, J_0(\beta) = 0 \tag{3}$$

where $\Theta = \frac{T - T_{bulk}}{T_{initial} - T_{bulk}}$, $\tau = \frac{\alpha t}{a^2}$, $\alpha = k/\rho C_p$, ρ is density, C_p is heat capacity, k is thermal conductivity, $Bi = ha/k$ is the Biot number, h is heat transfer coefficient, J_n is the Bessel function of the first kind of order n, r is radial position in a cylindrical coordinate system, and a is the cylinder radius. The characteristic equation $f(\beta) = 0$ has roots β_i for which we must solve before we can evaluate the summation in the expression for Θ; we cannot solve for β_i algebraically, but we can do so numerically.

Prandtl Correlation

To evaluate the Prandtl Correlation using Excel we set up the equation as follows.
1. Turn on recursive calculating: Office Button, Excel Options, Formulas, Calculation Options, Workbook Calculation, Enable iterative calculation.
2. (Optional) Set "Enter mode" so that cursor does not move when the Enter key is pressed: Office Button, Excel Options, Advanced, Editing Options, de-select button for "After pressing Enter, move selection down."
3. Enter the cells and formulas as shown in Figure 1. We have used Excel cell-labeling for clarity.
4. Close the loop of the recursion by going to the cell containing the first guess (B2 in this example) and enter the address of the new value of friction factor(B4). Excel will iterate until a final answer is obtained (Figure 2). If Excel is in "Manual recalculation" mode, press the F9 key to recalculate.

	A	B	C	D	E
1	Reynolds Number=	10000			
2	f_guess=	0.02			
3	RHS=	=4*LOG(Reynolds*SQRT(fguess))-0.4			
4	f_new=	0.006716			
5					

Figure 1: To evaluate the Prandtl correlation with Excel, enter a Reynolds number, a guess at the friction factor, and a cell that evaluates the righthand side (RHS) or the Prandtl correlation by pointing at the first two cells. The new friction factor value is now $1/RHS^2$. In the example we have named various cells (Reynolds, fguess) so that the formulas are easier to proofread.

	A	B	C	D	E
1	Reynolds Number=	10000			
2	f_guess=	=B4			
3	RHS=	12.20206			
4	f_new=	0.006716			
5					

Figure 2: To close the recursion loop, enter the formula in the f_guess cell that points to the new guess for friction factor. This is a circular definition; upon clicking Enter, Excel will perform iterative evaluation of the formulas in the loop until it converges to an answer or until the maximum number of iterations is reached. The answer is 0.007727.

Eigenvalue Evaluation

To evaluate Equation 3 for the roots β_i, it is best that we first plot $f(\beta)$; $f(\beta)$ is shown for various Biot numbers in Figure 3. The Bessel functions were evaluated with the Excel function BesselJ(x,n), where x is the argument of the Bessel function and *n* is the order of the Bessel function.

The eigenvalues are the roots of the characteristic equation, and thus we seek the values of β at which $f(\beta)$ crosses the x-axis. Choosing $Bi = 1$ for this illustration, we replot the data, zooming in on the first root (Figure 4). To solve for the value of β associated with this first root, we proceed as follows:

1. Begin with a guess for $\beta = x_1$ that is not too far away from the root.
2. If the value of $f(x_1)$ is not equal to zero, we find a next guess x_2 by following the slope of the line down to where it intersects the x-axis.
3. If the value of $f(x_2)$ is not equal to zero, we repeat step 2 until $f(x_i) = 0$.

To estimate the slope needed in step 2, we could rigorously calculate the slope from the function, but it is often just as effective to estimate the slope from a two-point calculation. Thus, for two points x and
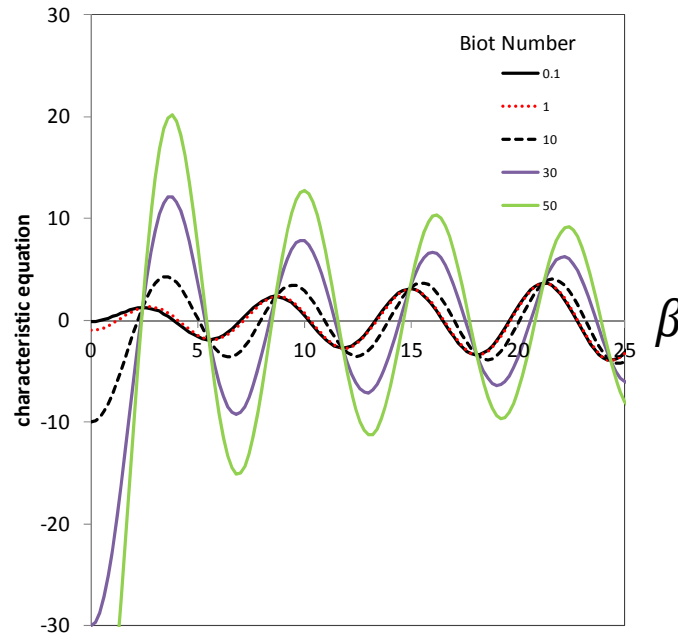
Figure 3: Equation 3 plotted for various Biot numbers.
The x-axis crossings vary with Biot number.

$x + x_{step}$, the function evaluates to and $y|_x$ and $y|_{x+x_{step}}$, and the slope and intercept of this secant line are

$$slope = \left(\frac{y|_{x+x_{step}} - y|_x}{x_{step}} \right)$$
$$intercept = y|_x - (x)(slope)$$

The new value of x_2 in our algorithm is thus

$$x_{new} = x_2 = -\left(\frac{intercept}{slope} \right)$$

We carried out this algorithm for the first root of the characteristic equation for $Bi = 1.0$ (Figure 5). To start the iteration, we create a circular reference between the cell labeled x_guess and that labeled x_new. We must first have turned on recursive calculations as explained in the first example. The root found is $x_{final} = 1.255784 = \beta_1$. Note that if our first guess is not close enough to the root, we may find instead the second or even higher roots (try an initial guess of 0.2). This difficulty can be addressed with more sophisticated root-seeking algorithms.
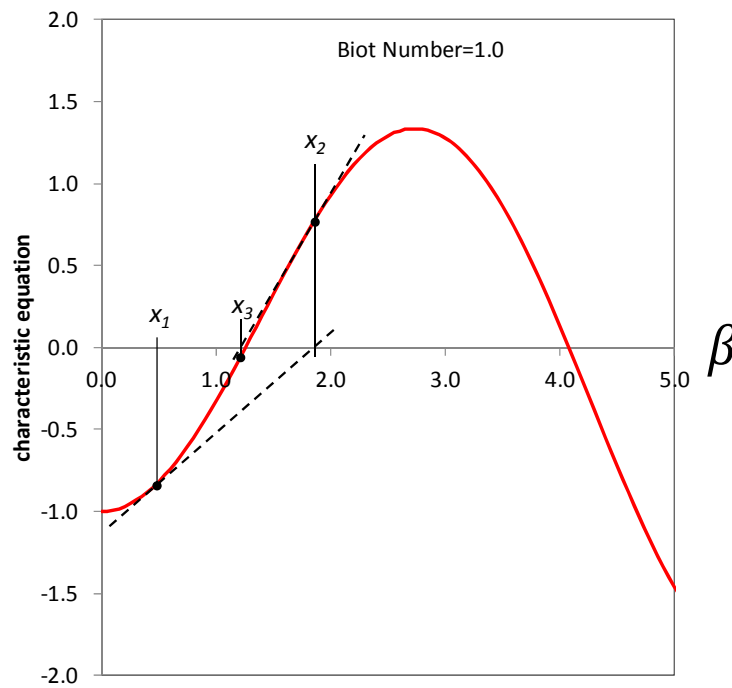
Figure 4: The root seeking secant algorithm described in the text. The first guess is x_1; the slope of the function at x_1 is estimated by a secant line at that point and the next guess for x is where the secant line intersects the x-axis. The algorithm then repeats.

	A	B	C
1	Biot Number=	1	
2	x_guess=	0.4	
3	x_step=	0.01	
4	f(x)=	-0.88199	
5	f(x+x_step)=	-0.87612	
6	slope=	0.586935	
7	intercept=	-1.11676	
8	x_new=	1.9027	
9			

	A	B	C
1	Biot Number=	1	
2	x_guess=	=x_new	
3	x_step=	0.01	
4	f(x)=	-0.88199	
5	f(x+x_step)=	-0.87612	
6	slope=	0.586935	
7	intercept=	-1.11676	
8	x_new=	1.9027	
9			

	A	B	C
1	Biot Number=	1	
2	x_guess=	1.255784	
3	x_step=	0.01	
4	f(x)=	0	
5	f(x+x_step)=	0.013205	
6	slope=	1.320543	
7	intercept=	-1.65832	
8	x_new=	1.255784	
9			

Figure 5: The secant algorithm is implemented in Excel, taking advantage of the recursive calculation option. At left, we enter the formulas described in the text. At the middle, we create a circular reference by making the contents of cell B2 equal to "=B8" or "=x_new" (the cell B8 is labeled x_new). At the right, we see the result of the iterative calculation: f(x)=0 and x_guess=x_new=1.255784.

Reference

H.S. Carslaw and J.C. Jaeger, *Conduction of Heat in Solids* (Oxford: Oxford University Press, 1946).