Final Exam
CM 4650 Polymer Rheology
27 April 2011

Please be neat.
Please write on only one side of each piece of paper in your solution.
This exam is closed book, closed notes.
Advice: Before handing in your exam, re-read the question and make sure you have followed all the directions.

1. (20 points) What is the 31-component of \( (\nabla \mathbf{v}) \cdot \mathbf{A} \)? Note that \( \mathbf{v} \) is a vector and \( \mathbf{A} \) is a tensor. Use Einstein notation and write out all summations in your final answer.

2. (20 points) Please answer all four parts:
   a. \( G'(\omega) \) is shown below for two different polymers. Which polymer (A or B) is entangled?
b. True or False: temperature has a strong effect on rheological properties?

c. \( \eta(\dot{\gamma}) \) is shown below for two different polymers. Which polymer (C or D) has higher zero shear viscosity?

d. For the polymer data shown in question c), which polymer (C or D) has higher molecular weight? Both are chemically the same type of polymer.

3. (20 points) What is the steady shear viscosity predicted by the model shown below? Show how you arrive at your final answer.

\[
\tau(t) = -\int_{-\infty}^{t} \frac{\eta_0}{\lambda^2} e^{-\frac{(t-t')}{\lambda}} \mathcal{C}^{-1}(t',t) dt'
\]
4. (30 points) Calculate the velocity profile for steady pressure-driven flow of a power-law, generalized Newtonian fluid flowing in the narrow gap between parallel plates that are long and wide (see figure). The top plate moves at velocity \( V \). You may neglect gravity, and the fluid is incompressible. Use the coordinate system given. Indicate your assumptions and show your work. Please be neat. The pressure gradient is low, and thus the fluid velocity increases from the bottom plate to the top plate (there is no velocity maximum within the gap). You may give your answer in terms of integration constants and boundary conditions (you do not need to do the final algebra).

![Diagram of steady pressure-driven flow](image)

5. (10 points) I propose a new material function for shear flow based on a time-dependent flow that experiences a small amount of shear just before time=0. The kinematics of my new shear material function are given below; \( t_0 \) is a positive, constant time parameter with units of time and \( \dot{\gamma}_0 \) is a positive, constant shear-rate parameter with units of 1/time.

\[
\dot{\gamma}(t^*) = \begin{cases} 
0 & t^* < -t_0 \\
\dot{\gamma}_0 & -t_0 \leq t^* \leq 0 \\
0 & t^* \geq 0
\end{cases}
\]

What is the shear strain function \( \gamma_{21}(t',t) = \int_{t'}^{t} \dot{\gamma}(t^*) dt^* \) in this experiment? As always, the time \( t \) varies and is always greater than zero; the time \( t' \) varies and may be less than or greater than zero and ranges from \(-\infty\) up to \( t \). Show explicitly how you obtain your answer. Please box your final answer.


Final Exam Formulas

Polymer Rheology
Prof. Faith Morrison

Rate of deformation: \( \dot{\gamma} = \left| \dot{\gamma} \right| \)

Tensor magnitude: \( A = |A| = +\sqrt{\frac{A \cdot A}{2}} \)

Navier-Stokes Equation: \( \rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g} \)

Momentum Equation: \( \rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p - \nabla \cdot \mathbf{t} + \rho \mathbf{g} \)

Continuity Equation: \( \frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}) \)

Newtonian, incompressible fluid: \( \mathbf{t} = -\mu \left( \nabla \mathbf{v} + (\nabla \mathbf{v})^T \right) \)

Generalized Newtonian fluid (GNF): \( \mathbf{t} = -\eta(\dot{\gamma}) \dot{\gamma} \)

Power-law GNF model: \( \eta(\dot{\gamma}) = m \dot{\gamma}^{n-1} \)

(Note that \( m \) and \( n \) are parameters of the model and are constants)

Carreau-Yasuda GNF model: \( \eta(\dot{\gamma}) = \eta_\infty + (\eta_0 - \eta_\infty) \left[ 1 + (\dot{\gamma} \lambda)^a \right] \frac{n-1}{a} \)

(Note that \( a, \lambda \) and \( n, \eta_0, \) and \( \eta_\infty \) are parameters of the model and are constants)

Generalized Linear Viscoelastic Fluid model (GLVE): \( \mathbf{t} = -\int_{-\infty}^{t} G(t-t') \mathbf{\gamma}(t')dt' \)

Maxwell model (differential version): \( \mathbf{t} + \lambda \frac{\partial \mathbf{t}}{\partial t} = -\eta_0 \dot{\gamma} \)

Maxwell model (integral version): \( \mathbf{t} = -\int_{-\infty}^{t} \frac{\eta_0}{\lambda} e^{\frac{(t-t')}{\lambda}} \mathbf{\gamma}(t')dt' \)

Generalized Maxwell model (GMM): \( \mathbf{t} = -\int_{-\infty}^{t} \sum_{k=1}^{N} \frac{\eta_k}{\lambda_k} e^{\frac{(t-t')}{\lambda_k}} \mathbf{\gamma}(t')dt' \)

Lodge model: \( \mathbf{t} = -\int_{-\infty}^{t} \frac{\eta_0}{\lambda} e^{\frac{(t-t')}{\lambda}} C^{-1}(t', t)dt' \)
Elongational flow (uniaxial, biaxial):

\[ \mathbf{\nu} = \begin{pmatrix} -\frac{\varepsilon(t)}{2} x_1 \\ \varepsilon(t) \frac{x_2}{2} \\ \varepsilon(t) x_3 \end{pmatrix}_{123} \]

Shear flow:

\[ \mathbf{\nu} = \begin{pmatrix} \dot{\gamma}(t) x_2 \\ 0 \\ 0 \end{pmatrix}_{123} \]

Steady shearing kinematics:
\[ \dot{\gamma}(t) = \dot{\gamma}_0 \]

Start-up of steady shearing kinematics:
\[ \dot{\gamma}(t) = \begin{cases} 0 & t < 0 \\ \gamma'_0 & t \geq 0 \end{cases} \]

Cessation of steady shearing kinematics:
\[ \dot{\gamma}(t) = \begin{cases} \gamma'_0 & t < 0 \\ 0 & t \geq 0 \end{cases} \]

Steady elongational kinematics:
\[ \dot{\varepsilon}(t) = \dot{\varepsilon}_0 \]

Start-up of steady elongation kinematics:
\[ \dot{\varepsilon}(t) = \begin{cases} 0 & t < 0 \\ \dot{\varepsilon}_0 & t \geq 0 \end{cases} \]

Cessation of steady elongation kinematics:
\[ \dot{\varepsilon}(t) = \begin{cases} \dot{\varepsilon}_0 & t < 0 \\ 0 & t \geq 0 \end{cases} \]

Shear viscosity:
\[ \eta = -\frac{(\tau_{21})}{\dot{\gamma}_0} \]

Shear normal stress coefficients:
\[ \Psi_1 = -\frac{(\tau_{11} - \tau_{22})}{\dot{\gamma}_0^2}, \Psi_2 = -\frac{(\tau_{22} - \tau_{33})}{\dot{\gamma}_0^2} \]

Elongational viscosity:
\[ \eta = -\frac{(\tau_{33} - \tau_{11})}{\dot{\varepsilon}_0} \]

Also: Table 9.3 (p 329)
<table>
<thead>
<tr>
<th>Tensor</th>
<th>Shear in 1-direction with gradient in 2-direction</th>
<th>Uniaxial elongation in 3-direction</th>
<th>CCW rotation around $\hat{e}_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbf{F}(t, t')$</td>
<td>$\begin{pmatrix} 1 &amp; 0 &amp; 0 \ -\gamma &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 1 \end{pmatrix}_{123}$</td>
<td>$\begin{pmatrix} e^{\gamma} &amp; 0 &amp; 0 \ 0 &amp; e^{\gamma} &amp; 0 \ 0 &amp; 0 &amp; e^{-\gamma} \end{pmatrix}_{123}$</td>
<td>$\begin{pmatrix} \cos \psi &amp; -\sin \psi &amp; 0 \ \sin \psi &amp; \cos \psi &amp; 0 \ 0 &amp; 0 &amp; 1 \end{pmatrix}_{123}$</td>
</tr>
<tr>
<td>$\mathbf{F}^{-1}(t, t')$</td>
<td>$\begin{pmatrix} 1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 1 \end{pmatrix}_{123}$</td>
<td>$\begin{pmatrix} e^{-\frac{\gamma}{2}} &amp; 0 &amp; 0 \ 0 &amp; e^{-\frac{\gamma}{2}} &amp; 0 \ 0 &amp; 0 &amp; e^{\frac{\gamma}{2}} \end{pmatrix}_{123}$</td>
<td>$\begin{pmatrix} \cos \psi &amp; \sin \psi &amp; 0 \ -\sin \psi &amp; \cos \psi &amp; 0 \ 0 &amp; 0 &amp; 1 \end{pmatrix}_{123}$</td>
</tr>
<tr>
<td>$\mathbf{C}(t, t')$</td>
<td>$\begin{pmatrix} 1 &amp; -\gamma &amp; 0 \ -\gamma &amp; 1 + \gamma^2 &amp; 0 \ 0 &amp; 0 &amp; 1 \end{pmatrix}_{123}$</td>
<td>$\begin{pmatrix} e^{\gamma} &amp; 0 &amp; 0 \ 0 &amp; e^{\gamma} &amp; 0 \ 0 &amp; 0 &amp; e^{-2\gamma} \end{pmatrix}_{123}$</td>
<td>$\mathbb{I}$</td>
</tr>
<tr>
<td>$\mathbf{C}^{-1}(t, t')$</td>
<td>$\begin{pmatrix} 1 + \gamma^2 &amp; 0 \ \gamma &amp; 1 \ 0 &amp; 0 &amp; 1 \end{pmatrix}_{123}$</td>
<td>$\begin{pmatrix} e^{-\gamma} &amp; 0 &amp; 0 \ 0 &amp; e^{-\gamma} &amp; 0 \ 0 &amp; 0 &amp; e^{2\gamma} \end{pmatrix}_{123}$</td>
<td>$\mathbb{I}$</td>
</tr>
<tr>
<td>$\mathbf{\gamma}_{[a]}(t, t')$</td>
<td>$\begin{pmatrix} 0 &amp; -\gamma &amp; 0 \ -\gamma &amp; 1 + \gamma^2 &amp; 0 \ 0 &amp; 0 &amp; 1 \end{pmatrix}_{123}$</td>
<td>$\begin{pmatrix} e^{\gamma} &amp; -1 &amp; 0 \ 0 &amp; e^{\gamma} &amp; -1 \ 0 &amp; 0 &amp; e^{-2\gamma} - 1 \end{pmatrix}_{123}$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\mathbf{\gamma}_{[\sigma]}(t, t')$</td>
<td>$\begin{pmatrix} -\gamma^2 &amp; 0 \ \gamma &amp; 0 \ 0 &amp; 0 \end{pmatrix}_{123}$</td>
<td>$\begin{pmatrix} e^{-\gamma} &amp; -1 &amp; 0 \ 0 &amp; e^{-\gamma} &amp; -1 \ 0 &amp; 0 &amp; e^{2\gamma} - 1 \end{pmatrix}_{123}$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

Table 9.3: Strain tensors for shear and extension in Cartesian coordinates.

For shear flows $\gamma = \gamma(t', t) = \int_{t'}^{t} \dot{\gamma}(t'') \, dt'' = \int_{t'}^{t} \dot{\gamma}_{[a]}(t'') \, dt''$ and for elongational flows $\varepsilon = \varepsilon(t', t) = \int_{t'}^{t} \dot{\varepsilon}(t'') \, dt''$. The angle $\psi$ is the angle from $x(t) = x$ to $x(t') = x'$ in counterclockwise (ccw) rotation around the $\hat{e}_3$-axis.

Errata: [www.chem.mtu.edu/~fmorriso/UKerrate.html](http://www.chem.mtu.edu/~fmorriso/UKerrate.html)