Midterm Exam I
CM 4650 Polymer Rheology
2 February 2012

Please be neat.
Please write on only one side of each piece of paper in your solution.
This exam is closed book, closed notes.

Navier-Stokes Equation (Gibbs notation): \[ \rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g} \]

Continuity Equation (Gibbs notation): \[ \frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}) \]

Newtonian Incompressible Constitutive Equation (Gibbs notation): \[ \mathbf{\tau} = -\mu [\nabla \mathbf{v} + (\nabla \mathbf{v})^T] \]

Fluid force on a surface \( S \): \[ \mathbf{f} = \iint_S \left[ \hat{n} \cdot \Pi \right]_{\text{surface}} dS \]

Flow rate through surface \( S \): \[ Q = \iint_S \left[ \hat{n} \cdot \mathbf{v} \right]_{\text{surface}} dS \]

Fluid torque on a surface \( S \): \[ \mathbf{T} = \iint_S (R \times \left[ \hat{n} \cdot \Pi \right]_{\text{surface}}) dS \]

<table>
<thead>
<tr>
<th>Coordinate system</th>
<th>surface differential ( dS )</th>
<th>coordinate system</th>
<th>volume differential ( dV )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cartesian (top, ( \hat{n} = \hat{e}_z ))</td>
<td>( dS = dx dy )</td>
<td>Cartesian</td>
<td>( dV = dx dy dz )</td>
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<tr>
<td>Cartesian (side a, ( \hat{n} = \hat{e}_y ))</td>
<td>( dS = dx dz )</td>
<td>cylindrical</td>
<td>( dV = r dr d\theta dz )</td>
</tr>
<tr>
<td>Cartesian (side b, ( \hat{n} = \hat{e}_x ))</td>
<td>( dS = dy dz )</td>
<td>spherical</td>
<td>( dV = r^2 \sin \theta dr d\theta d\phi )</td>
</tr>
</tbody>
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<tr>
<th>Coordinate system</th>
<th>coordinates</th>
<th>basis vectors</th>
</tr>
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<tr>
<td>spherical</td>
<td>( x = r \sin \theta \cos \phi ) ( y = r \sin \theta \sin \phi ) ( z = r \cos \theta )</td>
<td>( \hat{e}_r = (\sin \theta \cos \phi) \hat{e}_x + \sin \theta \sin \phi \hat{e}_y + \cos \theta \hat{e}<em>z ) ( \hat{e}</em>\phi = (\cos \theta \cos \phi) \hat{e}_x + (\cos \theta \sin \phi) \hat{e}_y + (-\sin \theta) \hat{e}<em>z ) ( \hat{e}</em>\theta = (-\sin \phi) \hat{e}_x + \cos \phi \hat{e}_y )</td>
</tr>
<tr>
<td>cylindrical</td>
<td>( x = r \cos \theta ) ( y = r \sin \theta ) ( z = z )</td>
<td>( \hat{e}_r = \cos \theta \hat{e}_x + \sin \theta \hat{e}<em>y ) ( \hat{e}</em>\theta = (-\sin \theta) \hat{e}_x + \cos \theta \hat{e}_y ) ( \hat{e}_z = \hat{e}_z )</td>
</tr>
</tbody>
</table>
1. (20 points) What is $\mathbf{A} \cdot \mathbf{m}$ in Einstein notation?

2. (20 points) How do we write the following Cartesian expression in Gibbs notation (Gibbs notation is the vector-tensor notation like $\mathbf{v}, \nabla \mathbf{p}$, etc.):

$$\left| \begin{array}{c}
\frac{\partial w_1}{\partial x_1} v_1 + \frac{\partial w_1}{\partial x_2} v_2 + \frac{\partial w_1}{\partial x_3} v_3 \\
\frac{\partial w_2}{\partial x_1} v_1 + \frac{\partial w_2}{\partial x_2} v_2 + \frac{\partial w_2}{\partial x_3} v_3 \\
\frac{\partial w_3}{\partial x_1} v_1 + \frac{\partial w_3}{\partial x_2} v_2 + \frac{\partial w_3}{\partial x_3} v_3 \\
\end{array} \right|_{123}$$

3. (20 points) What is $\nabla (\alpha \beta)$ in Einstein notation? What is it in matrix notation? $\alpha$, and $\beta$ are variables (not constants). For both parts, be sure to carry out the product rule of differentiation and carry out any summations.

4. (20 points) In rheology as in fluid mechanics, we are often interested in quantities like the flow rate, the force on a wall, or the torque to move a part. For steady, pressure-driven flow in a circular tube (total stress tensor shown below), calculate the $z$-component of the fluid force on the wall from the given stress tensor. Note that the stress tensor is given in cylindrical coordinates $(r, \theta, z)$, $R$ is the radius of the tube, and $L, p_0$, and $p_L$ are constants

$$\Pi = \left( \begin{array}{ccc}
\frac{(p_L - p_0)}{L} z + p_0 & 0 & \frac{(p_L - p_0)r}{2L} \\
0 & \frac{(p_L - p_0)}{L} z + p_0 & 0 \\
\frac{(p_L - p_0)r}{2L} & 0 & \frac{(p_L - p_0)}{L} z + p_0 \end{array} \right)_{r\theta z}$$

(problem 5 is on the next page)
5. (20 points) A steady flow of an incompressible, Newtonian fluid is created between two very wide, parallel plates as shown in the figure. The bottom plate is stationary and the top plate is pulled to the right at a speed \( V \). There is an opposing pressure gradient applied to the slit such that at the left edge the pressure is low (\( P_0 \)) and downstream a distance \( L \) the pressure is high (\( P_L \)). Answer the questions below. Please show your work and indicate your assumptions.

a. What is the differential equation for the velocity?

b. What are the boundary conditions on velocity? (express the boundary conditions mathematically)

c. What is the steady state velocity profile? Express your final answer only in terms of quantities given in the problem or figure.