Intro to strain (continued)

For steady shear, with $t_{ref}=0$:

$$
\gamma_{21}(0, t) \equiv \frac{\partial u_1}{\partial x_2} = t \dot{\gamma}_0
$$

*(short time interval)*

For a long time interval, we add up the strains over short time intervals.

$$
\gamma_{21}(t, t_{p+1}) = \dot{\gamma}_0 \Delta t
$$

$$
\gamma_{21}(0, t) = \sum_{p=0}^{N-1} \gamma_{21}(t, t_{p+1}) = (N \Delta t) \dot{\gamma}_0 = t \dot{\gamma}_0
$$

Same, because flow is steady.

---

For unsteady shear:

$$
\gamma_{21}(t, t_{p+1}) = \frac{\partial u_1}{\partial x_2} = \dot{\gamma}_{21}(t_{p+1}) \Delta t
$$

*(short time interval)*

For a long time interval, we add up the strains over short time intervals.

$$
\gamma_{21}(t, t_{p+1}) = \dot{\gamma}_{21}(t_{p+1}) \Delta t
$$

$$
\gamma_{21}(t_1, t_2) = \sum_{p=0}^{N-1} \gamma_{21}(t, t_{p+1}) = \sum_{p=0}^{N-1} \Delta t \dot{\gamma}_{21}(t_{p+1})
$$

Taking the limit as $\Delta t$ goes to zero,

$$
\gamma_{21}(t_1, t_2) = \lim_{\Delta t \to 0} \left[ \sum_{p=0}^{N-1} \Delta t \dot{\gamma}_{21}(t_{p+1}) \right] = \int_{t_1}^{t_2} \dot{\gamma}_{21}(t) dt
$$

Strain at $t_2$ with respect to fluid configuration at $t_1$ in unsteady shear flow.

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Creep Shear Flow Material Functions

**Kinematics:**

\[
\mathbf{v} = \begin{pmatrix} \dot{\gamma}_{21}(t) \chi_2 \\ 0 \\ 0 \end{pmatrix}, \quad \tau_{21}(t) = \begin{cases} 0 & t < 0 \\ \tau_0 & t \geq 0 \end{cases}
\]

**Material Functions:**

\[
J(t, \tau_0) = \frac{\gamma_{21}(0,t)}{-\tau_0}, \quad J_r(t', \tau_0) = \frac{\gamma_r(t')}{-\tau_0}
\]

Shear creep compliance
Recoverable creep compliance

Material functions predicted for creep of a Newtonian fluid

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Shear creep material functions

Steady-state compliance

\[ J_s(t, \tau_0) \]

constant slope = \( \frac{1}{\dot{\gamma}(\tau_0)} \)

Ultimate recoil function

\[ R(t', \tau_0), R(\tau_0) \]

0 \quad \quad t_2

\[ t, \text{ creep} \]

\[ -t_2 \quad 0 \quad t', \text{ recovery} \]

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Shear Creep

\[ \dot{\gamma}_{21}(t)x_2 \]

\[ \gamma_2(t) = \begin{cases} 0 & t < 0 \\ \tau_0 & t \geq 0 \end{cases} \]

\[ J(t, \tau_0) \equiv \frac{-\tau_2(0, t)}{-\tau_0} \]

Data have been corrected for vertical shift.

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At long times the creep compliance $J(t, \tau_0)$ becomes a straight line.

$$\frac{dJ}{dt}_{\text{steady state}} = \frac{d\gamma}{dt} \left( \frac{1}{-\tau_0} \right)$$

$$= \frac{\dot{\gamma}}{-\tau_0} = \frac{1}{\eta(\dot{\gamma}_\infty)}$$

the slope at steady state is the inverse of the steady viscosity

$$\frac{dJ}{dt}_{\text{steady state}} = \frac{1}{\eta(\dot{\gamma}_\infty)} \Rightarrow J(t)_{\text{steady state}} = \frac{1}{\eta(\dot{\gamma}_\infty)} t + C$$

Steady-state compliance $J_s(\tau_0)$
**Creep Recovery** - after creep, stop pulling forward and allow the flow to reverse

\[ \gamma_r(t) = \gamma_{21}(0,t_2) - \gamma_{21}(0,t) \]

- Recoverable strain
- Strain at the end of the forward motion
- Strain at the end of the recovery

\[ J_r(t', \tau_0) = \frac{\gamma_r(t')}{\tau_0} \]

- Recoverable creep compliance

---

**Linear Viscoelastic Creep**

\[ \gamma(t) = \gamma_r(t) + t\dot{\gamma}_\infty \]

- recoverable strain
- total strain
- non-recoverable strain

\[ J(t) = R(t) + \frac{t}{\eta_0} \]

*This is a way to get R(t) without measuring it*
Shear creep material functions

Linear-viscoelastic limit

Steady-state compliance

constant slope = \frac{1}{\eta_0}

Shear creep material functions

\begin{align*}
J(t) & = 0 \\
J_s & = \text{constant} \\
R(t') & = R_\infty = J_s^0 \\
\epsilon & = \text{constant} = \gamma_0
\end{align*}

Step Shear Strain Material Functions

Kinematics:

\[ v = \begin{pmatrix}
\dot{\gamma}(t) \gamma_2 \\
0 \\
0 \\
0
\end{pmatrix}_{123} \]

Material Functions:

\[ G(t, \gamma_0) = \frac{-\tau_{21}(t, \gamma_0)}{\gamma_0} \]

First normal-stress relaxation modulus

Relaxation modulus

\[ G_{Q_1} = \frac{-\tau_{11} - \tau_{22}}{\gamma_0^2} \]

Second normal-stress relaxation modulus

\[ G_{Q_2} = \frac{-\tau_{22} - \tau_{33}}{\gamma_0^2} \]
What is the strain in this flow?

\[ \gamma_{21}(-\infty, t) = \int_{-\infty}^{t} \gamma_{21}(t') dt' \]

\[ = \int_{-\infty}^{t} \lim_{\varepsilon \to 0} \begin{cases} 0 & t' < 0 \\ \frac{\gamma_0}{\varepsilon} & 0 \leq t' < \varepsilon \\ 0 & t \geq \varepsilon \end{cases} dt' \]

\[ = \lim_{\varepsilon \to 0} \int_{0}^{\varepsilon} \frac{\gamma_0}{\varepsilon} dt' \]

\[ = \gamma_0 \quad \text{The strain imposed is a constant} \]

Step shear strain - strain dependence

\( \text{Figure 6.57, p. 212} \)

Einaga et al.; PS soln

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Linear viscoelastic limit

\[
\lim_{\gamma_0 \to 0} G(t, \gamma_0) = G(t)
\]

At small strains the relaxation modulus is independent of strain.

Damping function, \(h\)

\[
h(\gamma_0) \equiv \frac{G(t, \gamma_0)}{G(t)}
\]

The damping function summarizes the non-linear effects as a function of strain amplitude.