Summary: Generalized Newtonian Fluid Constitutive Equations

**PRO:**
- A first constitutive equation
- Can match steady shearing data very well
- Simple to calculate with
- Found to predict pressure-drop/flow rate relationships well

**CON:**
- Fails to predict shear normal stresses
- Fails to predict start-up or cessation effects (time-dependence, memory) – only a function of instantaneous velocity gradient
- Derived ad hoc from shear observations; unclear of validity in non-shear flows

We now look to address this failing of GNF models by seeking to incorporate memory.

Rules for Constitutive Equations

\[ \tau(t) = f(\mathcal{V}, I_\mathcal{V}, II_\mathcal{V}, III_\mathcal{V}, \text{material info}) \]

The stress expression:

- **Must be of tensor order**
- **Must be a tensor (independent of coordinate system)**
- **Must be a symmetric tensor**
- **Must make predictions that are independent of the observer**
- **Should correctly predict observed flow/deformation behavior**
Rules for Constitutive Equations

\[ \tau(t) = f(\gamma, I, \Pi, III, \text{invariants}) \]

The stress expression:

- Must be of tensor order
- Must be a tensor (independent of coordinate system)
- Must be a symmetric tensor
- Must make predictions that are independent of the observer
- Should correctly predict observed flow/deformation behavior

Tensor invariants – scalars associated with a tensor that do not depend on coordinate system

\[ I_4 = \text{trace} A = \text{tr } A \]

For the tensor written in Cartesian coordinates:

\[ \text{trace } A = \sum_{p=1}^{3} A_{pp} = A_{11} + A_{22} + A_{33} \]

\[ II_4 = \text{trace}(A \cdot A) = A: A = \sum_{p=1}^{3} \sum_{k=1}^{3} A_{pk} A_{kp} \]

\[ III_4 = \text{trace}(A \cdot A \cdot A) = \sum_{p=1}^{3} \sum_{j=1}^{3} \sum_{h=1}^{3} A_{pj} A_{jh} A_{hp} \]

Note: the definitions of invariants written in terms of coefficients are only valid when the tensor is written in Cartesian coordinates.
We seek a constitutive equation that includes memory effects.

\[ \tau(t) = f(\dot{\gamma}, I_1, II, III, \text{material information}) \]

calculates the stress at a particular time, \( t \).

2 equations so far:

\[
\begin{align*}
\tau(t) &= -\mu \dot{\gamma}(t) \\
\tau(t) &= -\eta(\dot{\gamma}) \dot{\gamma}(t) \quad \dot{\gamma} = |\dot{\gamma}|
\end{align*}
\]

So far, stress at \( t \) depends on rate-of-deformation \textit{at} \( t \) only.

Current Constitutive Equations

Newtonian \[ \tau(t) = -\mu \dot{\gamma}(t) \]

Generalized Newtonian \[ \tau(t) = -\eta(\dot{\gamma}) \dot{\gamma}(t) \quad \dot{\gamma} = |\dot{\gamma}| \]

Neither can predict:

- Shear normal stresses - \textit{this will be wrong so long as we use constitutive equations proportional to} \( \dot{\gamma} \)
- Stress transients in shear (startup, cessation) - \textit{this flaw seems to be related to omitting fluid memory}

We will try to fix this now; we will address the first point when we discuss advanced constitutive equations.
Startup of Steady Shearing

\[ Y = \begin{cases} \ddot{\gamma} = (0)_{x_2} \\ 0 \\ 0 \\ \end{cases} \quad \ddot{\gamma}(t) = \begin{cases} 0 & t < 0 \\ \ddot{\gamma}_0 & t \geq 0 \end{cases} \]

\[ \eta^{+} = \frac{-\tau_{21}(t)}{\ddot{\gamma}_0} \]

Cessation of Steady Shearing

\[ Y = \begin{cases} \ddot{\gamma} = (0)_{x_2} \\ 0 \\ 0 \\ \end{cases} \quad \ddot{\gamma}(t) = \begin{cases} 0 & t < 0 \\ \ddot{\gamma}_0 & t \geq 0 \end{cases} \]

\[ \eta^{-} = \frac{-\tau_{21}(t)}{\ddot{\gamma}_0} \]

Figures 6.49, 6.50, p. 208 Menezes and Graessley, PB soln

Figures 6.51, 6.52, p. 209 Menezes and Graessley, PB soln
How can we incorporate time-dependent effects?

First we explore a simple memory fluid.

Let’s construct a new constitutive equation that remembers the stress at a time $t_0$ seconds ago

$$\tau(t) = -\tilde{\eta} \dot{\gamma}(t) - (0.8\tilde{\eta}) \ddot{\gamma}(t - t_0)$$

This is the rate-of-deformation tensor $t_0$ seconds before time $t$.

$\tilde{\eta}$ is a constant parameter of the model.

What does this model predict?

**Steady shear**

$\eta = ?$

$\Psi_1 = ?$

$\Psi_2 = ?$

**Shear start-up**

$\eta^+(t) = ?$

$\Psi_1^+(t) = ?$

$\Psi_2^+(t) = ?$
**Steady Shear Flow Material Functions**

**Kinematics:**

\[
\mathbf{v} = \begin{pmatrix}
\dot{\gamma}(t)x_2 \\
0 \\
0
\end{pmatrix}_{123}
\]

\[\dot{\gamma}(t) = \dot{\gamma}_0 = \text{constant}\]

**Material Functions:**

\[\eta \equiv -\frac{\tau_{21}}{\dot{\gamma}_0}\]

- Viscosity
- First normal-stress coefficient

\[\Psi_1 \equiv -\frac{(\tau_{11} - \tau_{22})}{\dot{\gamma}_0^2}\]

- Second normal-stress coefficient

\[\Psi_2 \equiv -\frac{(\tau_{22} - \tau_{33})}{\dot{\gamma}_0^2}\]


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**Start-up of Steady Shear Flow Material Functions**

**Kinematics:**

\[
\mathbf{v} = \begin{pmatrix}
\dot{\gamma}(t)x_2 \\
0 \\
0
\end{pmatrix}_{123}
\]

\[\dot{\gamma}(t) = \begin{cases} 
0 & t < 0 \\
\dot{\gamma}_0 & t \geq 0
\end{cases}\]

**Material Functions:**

\[\eta^+ \equiv -\frac{\tau_{21}(t)}{\dot{\gamma}_0}\]

- Shear stress growth function

\[\Psi_1^+ \equiv -\frac{(\tau_{11} - \tau_{22})}{\dot{\gamma}_0^2}\]

- First normal-stress growth function

\[\Psi_2^+ \equiv -\frac{(\tau_{22} - \tau_{33})}{\dot{\gamma}_0^2}\]

- Second normal-stress growth function

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**Cessation of Steady Shear Flow Material Functions**

**Kinematics:**
\[ \mathbf{v} = \begin{pmatrix} \dot{\gamma}(t)x_2 \\ 0 \\ 0 \end{pmatrix}_{123}, \quad \dot{\gamma}(t) = \begin{cases} \dot{\gamma}_0 & t < 0 \\ 0 & t \geq 0 \end{cases} \]

**Material Functions:**
- First normal-stress decay function: \( \Psi_1^- = -\left(\frac{\tau_{11} - \tau_{22}}{\dot{\gamma}_0^2}\right) \)
- Second normal-stress decay function: \( \Psi_2^- = -\left(\frac{\tau_{22} - \tau_{33}}{\dot{\gamma}_0^2}\right) \)

\[ \eta^- \equiv -\frac{\tau_{21}(t)}{\dot{\gamma}_0} \]

\[ \eta^{+}(t) = \begin{cases} 0 & t < 0 \\ \eta & 0 \leq t \leq t_0 \\ 1.8\eta & t \geq t_0 \end{cases} \]

\[ \Psi_1^{+}(t) = \Psi_2^{+}(t) = 0 \]

**Predictions of the simple memory fluid**
\[ \tau(t) = -\tilde{\eta} \dot{\gamma}(t) - (0.8\tilde{\eta})\dot{\gamma}(t - t_0) \]

- **Steady shear**
  - \( \eta = 1.8\tilde{\eta} \)
  - \( \Psi_1 = \Psi_2 = 0 \)
  - The steady viscosity reflects contributions from what is currently happening and contributions from what happened \( t_0 \) seconds ago.

- **Shear start-up**
  - \( \eta^{+}(t) = \begin{cases} 0 & t < 0 \\ \tilde{\eta} & 0 \leq t \leq t_0 \\ 1.8\tilde{\eta} & t \geq t_0 \end{cases} \)
  - \( \Psi_1^{+}(t) = \Psi_2^{+}(t) = 0 \)

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Predictions of the simple memory fluid

Shear start-up

\[ \eta^*(t) = \begin{cases} 
0 & t < 0 \\
\tilde{\eta} & 0 \leq t \leq t_0 \\
1.8\tilde{\eta} & t \geq t_0 
\end{cases} \]

\[ \Psi_1^*(t) = \Psi_2^*(t) = 0 \]

Figures 6.49, 6.50, p. 208 Menezes and Graessley, PB soln

What the data show:

What the GNF models predict:

What the simple memory fluid model predict:
Predictions of the simple memory fluid

**Shear start-up**

What the data show:

What the GNF model predict:

Adding that contribution from the past introduces the observed “build-up” effect to the predicted start-up material functions.

We can make the stress rise smoother by adding more fading memory terms.

\[ \tau(t) = -\eta \gamma(t) - (0.8\eta)\gamma(t-t_0) - (0.6\eta)\gamma(t-2t_0) \]

The memory is fading

Newtonian contribution

contribution from \( t_0 \) seconds ago

contribution from \( 2t_0 \) seconds ago

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The fit can be made to be perfectly smooth by using a sum of exponentially decaying terms as the weighting functions.

\[
\tau(t) = -\tilde{\eta} \left[ \dot{\gamma}(t) + (0.37) \dot{\gamma}(t-t_0) + (0.14) \dot{\gamma}(t-2t_0) + (0.05) \dot{\gamma}(t-3t_0) + \ldots \right] 
\]

\[
= -\tilde{\eta} \sum_{p=0}^{\infty} e^{-p} \dot{\gamma}(t - pt_0) 
\]

This sum can be approximated by an integral.

\[
\tau(t) = -\tilde{\eta} \sum_{p=0}^{\infty} e^{-p} \gamma(t - pt_0) 
\]

\[
= -\tilde{\eta} \sum_{p=0}^{\infty} e^{-(t-t'_p)/t_0} \gamma(t'_p) = -\tilde{\eta} \frac{\text{area}}{(-t_0)} 
\]

\[
\gamma(t'_p) = t - pt_0 
\]

\[
e^{-(t-t'_p)/t_0} \gamma(t'_p) 
\]

\[
(-t_0) \frac{\gamma(t)}{2} + (-t_0)e^{-(t-t_0)/t_0} \frac{\gamma(t-t_0)}{2} + (-t_0)e^{-(t-2t_0)/t_0} \frac{\gamma(t-2t_0)}{2} + \ldots 
\]

\[
t' = t - pt_0 
\]
This sum can be approximated by this integral.

\[ T(t) = -\bar{n} \left( \frac{\text{area}}{-t_0} \right) = -\bar{n} \int_{t_0}^{\infty} e^{-(t-t'/t_0)} \gamma(t') \, dt' \]

\[ T(t) = -\bar{n} \int_{-\infty}^{t} e^{-(t-t'/t_0)} \gamma(t') \, dt' \]

(Note: this is an underestimate of our previous decaying function, but the choice of function is arbitrary)
Maxwell Model (integral version)

\[ \tau(t) = -\int_{-\infty}^{t} \left( \frac{\eta}{t_0} \right) e^{-(t-t')/\lambda} \gamma(t') \, dt' \]

Two parameters:
- Zero-shear viscosity \( \eta_0 \) – gives the value of the steady shear viscosity
- Relaxation time \( \lambda \) - quantifies how fast memory fades

\( \tau(t) = -\int_{-\infty}^{t} \left( \frac{\eta_0}{\lambda} \right) e^{-(t-t')/\lambda} \gamma(t') \, dt' \)