3. Tensor – (continued)

Definitions

Scalar product of two tensors

\[ A : M = A_{ip} \hat{e}_i \cdot \hat{e}_p \cdot M_{km} \hat{e}_k \hat{e}_m \]

- carry out the dot products indicated

\[ = A_{ip} M_{km} \hat{e}_i \cdot \hat{e}_k \cdot \hat{e}_m \]

- “p” becomes “k”

\[ = A_{iq} M_{kn} \delta_{pk} \delta_{im} \]

- “i” becomes “m”

\[ = A_{mk} M_{km} \]

But, what is a tensor really?

A tensor is a handy representation of a Linear Vector Function

scalar function: \[ y = f(x) = x^2 + 2x + 3 \]

a mapping of values of \( x \) onto values of \( y \)

vector function: \[ w = f(v) \]

a mapping of vectors of \( v \) into vectors \( w \)

How do we express a vector function?
**What is a linear function?**

*Linear, in this usage, has a precise, mathematical definition.*

Linear functions (scalar and vector) have the following two properties:

\[
\begin{align*}
   f(\lambda x) &= \lambda f(x) \\
   f(x + w) &= f(x) + f(w)
\end{align*}
\]

It turns out . . . Multiplying vectors and tensors is a convenient way of representing the actions of a linear vector function (as we will now show).

---

**Tensors are Linear Vector Functions**

Let \( f(a) = b \) be a linear vector function.

We can write \( a \) in Cartesian coordinates.

\[
\begin{align*}
   a &= a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3 \\
   f(a) &= f(a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3) = b
\end{align*}
\]

Using the linear properties of \( f \), we can distribute the function action:

\[
\begin{align*}
   f(a) &= a_1 f(\hat{e}_1) + a_2 f(\hat{e}_2) + a_3 f(\hat{e}_3) = b
\end{align*}
\]

These results are just vectors, we will name them \( v \), \( w \), and \( m \).
Tensors are *Linear Vector Functions* (continued)

\[
f(a) = a_1 f(\hat{e}_1) + a_2 f(\hat{e}_2) + a_3 f(\hat{e}_3) = b
\]

\[
f(a) = a_1 \mathbf{v} + a_2 \mathbf{w} + a_3 \mathbf{m} = b
\]

Now we note that the coefficients \( a_i \) may be written as,

\[
a_1 = a \cdot \hat{e}_1, \quad a_2 = a \cdot \hat{e}_2, \quad a_3 = a \cdot \hat{e}_3
\]

Substituting,

\[
f(a) = a \cdot \hat{e}_1 \mathbf{v} + a \cdot \hat{e}_2 \mathbf{w} + a \cdot \hat{e}_3 \mathbf{m} = b
\]

The indeterminate vector product has appeared!

Using the distributive law, we can factor out the dot product with \( a \):

\[
f(a) = a \cdot (\hat{e}_1 \mathbf{v} + \hat{e}_2 \mathbf{w} + \hat{e}_3 \mathbf{m}) = b
\]

This is just a tensor (the sum of dyadic products of vectors)

\[
(\hat{e}_1 \mathbf{v} + \hat{e}_2 \mathbf{w} + \hat{e}_3 \mathbf{m}) \equiv \mathbf{M}
\]

\[
f(a) = a \cdot \mathbf{M} = b
\]

**CONCLUSION:** Tensor operations are convenient to use to express linear vector functions.

© Faith A. Morrison, Michigan Tech U.
3. Tensor – (continued)

More Definitions

Identity Tensor

\[ \mathbb{I} = \hat{e}_i \hat{e}_i = \hat{e}_1 \hat{e}_1 + \hat{e}_2 \hat{e}_2 + \hat{e}_3 \hat{e}_3 \]

\[ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ \mathbb{A} \cdot \mathbb{I} = A_{ip} \hat{e}_p \cdot \hat{e}_k \]

\[ = A_{ip} \hat{e}_p \delta_{pk} \hat{e}_k \]

\[ = A_{ik} \hat{e}_k \hat{e}_k \]

\[ = \mathbb{A} \]

Zero Tensor

\[ \mathbb{0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

Magnitude of a Tensor

\[ |\mathbb{A}| = \sqrt{\frac{\mathbb{A} : \mathbb{A}}{2}} \]

\[ \mathbb{A} : \mathbb{A} = A_{ip} \hat{e}_p \cdot \hat{e}_m A_{km} \hat{e}_k \hat{e}_m \]

\[ = A_{ip} A_{km} (\hat{e}_p \cdot \hat{e}_m) (\hat{e}_i \cdot \hat{e}_m) \]

\[ = A_{ik} A_{km} \]

products across the diagonal
Mathematics Review

3. Tensor – (continued) More Definitions

Tensor Transpose

\[ M^T = (M_{ik} \hat{e}_k)^T = M_{ik} \hat{e}_i \]

Exchange the coefficients across the diagonal

CAUTION:

\[ \left( \mathbf{A} \cdot \mathbf{C} \right)^T = \left( A_{ik} \hat{e}_k \cdot C_{pj} \hat{e}_j \right)^T = \left( A_{ik} C_{pj} \hat{e}_k \delta_{kp} \right)^T \]
\[ = \left( A_{ip} C_{pj} \hat{e}_i \hat{e}_j \right)^T = A_{ip} C_{pj} \hat{e}_i \hat{e}_j \]

It is not equal to:

\[ \left( \mathbf{A} \cdot \mathbf{C} \right)^T = \left( A_{ip} C_{pj} \hat{e}_i \hat{e}_j \right)^T \neq A_{ip} C_{pj} \hat{e}_i \hat{e}_j \]

I recommend you always interchange the indices on the basis vectors rather than on the coefficients.

Symmetric Tensor

\[ M = \frac{\mathbf{M}}{M_{ik} = M_{ki}} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}_{123} \]

Antisymmetric Tensor

\[ M = \frac{\mathbf{M}}{-M_{ik} = -M_{ki}} \begin{pmatrix} 0 & -2 & -3 \\ 2 & 0 & -5 \\ 3 & 5 & 0 \end{pmatrix}_{123} \]
3. Tensor – (continued) More Definitions

Tensor order

Scalars, vectors, and tensors may all be considered to be tensors (entities that exist independent of coordinate system). They are tensors of different orders, however.

order = degree of complexity

<table>
<thead>
<tr>
<th>Scalars</th>
<th>0th -order tensors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vectors</td>
<td>1st -order tensors</td>
</tr>
<tr>
<td>Tensors</td>
<td>2nd -order tensors</td>
</tr>
<tr>
<td>Higher-order tensors</td>
<td>3rd -order tensors</td>
</tr>
</tbody>
</table>

Number of coefficients needed to express the tensor in 3D space

Tensor Invariants

Scalars that are associated with tensors; these are numbers that are independent of coordinate system.

Vectors: \[ V = \overrightarrow{V} \] The magnitude of a vector is a scalar associated with the vector. It is independent of coordinate system, i.e. it is an invariant.

Tensors: \[ \mathbf{A} \] There are three invariants associated with a second-order tensor.
Tensor Invariants

\[ I_A \equiv tr A = tr A \]

For the tensor written in Cartesian coordinates:

\[ tr A = A_{pp} = A_{11} + A_{22} + A_{33} \]

\[ II_A \equiv tr(A \cdot A) = A : A = A_{pk}A_{kp} \]

\[ III_A \equiv tr((A \cdot A) \cdot A) = A_{pj}A_{jk}A_{hp} \]

Note: the definitions of invariants written in terms of coefficients are only valid when the tensor is written in Cartesian coordinates.