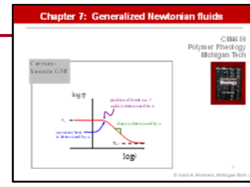
Done with Inelastic models (GNF).



Summary:

Summary: *Generalized Newtonian Fluid Constitutive Equations*

PRO:
- A first constitutive equation
- Can match steady shearing data very well
- Simple to calculate with
- Found to predict pressure-drop/flow rate relationships well

CON:
- Fails to predict shear normal stresses
- Fails to predict start-up or cessation effects (time-dependence, memory) – only a function of instantaneous velocity gradient
- Derived ad hoc from shear observations; unclear of validity in non-shear flows

Innovate and improve constitutive equations

What we know so far…

Rules for Constitutive Equations

$$\underline{\underline{\tau}}(t) = f\left(\dot{\underline{\underline{\gamma}}}, I_{\dot{\underline{\underline{\gamma}}}}, II_{\dot{\underline{\underline{\gamma}}}}, III_{\dot{\underline{\underline{\gamma}}}}, \text{material info}\right)$$

The stress expression:

- ***Must be of tensor order***
- ***Must be a tensor (independent of coordinate system)***
- ***Must be a symmetric tensor***
- ***Must make predictions that are independent of the observer***
- ***Should correctly predict observed flow/deformation behavior***

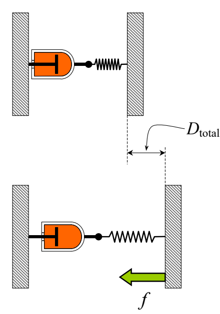
Let's move on to Linear Elastic models

1

Chapter 8: Memory Effects: GLVE

Maxwell's model combines viscous and elastic responses in series

Spring (elastic) and dashpot (viscous) in series:



Displacements are additive:

$$D_{total} = D_{spring} + D_{dashpot}$$

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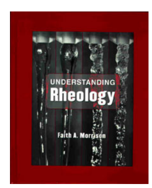


Professor Faith A. Morrison
Department of Chemical Engineering
Michigan Technological University

2

Fluids with Memory - Chapter 8

We seek a constitutive equation that includes memory effects.

$$\underline{\underline{\tau}}(t) = f\left(\underline{\underline{\dot{\gamma}}}, I_{\underline{\underline{\dot{\gamma}}}}, II_{\underline{\underline{\dot{\gamma}}}}, III_{\underline{\underline{\dot{\gamma}}}}, \text{material info}\right)$$

calculates the stress at a particular time, t

2 Constitutive Equations so far:

$$\underline{\underline{\tau}} = -\mu\underline{\underline{\dot{\gamma}}}(t)$$

$$\underline{\underline{\tau}} = -\eta(\dot{\gamma})\underline{\underline{\dot{\gamma}}}(t) \qquad \dot{\gamma} = \left|\underline{\underline{\dot{\gamma}}}\right|$$

So far, stress at t depends on rate-of-deformation ***at t only***

3

Current Constitutive Equations

Newtonian	$\underline{\underline{\tau}} = -\mu\underline{\underline{\dot{\gamma}}}(t)$
Generalized Newtonian	$\underline{\underline{\tau}} = -\eta(\dot{\gamma})\underline{\underline{\dot{\gamma}}}(t) \qquad \dot{\gamma} = \left\|\underline{\underline{\dot{\gamma}}}\right\|$

Neither can predict:

- Shear normal stresses - *this will be wrong so long as we use constitutive equations proportional to $\underline{\underline{\dot{\gamma}}}$*
- stress transients in shear (startup, cessation) - *this flaw seems to be related to omitting fluid memory*

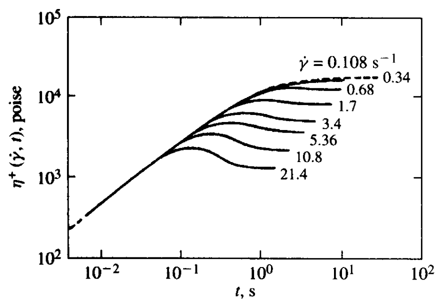
We will try to fix this now; we will address the first point when we discuss advanced constitutive equations

4

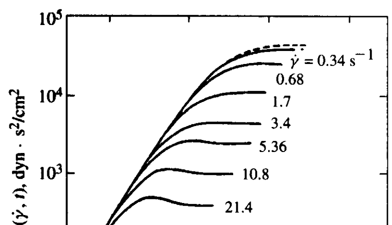
Fluids with Memory – Chapter 8

Startup of Steady Shearing

$$\underline{v} \equiv \begin{pmatrix} \dot{\varsigma}(t)x_2 \\ 0 \\ 0 \end{pmatrix}_{123} \qquad \dot{\varsigma}(t) = \begin{cases} 0 & t < 0 \\ \dot{\gamma}_0 & t \geq 0 \end{cases}$$



$$\eta^+ \equiv \frac{-\tau_{21}(t)}{\dot{\gamma}_0}$$



$$\Psi_1^+ \equiv \frac{-(\tau_{11} - \tau_{22})}{\dot{\gamma}_0^2}$$

Figures 6.49, 6.50, p. 208 Menezes and Graessley, PB soln

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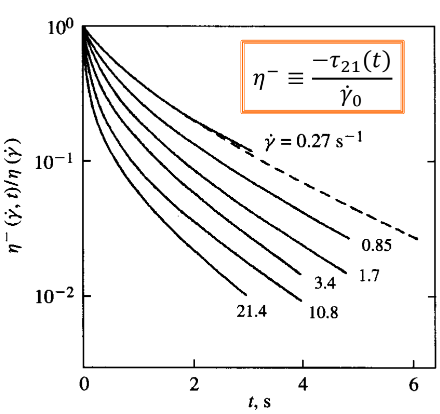
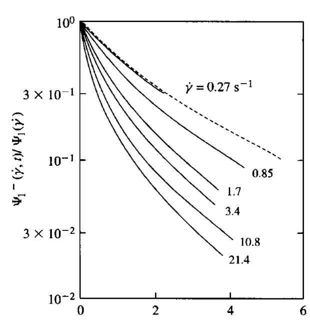
Fluids with Memory – Chapter 8

Cessation of Steady Shearing

$$\underline{v} \equiv \begin{pmatrix} \dot{\varsigma}(t)x_2 \\ 0 \\ 0 \end{pmatrix}_{123} \qquad \dot{\varsigma}(t) = \begin{cases} \dot{\gamma}_0 & t < 0 \\ 0 & t \geq 0 \end{cases}$$



$$\eta^- \equiv \frac{-\tau_{21}(t)}{\dot{\gamma}_0}$$



$$\Psi_1^- \equiv \frac{-(\tau_{11} - \tau_{22})}{\dot{\gamma}_0^2}$$

Figures 6.51, 6.52, p. 209 Menezes and Graessley, PB soln

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Fluids with Memory – Chapter 8

How can we incorporate time-dependent effects?

First, we explore a simple memory fluid.

Let's construct a new constitutive equation that remembers the stress at a time t_0 seconds ago

$$-\underline{\underline{\tau}}(t) = \tilde{\eta}\dot{\underline{\underline{\gamma}}}(t) + (0.8\tilde{\eta})\dot{\underline{\underline{\gamma}}}(t - t_0)$$

"Newtonian" contribution (first term)

contribution from fluid memory (second term)

This is the rate-of-deformation tensor t_0 seconds before time t

$\tilde{\eta}$ is a parameter of the model (it is constant)

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Fluids with Memory – Chapter 8

What does this model predict?

Simple memory fluid

$$-\underline{\underline{\tau}}(t) = \tilde{\eta}\dot{\underline{\underline{\gamma}}}(t) + (0.8\tilde{\eta})\dot{\underline{\underline{\gamma}}}(t - t_0)$$

Steady shear

$\eta = ?$

$\Psi_1 = ?$

$\Psi_2 = ?$

Shear start-up

$\eta^+(t) = ?$

$\Psi_1^+(t) = ?$

$\Psi_2^+(t) = ?$

8

Fluids with Memory – Chapter 8

What does this model predict?

Simple memory fluid

$$-\underline{\underline{\tau}}(t) = \tilde{\eta}\dot{\underline{\underline{\gamma}}}(t) + (0.8\tilde{\eta})\dot{\underline{\underline{\gamma}}}(t - t_0)$$

Steady shear

$\eta = ?$

$\Psi_1 = ?$

$\Psi_2 = ?$

Let's try.

Shear start-up

$\eta^+(t) = ?$

$\Psi_1^+(t) = ?$

$\Psi_2^+(t) = ?$

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Steady Shear Flow Material Functions

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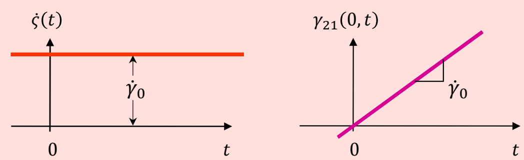
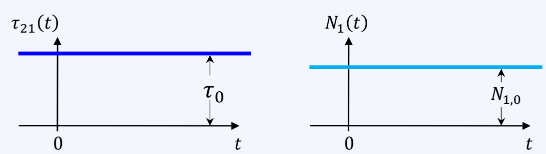
Imposed Kinematics:

$$\underline{v} \equiv \begin{pmatrix} \dot{\varsigma}(t) x_2 \\ 0 \\ 0 \end{pmatrix}_{123}$$

$\dot{\varsigma}(t) = \dot{\gamma}_0 = \text{constant}$



Material Stress Response:



Material Functions:

Viscosity $\eta(\dot{\gamma}_0) \equiv \frac{\tilde{\tau}_{21}}{\dot{\gamma}_0} = \frac{-\tau_{21}}{\dot{\gamma}_0}$

First normal-stress coefficient $\Psi_1(\dot{\gamma}_0) \equiv \frac{\tilde{\tau}_{11} - \tilde{\tau}_{22}}{\dot{\gamma}_0^2} = \frac{-(\tau_{11} - \tau_{22})}{\dot{\gamma}_0^2}$

Second normal-stress coefficient $\Psi_2(\dot{\gamma}_0) \equiv \frac{\tilde{\tau}_{22} - \tilde{\tau}_{33}}{\dot{\gamma}_0^2} = \frac{-(\tau_{22} - \tau_{33})}{\dot{\gamma}_0^2}$

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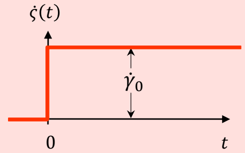
Start-up of Steady Shear Flow Material Functions

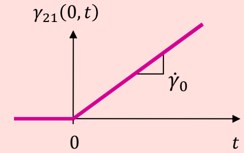
$\Psi_2^+(t,\dot{\gamma}_0) \equiv \frac{\tilde{\tau}_{22}-\tilde{\tau}_{33}}{\dot{\gamma}_0^2}$

Imposed Kinematics:

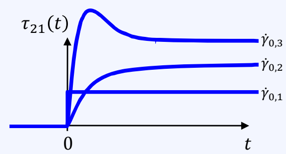
$$\underline{v} \equiv \begin{pmatrix} \dot{\varsigma}(t)x_2 \\ 0 \\ 0 \end{pmatrix}_{123}$$

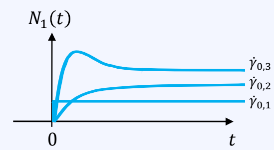
$$\dot{\varsigma}(t) = \begin{cases} 0 & t < 0 \\ \dot{\gamma}_0 & t \geq 0 \end{cases}$$





Material Stress Response:





Material Functions:

Shear stress growth function
$$\eta^+(t,\dot{\gamma}_0) \equiv \frac{\tilde{\tau}_{21}(t)}{\dot{\gamma}_0} = \frac{-\tau_{21}(t)}{\dot{\gamma}_0}$$

First normal-stress growth coefficient
$$\Psi_1^+(t,\dot{\gamma}_0) \equiv \frac{\tilde{\tau}_{11}-\tilde{\tau}_{22}}{\dot{\gamma}_0^2}$$

Second normal-stress growth coefficient
$$\Psi_2^+(t,\dot{\gamma}_0) \equiv \frac{\tilde{\tau}_{22}-\tilde{\tau}_{33}}{\dot{\gamma}_0^2}$$

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Fluids with Memory – Chapter 8

Predictions of the simple memory fluid

$$-\underline{\underline{\tau}}(t) = \tilde{\eta}\underline{\underline{\dot{\gamma}}}(t) + (0.8\tilde{\eta})\underline{\underline{\dot{\gamma}}}(t-t_0)$$

Steady shear

$$\eta = 1.8\tilde{\eta}$$
$$\Psi_1 = \Psi_2 = 0$$

The steady viscosity reflects contributions from what is currently happening and contributions from what happened t_0 seconds ago.

Shear start-up

$$\eta^+(t) = \begin{cases} 0 & t < 0 \\ \tilde{\eta} & 0 \leq t \leq t_0 \\ 1.8\tilde{\eta} & t \geq t_0 \end{cases}$$

$$\Psi_1^+(t) = \Psi_2^+(t) = 0$$

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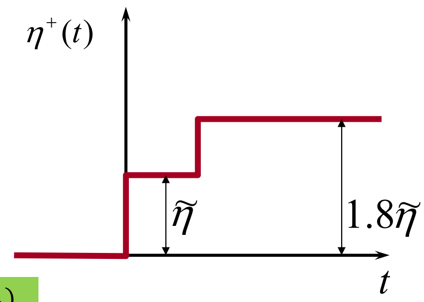
Predictions of the simple memory fluid

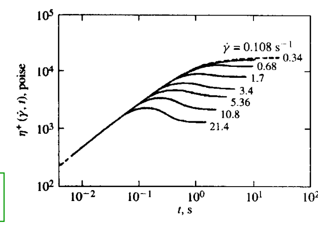
Shear start-up

$$\eta^+(t) = \begin{cases} 0 & t < 0 \\ \tilde{\eta} & 0 \le t \le t_0 \\ 1.8\tilde{\eta} & t \ge t_0 \end{cases}$$

$$\Psi_1^+(t) = \Psi_2^+(t) = 0$$

$$-\underline{\underline{\tau}}(t) = \tilde{\eta}\dot{\underline{\underline{\gamma}}}(t) + (0.8\tilde{\eta})\dot{\underline{\underline{\gamma}}}(t - t_0)$$



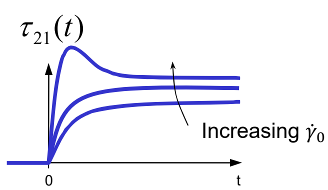


Figures 6.49, 6.50, p. 208 Menezes and Graessley, PB soln

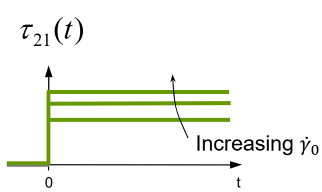
$$-\underline{\underline{\tau}}(t) = \tilde{\eta}\dot{\underline{\underline{\gamma}}}(t) + (0.8\tilde{\eta})\dot{\underline{\underline{\gamma}}}(t - t_0)$$

Predictions of the simple memory fluid

What the data show:



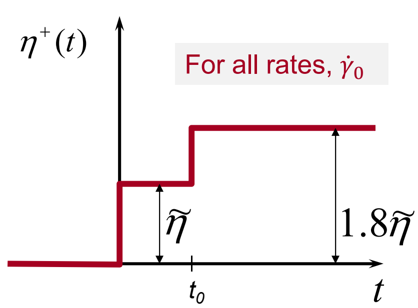
What the GNF models predict:

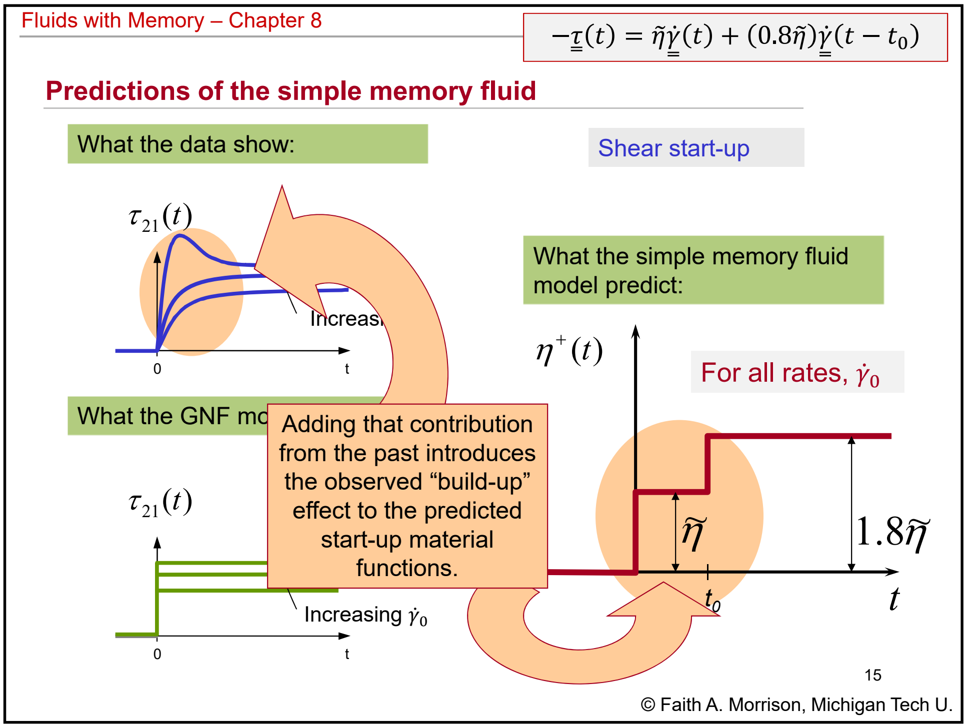


Shear start-up

What the simple memory fluid model predict:

For all rates, $\dot{\gamma}_0$




Fluids with Memory – Chapter 8
$-\underline{\underline{\tau}}(t) = \tilde{\eta}\underline{\underline{\dot{\gamma}}}(t) + (0.8\tilde{\eta})\underline{\underline{\dot{\gamma}}}(t-t_0)$
Predictions of the simple memory fluid
What the data show:
Shear start-up
$\tau_{21}(t)$
Increas
What the simple memory fluid model predict:
$\eta^+(t)$
For all rates, $\dot{\gamma}_0$
What the GNF mo
Adding that contribution from the past introduces the observed "build-up" effect to the predicted start-up material functions.
$\tau_{21}(t)$
Increasing $\dot{\gamma}_0$
$\tilde{\eta}$
$1.8\tilde{\eta}$
t_0
t
15
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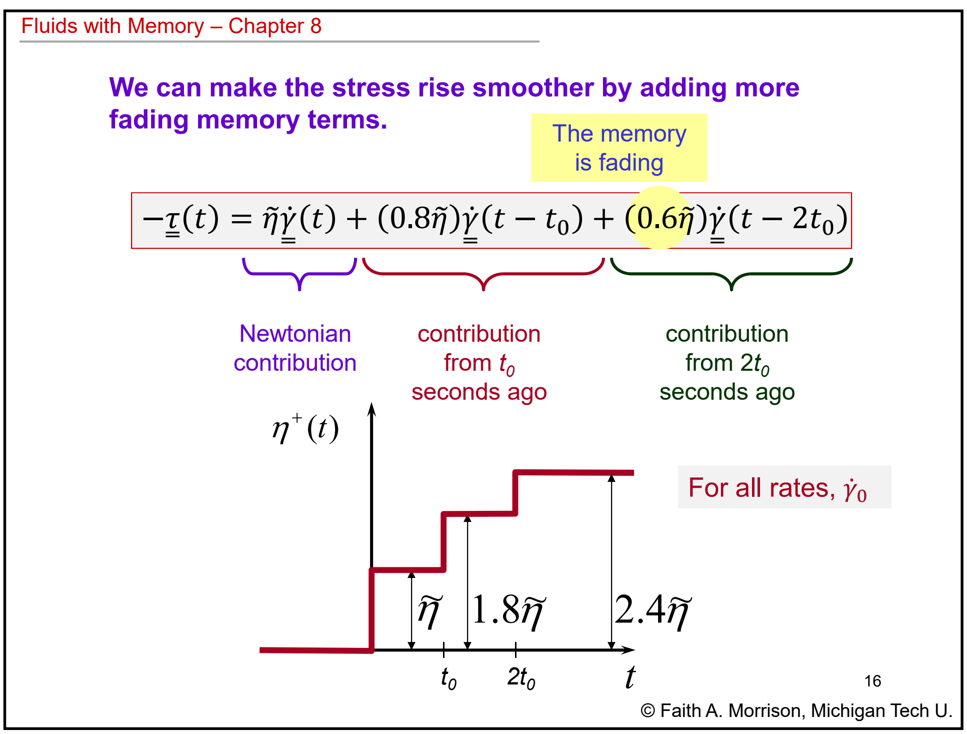
Fluids with Memory – Chapter 8
We can make the stress rise smoother by adding more fading memory terms.
The memory is fading
$-\underline{\underline{\tau}}(t) = \tilde{\eta}\underline{\underline{\dot{\gamma}}}(t) + (0.8\tilde{\eta})\underline{\underline{\dot{\gamma}}}(t-t_0) + (0.6\tilde{\eta})\underline{\underline{\dot{\gamma}}}(t-2t_0)$
Newtonian contribution
contribution from t_0 seconds ago
contribution from $2t_0$ seconds ago
$\eta^+(t)$
For all rates, $\dot{\gamma}_0$
$\tilde{\eta}$
$1.8\tilde{\eta}$
$2.4\tilde{\eta}$
t_0
$2t_0$
t
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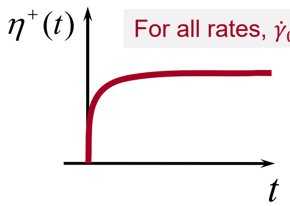
Fluids with Memory – Chapter 8

The fit can be made to be perfectly smooth by using a sum of exponentially decaying terms as the weighting functions.

$$-\underline{\underline{\tau}}(t) = \tilde{\eta}\left[\dot{\underline{\underline{\gamma}}}(t) + (0.37)\dot{\underline{\underline{\gamma}}}(t-t_0) + (0.14)\dot{\underline{\underline{\gamma}}}(t-2t_0) + (0.05)\dot{\underline{\underline{\gamma}}}(t-3t_0) + \cdots\right.$$
$$= \tilde{\eta}\sum_{p=0}^{\infty} e^{-pt_0/\lambda}\dot{\underline{\underline{\gamma}}}(t-pt_0)$$

start-up prediction:



(t_0/λ scales the decay)

$\frac{pt_0}{\lambda}$	$e^{-pt_0/\lambda}$
0	1.00
1	0.37
2	0.14
3	0.05
4	0.02

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Fluids with Memory – Chapter 8

New model: $-\underline{\underline{\tau}}(t) = \tilde{\eta}\sum_{p=0}^{\infty} e^{-\frac{pt_0}{\lambda}}\dot{\underline{\underline{\gamma}}}(t-pt_0)$

This sum can be rewritten as an integral.

$$I = \int_a^b f(x)dx \equiv \lim_{N\to\infty}\left[\sum_{i=1}^{N} f(a+i\Delta x)\Delta x\right], \quad \Delta x = \frac{b-a}{N}$$

$a \to t$

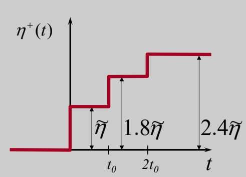
$x \to -t'$

$\Delta x \to -\Delta t'$

$i\Delta x \to -pt_0 = -p\Delta t'$

$f(a+i\Delta x) \to e^{-p\Delta t'}\dot{\underline{\underline{\gamma}}}(t-p\Delta t')$

(Actually, it takes a bit of renormalizing to make this transformation actually work.)



In the current formulation, $\eta^+(t)$ grows as N goes to infinity.

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With proper reformulation, we obtain:

Maxwell Model (integral version)

$$\underline{\underline{\tau}}(t) = -\int_{-\infty}^{t} \frac{\eta_0}{\lambda} e^{\frac{-(t-t')}{\lambda}} \dot{\underline{\underline{\gamma}}}(t')dt'$$

Two parameters:

Zero-shear viscosity η_0 – gives the value of the steady shear viscosity

Relaxation time λ - quantifies how fast memory fades

Often we write in terms of a *modulus* parameter, $G = \frac{\eta_0}{\lambda}$

Steps to arrive here:
•Add information about past deformations
•Make memory fade

We've seen that including terms that invoke past deformations (fluid memory) can improve the constitutive predictions we make.

This same class of models can be derived in differential form, beginning with the idea of combining viscous and elastic effects.

Fluids with Memory – Maxwell Models

The Maxwell Models

The basic Maxwell model is based on the observation that at long times viscoelastic materials behave like Newtonian liquids, while at short times they behave like elastic solids.

Hooke's Law for elastic solids $\tau_{21} = -G\gamma_{21}$

Newton's Law for viscous liquids $\tau_{21} = -\eta\dot{\gamma}_{21}$

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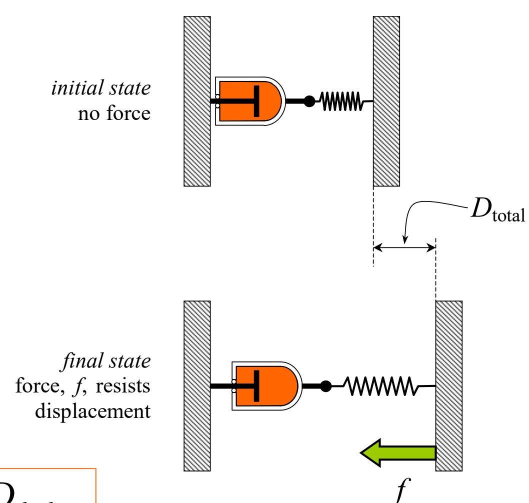
Fluids with Memory – Maxwell Models

Maxwell's model combines viscous and elastic responses in series

Spring (elastic) and dashpot (viscous) in series:



Displacements are additive:

$$D_{total} = D_{spring} + D_{dashpot}$$

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Fluids with Memory – Maxwell Models

*(proportional to **displacement**)* **In the spring:** $f = -G_{sp} D_{spring}$

*(proportional to **rate**)* **In the dashpot:** $f = -\mu \frac{dD_{dash}}{dt}$

$$D_{total} = D_{spring} + D_{dash}$$

$$\frac{dD_{total}}{dt} = \frac{dD_{spring}}{dt} + \frac{dD_{dash}}{dt}$$

$$= -\frac{1}{G_{sp}}\frac{df}{dt} - \frac{1}{\mu} f$$

$$f + \frac{\mu}{G_{sp}}\frac{df}{dt} = -\mu \frac{dD_{total}}{dt}$$

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Fluids with Memory – Maxwell Models

$$f + \frac{\mu}{G_{sp}}\frac{df}{dt} = -\mu \frac{dD_{total}}{dt}$$

By analogy:

$$\tau_{21} + \frac{\eta_0}{G}\frac{\partial \tau_{21}}{\partial t} = -\eta_0 \dot{\gamma}_{21} \quad \text{shear}$$

Also by analogy: *(we're taking a leap here)*

$$\underline{\underline{\tau}} = + \frac{\eta_0}{G}\frac{\partial \underline{\underline{\tau}}}{\partial t} = -\eta_0 \underline{\underline{\dot{\gamma}}} \quad \text{all flows}$$

Two parameter model:

$\lambda = \frac{\eta_0}{G}$ *Relaxation time*

η_0 *Viscosity*

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The Maxwell Model

$$\underline{\underline{\tau}} + \frac{\eta_0}{G}\frac{\partial \underline{\underline{\tau}}}{\partial t} = -\eta_0 \dot{\underline{\underline{\gamma}}}$$

Two parameter model:	$\lambda = \frac{\eta_0}{G}$	*Relaxation time*
	η_0	*Viscosity*

The Maxwell Model

$$\tau_{21} + \frac{\eta_0}{G}\frac{\partial \tau_{21}}{\partial t} = -\eta_0 \dot{\gamma}_{21}$$

How does the Maxwell model behave at steady state? For short time deformations?

Fluids with Memory – Maxwell Models

Example: Solve the Maxwell Model for an expression explicit in the stress tensor

$$\underline{\underline{\tau}} + \frac{\eta_0}{G}\frac{\partial \underline{\underline{\tau}}}{\partial t} = -\eta_0 \dot{\underline{\underline{\gamma}}}$$

Fluids with Memory – Maxwell Models

Recall:

To solve first-order, linear differential equations:

$$\frac{d\boldsymbol{y}}{dx} + a(x)\boldsymbol{y} + b(x) = 0$$

Use an ***integrating function***, $u(x)$

$$u(x) = e^{\int a(x')dx'}$$

Fluids with Memory – Maxwell Models

Example: Solve the Maxwell Model for an expression explicit in the stress tensor

$$\underline{\underline{\tau}} + \frac{\eta_0}{G}\frac{\partial \underline{\underline{\tau}}}{\partial t} = -\eta_0 \dot{\underline{\underline{\gamma}}}$$

Let's try.

Fluids with Memory – Maxwell Models

Maxwell Model (integral version)

$$\underline{\underline{\tau}}(t) = -\int_{-\infty}^{t} \frac{\eta_0}{\lambda} e^{-\frac{(t-t')}{\lambda}} \dot{\underline{\underline{\gamma}}}(t')dt'$$

This is the same equation we "made up" when starting from the simple memory model!

We arrived at this equation following **two different** paths:

•Add up fading contributions of past deformations
•Add viscous and elastic effects in series

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What now?

What are the predictions of the Maxwell model?

Need to check the predictions to see if what we have done is worth keeping.

Predictions:

- Steady shear
- Steady elongation
- Start-up of steady shear
- Step shear strain
- Small-amplitude oscillatory shear

What now?

What are the predictions of the Maxwell model?

Need to check the predictions to see if what we have done is worth keeping.

Predictions:

- Steady shear
- Steady elongation
- Start-up of steady shear
- Step shear strain
- Small-amplitude oscillatory shear

Let's get to work.

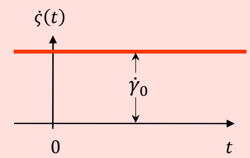
Steady Shear Flow Material Functions

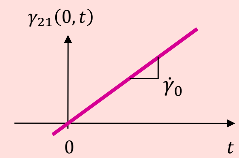
Michigan Tech

Imposed Kinematics:

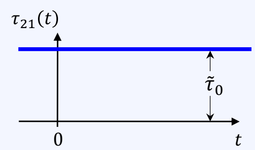
$$\underline{v} \equiv \begin{pmatrix} \dot{\varsigma}(t) x_2 \\ 0 \\ 0 \end{pmatrix}_{123}$$

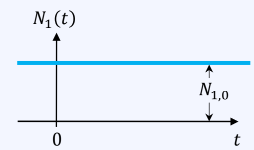
$$\dot{\varsigma}(t) = \dot{\gamma}_0 = \text{constant}$$





Material Stress Response:





Material Functions:

Viscosity $\eta(\dot{\gamma}_0) \equiv \dfrac{\tilde{\tau}_{21}}{\dot{\gamma}_0} = \dfrac{-\tau_{21}}{\dot{\gamma}_0}$

First normal-stress coefficient $\Psi_1(\dot{\gamma}_0) \equiv \dfrac{\tilde{\tau}_{11} - \tilde{\tau}_{22}}{\dot{\gamma}_0^2} = \dfrac{-(\tau_{11} - \tau_{22})}{\dot{\gamma}_0^2}$

Second normal-stress coefficient $\Psi_2(\dot{\gamma}_0) \equiv \dfrac{\tilde{\tau}_{22} - \tilde{\tau}_{33}}{\dot{\gamma}_0^2} = \dfrac{-(\tau_{22} - \tau_{33})}{\dot{\gamma}_0^2}$

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Fluids with Memory – Chapter 8

Predictions of the (single-mode) Maxwell Model

$$\underline{\underline{\tau}} + \frac{\eta_0}{G}\frac{\partial \underline{\underline{\tau}}}{\partial t} = -\eta_0 \underline{\underline{\dot{\gamma}}}$$

$$\underline{\underline{\tau}}(t) = -\int_{-\infty}^{t} \left[\frac{\eta_0}{\lambda} e^{\frac{-(t-t')}{\lambda}} \right] \underline{\underline{\dot{\gamma}}}(t') dt'$$

Steady shear	$\eta = \eta_0$ $\Psi_1 = \Psi_2 = 0$	Fails to predict shear normal stresses. Fails to predict shear-thinning.
Steady elongation	$\bar{\eta} = 3\eta_0$	Trouton's rule

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Fluids with Memory – Chapter 8

Steady shear viscosity and first normal stress coefficient

BOGER FLUIDS

There are some systems with a constant viscosity but still start-up effects.

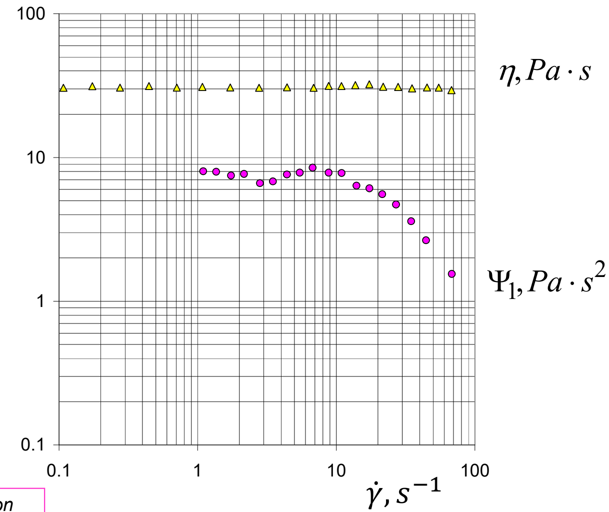

Figure 6.5, p. 173 Binnington and Boger; PIB soln

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Fluids with Memory – Chapter 8

Steady shear viscosity and first and ***second*** normal stress coefficient

BOGER FLUIDS

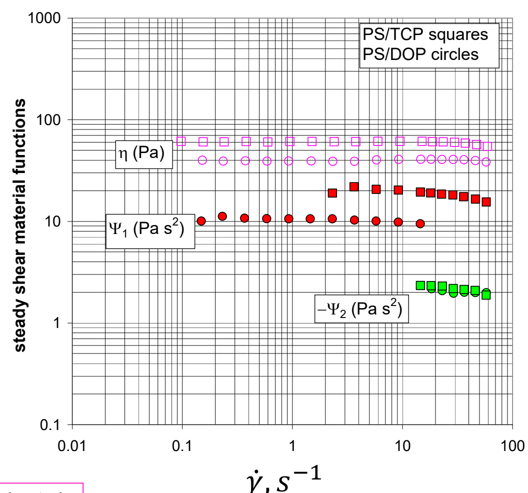

Figure 6.6, p. 174 Magda et al.; PS solns

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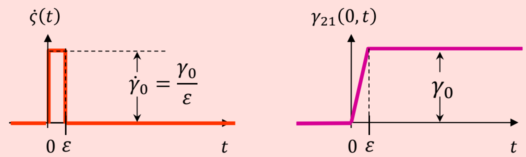
Step Strain Shear Flow Material Functions

Michigan Tech

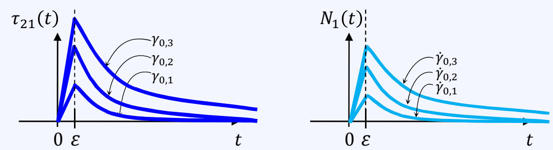
Imposed Kinematics:

$$\underline{v} \equiv \begin{pmatrix} \dot{\varsigma}(t) x_2 \\ 0 \\ 0 \end{pmatrix}_{123}$$

$$\dot{\varsigma}(t) = \lim_{\varepsilon \to 0} \begin{cases} 0 & t \le 0 \\ \gamma_0/\varepsilon & 0 < t < \varepsilon \\ 0 & t \ge \varepsilon \end{cases}$$



Material Stress Response:



Material Functions:

Relaxation modulus $G(t,\gamma_0) \equiv \frac{\tilde{\tau}_{21}(t,\gamma_0)}{\gamma_0} = \frac{-\tau_{21}(t,\gamma_0)}{\gamma_0}$

First normal-stress relaxation modulus $G_{\Psi_1}(t,\gamma_0) \equiv \frac{\tilde{\tau}_{11} - \tilde{\tau}_{22}}{\gamma_0^2}$

Second normal-stress relaxation modulus $G_{\Psi_2}(t,\gamma_0) \equiv \frac{\tilde{\tau}_{22} - \tilde{\tau}_{33}}{\gamma_0^2}$

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Fluids with Memory – Chapter 8

Predictions of the (single-mode) Maxwell Model

$$\underline{\underline{\tau}} + \frac{\eta_0}{G}\frac{\partial \underline{\underline{\tau}}}{\partial t} = -\eta_0 \dot{\underline{\underline{\gamma}}}$$

$$\underline{\underline{\tau}}(t) = -\int_{-\infty}^{t} \left[\frac{\eta_0}{\lambda} e^{\frac{-(t-t')}{\lambda}} \right] \dot{\underline{\underline{\gamma}}}(t') dt'$$

Shear start-up	$\eta^+(t) = \eta_0\left(1 - e^{-t/\lambda}\right)$ $\Psi_1^+(t) = \Psi_2^+(t) = 0$	**Does** predict a gradual build-up of stresses on start-up.
Step shear strain	$G(t) = \frac{\eta_0}{\lambda} e^{-t/\lambda}$ $G_{\Psi 1} = G_{\Psi 2} = 0$	**Does** predict a reasonable relaxation function in step strain (but no normal stresses again).

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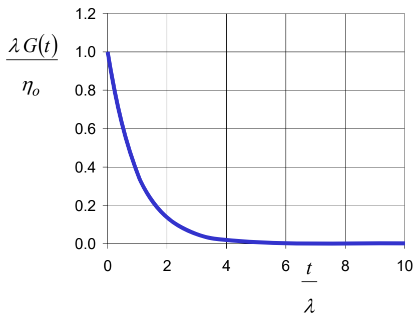
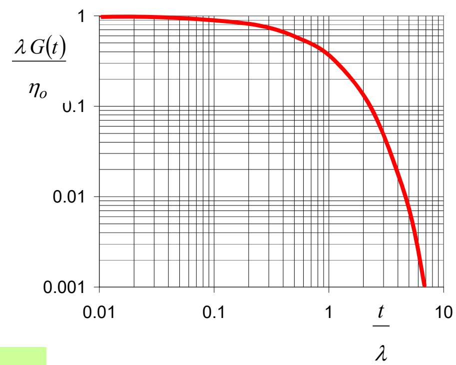
Fluids with Memory – Chapter 8

Step-Shear-Strain Material Function *G(t)* for Maxwell Model





$$\underline{\underline{\tau}}(t) = -\int_{-\infty}^{t} \left[\frac{\eta_0}{\lambda} e^{\frac{-(t-t')}{\lambda}} \right] \underline{\underline{\dot{\gamma}}}(t')dt'$$

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Fluids with Memory – Chapter 8

Comparison to experimental data

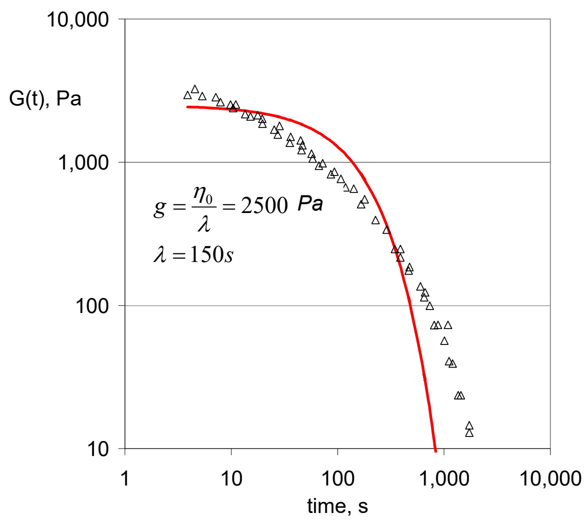

Figure 8.4, p. 274 data from Einaga et al., PS 20% soln in chlorinated diphenyl

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Fluids with Memory – Chapter 8

We can improve this fit by adjusting the Maxwell model to allow multiple relaxation modes

$$\underline{\underline{\tau}}_{(k)} = -\int_{-\infty}^{t} \left(\frac{\eta_k}{\lambda_k}\right) e^{-(t-t')/\lambda_k} \underline{\underline{\dot{\gamma}}}(t')dt'$$

$$\underline{\underline{\tau}}(t) = \sum_{k=1}^{N} \underline{\underline{\tau}}_{(k)}$$

Generalized Maxwell Model

$$\underline{\underline{\tau}}(t) = -\int_{-\infty}^{t} \left[\sum_{k=1}^{N} \frac{\eta_k}{\lambda_k} e^{\frac{-(t-t')}{\lambda_k}}\right] \underline{\underline{\dot{\gamma}}}(t')dt'$$

$2N$ parameters (can fit *anything*)

$2N$ model parameters:
g_k, λ_k (constants)
($\eta_k = g_k \lambda_k$)

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Fluids with Memory – Chapter 8

Generalized Maxwell Model

$$\underline{\underline{\tau}}(t) = -\int_{-\infty}^{t} \left[\sum_{k=1}^{N} \frac{\eta_k}{\lambda_k} e^{\frac{-(t-t')}{\lambda_k}}\right] \underline{\underline{\dot{\gamma}}}(t')dt'$$

What are the predictions of the Generalized Maxwell model?

- Steady shear
- Steady elongation
- Start-up of steady shear
- Step shear strain
- Small-amplitude oscillatory shear

Let's try.

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Fluids with Memory – Chapter 8

Generalized Maxwell Model

$$\underline{\underline{\tau}}(t) = -\int_{-\infty}^{t} \left[\sum_{k=1}^{N} \frac{\eta_k}{\lambda_k} e^{\frac{-(t-t')}{\lambda_k}} \right] \underline{\underline{\dot{\gamma}}}(t')dt'$$

What are the predictions of the **Generalized** Maxwell model?

- Steady shear
- Steady elongation
- Start-up of steady shear
- Step shear strain
- Small-amplitude oscillatory shear

Let's try.

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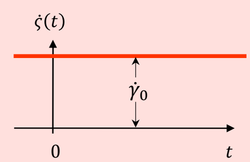
Steady Shear Flow Material Functions

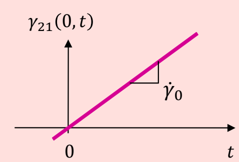
Michigan Tech

Imposed Kinematics:

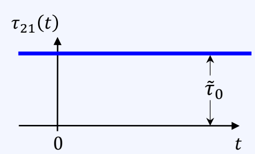
$$\underline{v} \equiv \begin{pmatrix} \dot{\varsigma}(t)x_2 \\ 0 \\ 0 \end{pmatrix}_{123}$$

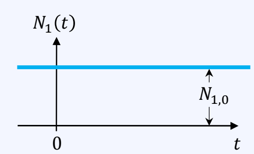
$$\dot{\varsigma}(t) = \dot{\gamma}_0 = \text{constant}$$





Material Stress Response:





Material Functions:

Viscosity $\eta(\dot{\gamma}_0) \equiv \frac{\tilde{\tau}_{21}}{\dot{\gamma}_0} = \frac{-\tau_{21}}{\dot{\gamma}_0}$

First normal-stress coefficient $\Psi_1(\dot{\gamma}_0) \equiv \frac{\tilde{\tau}_{11}-\tilde{\tau}_{22}}{\dot{\gamma}_0^2} = \frac{-(\tau_{11}-\tau_{22})}{\dot{\gamma}_0^2}$

Second normal-stress coefficient $\Psi_2(\dot{\gamma}_0) \equiv \frac{\tilde{\tau}_{22}-\tilde{\tau}_{33}}{\dot{\gamma}_0^2} = \frac{-(\tau_{22}-\tau_{33})}{\dot{\gamma}_0^2}$

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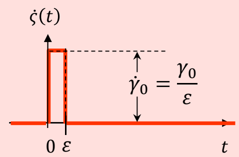
Step Strain Shear Flow Material Functions

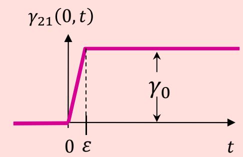
Michigan Tech

Imposed Kinematics:

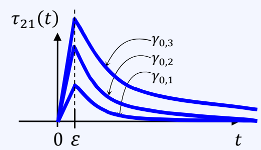
$$\underline{v} \equiv \begin{pmatrix} \dot{\varsigma}(t)x_2 \\ 0 \\ 0 \end{pmatrix}_{123}$$

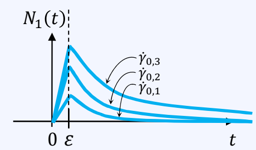
$$\dot{\varsigma}(t) = \lim_{\varepsilon \to 0} \begin{cases} 0 & t \le 0 \\ \gamma_0/\varepsilon & 0 < t < \varepsilon \\ 0 & t \ge \varepsilon \end{cases}$$





Material Stress Response:





Material Functions:

Relaxation modulus $G(t,\gamma_0) \equiv \dfrac{\tilde{\tau}_{21}(t,\gamma_0)}{\gamma_0} = \dfrac{-\tau_{21}(t,\gamma_0)}{\gamma_0}$

First normal-stress relaxation modulus $G_{\Psi_1}(t,\gamma_0) \equiv \dfrac{\tilde{\tau}_{11}-\tilde{\tau}_{22}}{\gamma_0^2}$

Second normal-stress relaxation modulus $G_{\Psi_2}(t,\gamma_0) \equiv \dfrac{\tilde{\tau}_{22}-\tilde{\tau}_{33}}{\gamma_0^2}$

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Fluids with Memory – Chapter 8

Predictions of the Generalized Maxwell Model

$2N$ model parameters:
g_k, λ_k (constants)
$(\eta_k = g_k \lambda_k)$

$$\underline{\underline{\tau}}(t) = -\int_{-\infty}^{t} \left[\sum_{k=1}^{N} \frac{\eta_k}{\lambda_k} e^{\frac{-(t-t')}{\lambda_k}} \right] \underline{\underline{\dot{\gamma}}}(t')dt'$$

Steady shear	$\eta = \sum_{k=1}^{N} \eta_k$ $\Psi_1 = \Psi_2 = 0$	Fails to predict shear normal stresses Fails to predict shear-thinning
Step shear strain	$G(t) = \sum_{k=1}^{N} \frac{\eta_k}{\lambda_k} e^{-t/\lambda_k}$ $G_{\Psi 1} = G_{\Psi 2} = 0$	This function can fit ***any*** observed data (we show how) Note that the GMM does not predict shear normal stresses.

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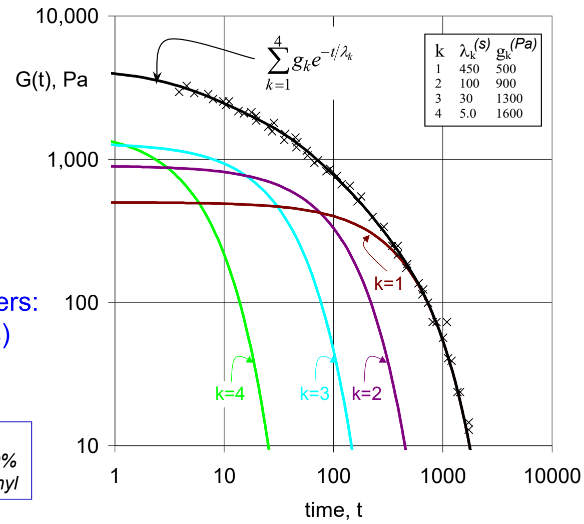
Fluids with Memory – Chapter 8

Fitting $G(t)$ to Generalized Maxwell Model



$2N$ model parameters:
g_k, λ_k (constants)
$(\eta_k = g_k \lambda_k)$

Figure 8.4, p. 274 data from Einaga et al., PS 20% soln in chlorinated diphenyl

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Fluids with Memory – Chapter 8

The Linear-Viscoelastic Models

Differential Maxwell (one mode):

$$\underline{\underline{\tau}} + \frac{\eta_0}{G}\frac{\partial \underline{\underline{\tau}}}{\partial t} = -\eta_0 \dot{\underline{\underline{\gamma}}}$$

Integral Maxwell (one mode):

$$\underline{\underline{\tau}}(t) = -\int_{-\infty}^{t} \left[\frac{\eta_0}{\lambda} e^{\frac{-(t-t')}{\lambda}}\right] \dot{\underline{\underline{\gamma}}}(t')dt'$$

Generalized Maxwell model (N modes):

$$\underline{\underline{\tau}}(t) = -\int_{-\infty}^{t} \left[\sum_{k=1}^{N} \frac{\eta_k}{\lambda_k} e^{\frac{-(t-t')}{\lambda_k}}\right] \dot{\underline{\underline{\gamma}}}(t')dt'$$

$2N$ model parameters:
g_k, λ_k (constants)
$(\eta_k = g_k \lambda_k)$

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Predictions of the Generalized Maxwell Model

$$\underline{\underline{\tau}}(t) = -\int_{-\infty}^{t} \left[\sum_{k=1}^{N} \frac{\eta_k}{\lambda_k} e^{\frac{-(t-t')}{\lambda_k}} \right] \dot{\underline{\underline{\gamma}}}(t')dt'$$

Note: The time-dependent function in the integral of the GMM is the same form as the step strain result:

Step shear strain	$G(t) = \sum_{k=1}^{N} \frac{\eta_k}{\lambda_k} e^{-t/\lambda_k}$ $G_{\Psi 1} = G_{\Psi 2} = 0$

The Linear-Viscoelastic Models

Differential Maxwell (one mode):

$$\underline{\underline{\tau}} + \frac{\eta_0}{G}\frac{\partial \underline{\underline{\tau}}}{\partial t} = -\eta_0 \dot{\underline{\underline{\gamma}}}$$

Integral Maxwell (one mode):

$$\underline{\underline{\tau}}(t) = -\int_{-\infty}^{t} \left[\frac{\eta_0}{\lambda} e^{\frac{-(t-t')}{\lambda}} \right] \dot{\underline{\underline{\gamma}}}(t')dt'$$

Generalized Maxwell model (N modes):

$$\underline{\underline{\tau}}(t) = -\int_{-\infty}^{t} \left[\sum_{k=1}^{N} \frac{\eta_k}{\lambda_k} e^{\frac{-(t-t')}{\lambda_k}} \right] \dot{\underline{\underline{\gamma}}}(t')dt'$$

Since the term in brackets is just the predicted relaxation modulus *G(t)*, we can write an even more ***general linear viscoelastic model*** by leaving this function unspecified.

Fluids with Memory – Chapter 8

The Linear-Viscoelastic Models

Differential Maxwell (one mode):

$$\underline{\underline{\tau}} + \frac{\eta_0}{G}\frac{\partial \underline{\underline{\tau}}}{\partial t} = -\eta_0 \dot{\underline{\underline{\gamma}}}$$

Integral Maxwell (one mode):

$$\underline{\underline{\tau}}(t) = -\int_{-\infty}^{t} \left[\frac{\eta_0}{\lambda} e^{\frac{-(t-t')}{\lambda}}\right] \dot{\underline{\underline{\gamma}}}(t')dt'$$

Generalized Maxwell model (N modes):

$$\underline{\underline{\tau}}(t) = -\int_{-\infty}^{t} \left[\sum_{k=1}^{N} \frac{\eta_k}{\lambda_k} e^{\frac{-(t-t')}{\lambda_k}}\right] \dot{\underline{\underline{\gamma}}}(t')dt'$$

Generalized Linear-Viscoelastic Model:

$$\underline{\underline{\tau}}(t) = -\int_{-\infty}^{t} G(t-t')\dot{\underline{\underline{\gamma}}}(t')dt'$$

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Fluids with Memory – Chapter 8

Generalized Maxwell Model

$$\underline{\underline{\tau}}(t) = -\int_{-\infty}^{t} \left[\sum_{k=1}^{N} \frac{\eta_k}{\lambda_k} e^{\frac{-(t-t')}{\lambda_k}}\right] \dot{\underline{\underline{\gamma}}}(t')dt'$$

Generalized Linear-Viscoelastic Model

$$\underline{\underline{\tau}}(t) = -\int_{-\infty}^{t} G(t-t')\dot{\underline{\underline{\gamma}}}(t')dt'$$

What are the predictions of these models?

- Steady shear
- Steady elongation
- Start-up of steady shear
- Step shear strain
- Small-amplitude oscillatory shear

Let's try.

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Small-Amplitude Oscillatory Shear Material Functions

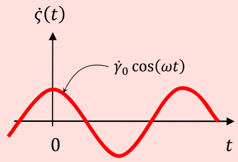
Michigan Tech

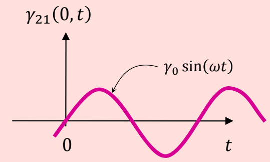
Imposed Kinematics:

$$\underline{v} \equiv \begin{pmatrix} \dot{\varsigma}(t) x_2 \\ 0 \\ 0 \end{pmatrix}_{123}$$

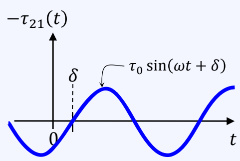
$$\dot{\varsigma}(t) = \dot{\gamma}_0 \cos(\omega t)$$

$$\gamma_0 = \frac{\dot{\gamma}_0}{\omega}$$





Material Stress Response:



δ = phase difference between stress and strain waves

$N_1(t) = N_2(t) = 0$
(linear viscoelastic regime)

Material Functions:

SAOS stress

$$\frac{\tilde{\tau}_{21}(t,\gamma_0)}{\gamma_0} = \frac{-\tau_{21}(t,\gamma_0)}{\gamma_0} = \tilde{\tau}_0 \sin(\omega t + \delta) = G' \sin(\omega t) + G'' \cos(\omega t)$$

Storage modulus $G'(\omega) \equiv \frac{-\tau_0}{\gamma_0} \cos(\delta)$

Loss modulus $G''(\omega) \equiv \frac{-\tau_0}{\gamma_0} \sin(\delta)$

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Fluids with Memory – Chapter 8

Predictions of the Generalized Maxwell Model (GMM) and Generalized Linear-Viscoelastic Model (GLVE)

$$\underline{\underline{\tau}}(t) = -\int_{-\infty}^{t} G(t-t') \dot{\underline{\underline{\gamma}}}(t') dt'$$

$$\underline{\underline{\tau}}(t) = -\int_{-\infty}^{t} \left[\sum_{k=1}^{N} \frac{\eta_k}{\lambda_k} e^{\frac{-(t-t')}{\lambda_k}} \right] \dot{\underline{\underline{\gamma}}}(t') dt'$$

Small-amplitude oscillatory shear

GLVE

$$G'(\omega) = \omega \int_0^{\infty} G(s) \cos \omega s \, ds$$

$$G''(\omega) = \omega \int_0^{\infty} G(s) \sin \omega s \, ds$$

GMM

$$G'(\omega) = \sum_{k=1}^{N} \frac{\eta_k \lambda_k \omega^2}{1 + (\lambda_k \omega)^2}$$

$$G''(\omega) = \sum_{k=1}^{N} \frac{\eta_k \omega}{1 + (\lambda_k \omega)^2}$$

$2N$ model parameters:
g_k, λ_k (constants)
$(\eta_k = g_k \lambda_k)$

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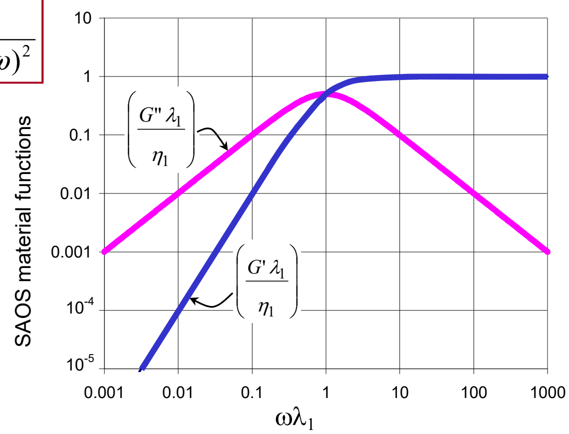
Fluids with Memory – Chapter 8

Predictions of (single-mode) Maxwell Model in SAOS

$$G'(\omega)=\frac{g_1\lambda_1^2\omega^2}{1+(\lambda_1\omega)^2}=\frac{\eta_1\lambda_1\omega^2}{1+(\lambda_1\omega)^2}$$

$$G''(\omega)=\frac{g_1\lambda_1\omega}{1+(\lambda_1\omega)^2}=\frac{\eta_1\omega}{1+(\lambda_1\omega)^2}$$

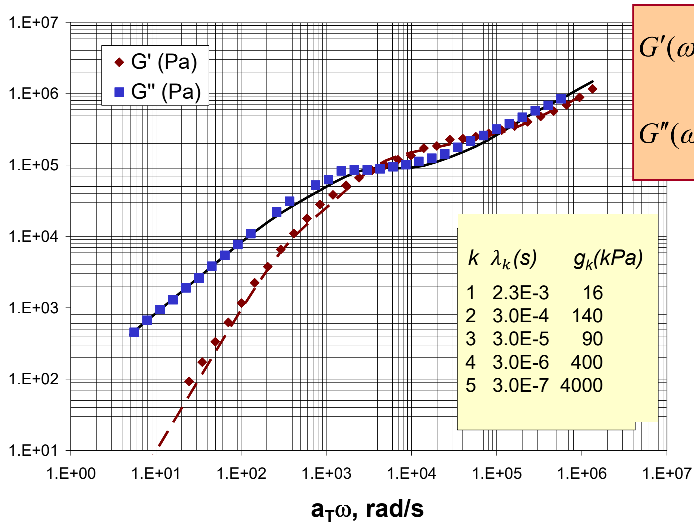
$$\underline{\underline{\tau}}(t)=-\int_{-\infty}^{t}\left[\frac{\eta_1}{\lambda_1}e^{\frac{-(t-t')}{\lambda_1}}\right]\underline{\underline{\dot{\gamma}}}(t')dt'$$



Two model parameters:
g_1, λ_1 (constants)
$(\eta_1 = g_1\lambda_1)$

Fluids with Memory – Chapter 8

Predictions of (multi-mode) Maxwell Model in SAOS



$$G'(\omega)=\sum_{k=1}^{N}\frac{\eta_k\lambda_k\omega^2}{1+(\lambda_k\omega)^2}$$

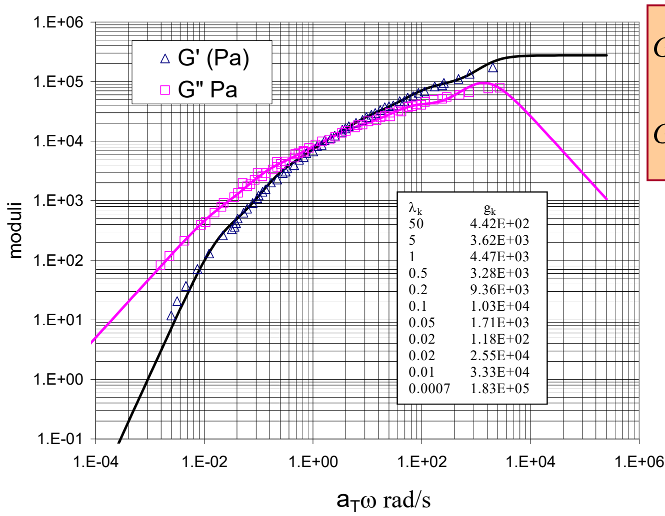
$$G''(\omega)=\sum_{k=1}^{N}\frac{\eta_k\omega}{1+(\lambda_k\omega)^2}$$

k	$\lambda_k(s)$	$g_k(kPa)$
1	2.3E-3	16
2	3.0E-4	140
3	3.0E-5	90
4	3.0E-6	400
5	3.0E-7	4000

Figure 8.8, p. 284 data from Vinogradov, PS melt

Fluids with Memory – Chapter 8

Predictions of (multi-mode) Maxwell Model in SAOS



λ_k	g_k
50	4.42E+02
5	3.62E+03
1	4.47E+03
0.5	3.28E+03
0.2	9.36E+03
0.1	1.03E+04
0.05	1.71E+03
0.02	1.18E+02
0.02	2.55E+04
0.01	3.33E+04
0.0007	1.83E+05

$$G'(\omega) = \sum_{k=1}^{N} \frac{\eta_k \lambda_k \omega^2}{1+(\lambda_k \omega)^2}$$

$$G''(\omega) = \sum_{k=1}^{N} \frac{\eta_k \omega}{1+(\lambda_k \omega)^2}$$

Figure 8.10, p. 286 data from Laun, PE melt

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Fluids with Memory – Chapter 8

1. *Shear*		
Startup	$\eta^+(t,\dot\gamma)$	$\int_0^t G(s)\,ds$
	$\Psi_1^+(t,\dot\gamma)$	0
	$\Psi_2^+(t,\dot\gamma)$	0
Steady	$\eta(\dot\gamma)$	$\int_0^\infty G(s)\,ds$
	$\Psi_1(\dot\gamma)$	0
	$\Psi_2(\dot\gamma)$	0
Cessation	$\eta^-(t,\dot\gamma)$	$\int_t^\infty G(s)\,ds$
	$\Psi_1^-(t,\dot\gamma)$	0
	$\Psi_2^-(t,\dot\gamma)$	0
SAOS	$G'(\omega)$	$\omega\int_0^\infty G(s)\sin\omega s\,ds$
	$G''(\omega)$	$\omega\int_0^\infty G(s)\cos\omega s\,ds$
Step shear strain	$G(t,\gamma_0)$	$G(t)$
	$G_{\Psi_1}(t,\gamma_0)$	0
	$G_{\Psi_2}(t,\gamma_0)$	0
2. *Extension*		
Startup		
Uniaxial ($b=0, \dot\epsilon_0>0$)	$\bar\eta^+(t,\dot\epsilon_0)$	$3\int_0^t G(s)\,ds$
or biaxial ($b=0, \dot\epsilon_0<0$)	$\bar\eta_B(t,\dot\epsilon_0)$	
Planar ($b=1, \dot\epsilon_0>0$)	$\bar\eta_{P_1}^+(t,\dot\epsilon_0)$	$4\int_0^t G(s)\,ds$
	$\bar\eta_{P_2}^+(t,\dot\epsilon_0)$	$2\int_0^t G(s)\,ds$
Steady		
Uniaxial ($b=0, \dot\epsilon_0>0$)	$\bar\eta^(\dot\epsilon_0)$	$3\int_0^\infty G(s)\,ds = 3\eta$
or biaxial ($b=0, \dot\epsilon_0<0$)	$\bar\eta_B(t,\dot\epsilon_0)$	
Planar ($b=1, \dot\epsilon_0>0$)	$\bar\eta_{P_1}(\dot\epsilon_0)$	$4\int_0^\infty G(s)\,ds$
	$\bar\eta_{P_2}(\dot\epsilon_0)$	$2\int_0^\infty G(s)\,ds$

Table 1: Predictions of Generalized Linear Viscoelastic Model in Shear and Extensional Flows

Predictions of the Generalized Linear Viscoelastic Model

(the "report card")

http://pages.mtu.edu/~fmorriso/cm4650/PredictionsGLVE.pdf

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Steady shear viscosity and first and ***second*** normal stress coefficient

BOGER FLUIDS

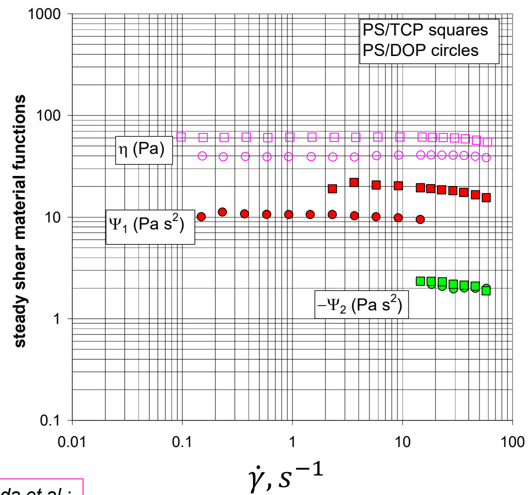
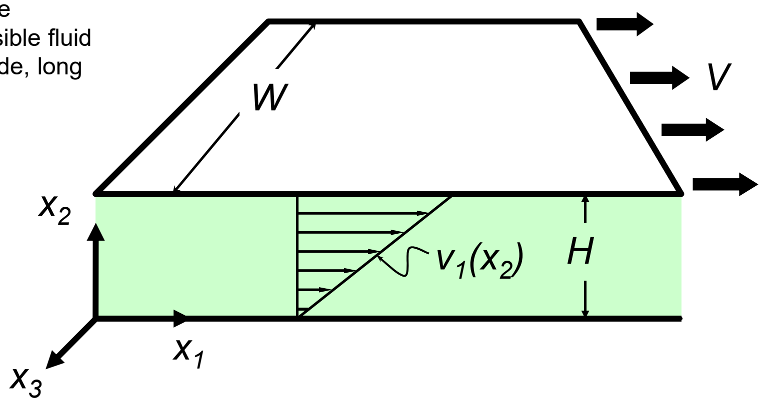


Figure 6.6, p. 174 Magda et al.; PS solns

$$\underline{\underline{\tau}}(t) = -\int_{-\infty}^{t} G(t-t')\dot{\underline{\underline{\gamma}}}(t')dt'$$

<u>EXAMPLE</u>: Drag flow of a Generalized Linear-Viscoelastic fluid between infinite parallel plates

- •steady state
- •incompressible fluid
- •infinitely wide, long



Fluids with Memory – Chapter 8

Limitations of the GLVE Models

$$\underline{\underline{\tau}}(t) = -\int_{-\infty}^{t} G(t-t')\dot{\underline{\underline{\gamma}}}(t')dt'$$

- Predicts constant shear viscosity
- Only valid in regime where strain is additive (small-strain, low rates)
- All stresses are proportional to the deformation-rate tensor; thus shear normal stresses cannot be predicted
- Cannot describe flows with a superposed rigid rotation (as we will now discuss; see Morrison p296)

What else can we try?

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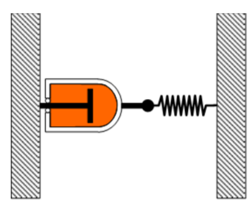
Advanced Constitutive Modeling – Chapter 9

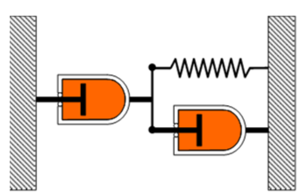
Maxwell Model - Mechanical Analog

$$\tau_{21}(t) + \lambda \frac{\partial \tau_{21}}{\partial t} = -\eta_0 \dot{\gamma}_{21}$$



Jeffreys Model - Mechanical Analog

$$\tau_{21}(t) + \lambda_1 \frac{\partial \tau_{21}}{\partial t} = -\eta_0 \left(\dot{\gamma}_{21} + \lambda_2 \frac{\partial \dot{\gamma}_{21}}{\partial t} \right)$$



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Advanced Constitutive Modeling – Chapter 9

Jeffreys Model

$$\underline{\underline{\tau}} + \lambda_1 \frac{\partial \underline{\underline{\tau}}}{\partial t} = -\eta_0 \left(\dot{\gamma}_{21} + \lambda_2 \frac{\partial \underline{\underline{\dot{\gamma}}}}{\partial t} \right)$$

Now, solving for $\underline{\underline{\tau}}$ explicitly we obtain,

$$\underline{\underline{\tau}}(t) = -\int_{-\infty}^{t} \left[\frac{\eta_0}{\lambda_1}\left(1 - \frac{\lambda_2}{\lambda_1}\right) e^{\frac{-(t-t')}{\lambda_1}} + \frac{2\eta_0\lambda_2}{\lambda_1}\delta(t-t') \right] \underline{\underline{\dot{\gamma}}}(t')dt'$$

Unfortunately, this change only modifies $G(t-t')$;
the Jeffreys Model is a GLVE model

Other modifications of the Maxwell model motivated by springs and dashpots in series and parallel modify $G(t-t')$ but do not otherwise introduce new behavior.

(Might as well use the Generalized Maxwell model)

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Advanced Constitutive Modeling – Chapter 9

Where do we go from here?

Fluids with Memory – Chapter 8

Limitations of the GLVE Models

$$\underline{\underline{\tau}}(t) = -\int_{-\infty}^{t} G(t-t')\underline{\underline{\dot{\gamma}}}(t')dt'$$

- Predicts constant shear viscosity
- Only valid in regime where strain is additive (small-strain, low rates)
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What else can we try?

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Where do we go from here?

Fluids with Memory – Chapter 8

Limitations of the GLVE Models

$$\underline{\underline{\tau}}(t) = -\int_{-\infty}^{t} G(t-t')\dot{\underline{\underline{\gamma}}}(t')dt'$$

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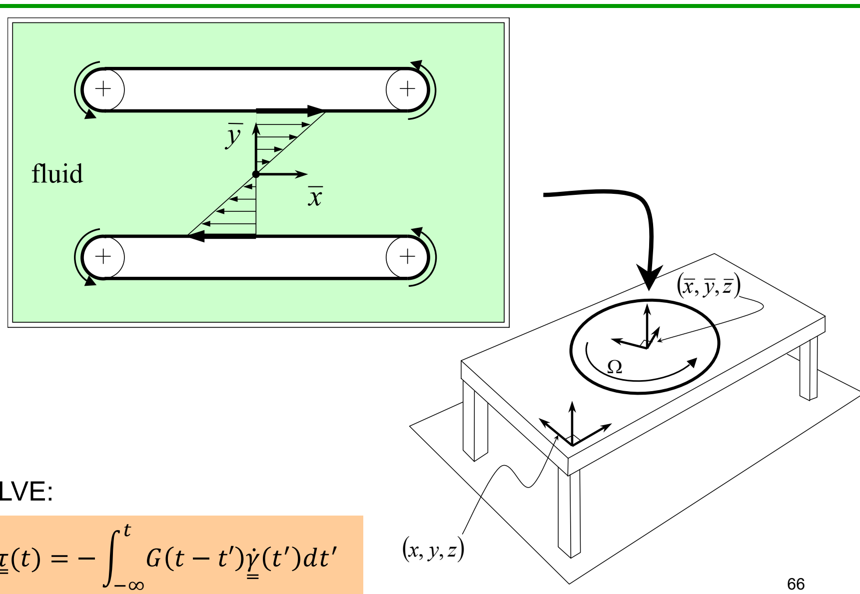
Let's talk about this issue.

What else can we try?

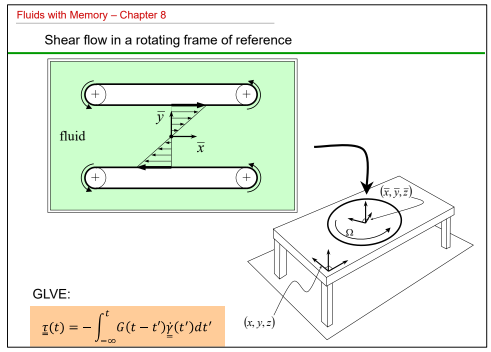
Shear flow in a rotating frame of reference



GLVE:

$$\underline{\underline{\tau}}(t) = -\int_{-\infty}^{t} G(t-t')\dot{\underline{\underline{\gamma}}}(t')dt'$$

The choice of coordinate system should make no difference. Let's check.



x, y, z coordinate system=stationary
$\bar{x}, \bar{y}, \bar{z}$ coordinate system=rotating at Ω

Predict: η

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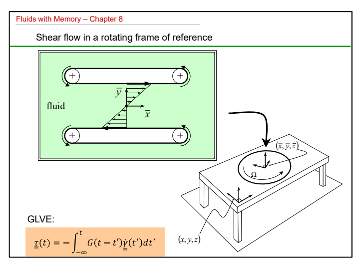
x, y, z coordinate system = stationary
$\bar{x}, \bar{y}, \bar{z}$ coordinate system = rotating at Ω



We write the fluid velocity two ways:

$$\underline{v}_{\substack{wrt\\stationary\\frame}} = \underline{v}_{\substack{wrt\\rotating\\frame}} + \underline{v}_{frame}$$

$$= \dot{\gamma}_0 \bar{y} \hat{e}_{\bar{x}} + \underline{v}_{frame}$$

GLVE:

$$\underline{\underline{\tau}}(t) = -\int_{-\infty}^{t} G(t-t')\dot{\underline{\underline{\gamma}}}(t')dt'$$

Both ways we should get the same answer for $\underline{\underline{\tau}}$

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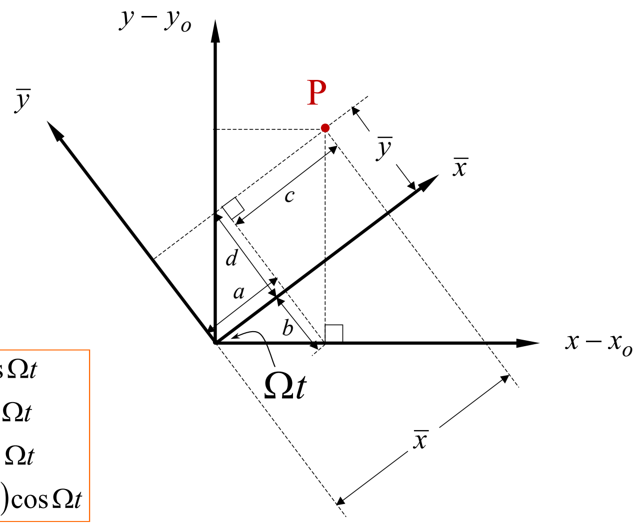
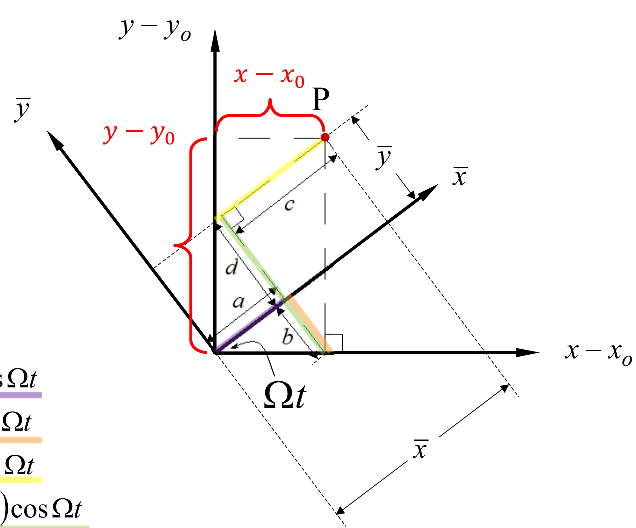
Shear flow in a rotating frame of reference



$$a = (x - x_o)\cos\Omega t$$
$$b = (x - x_o)\sin\Omega t$$
$$c = (y - y_o)\sin\Omega t$$
$$d + b = (y - y_o)\cos\Omega t$$

Shear flow in a rotating frame of reference

Location of P:
$x - x_0, y - y_0$

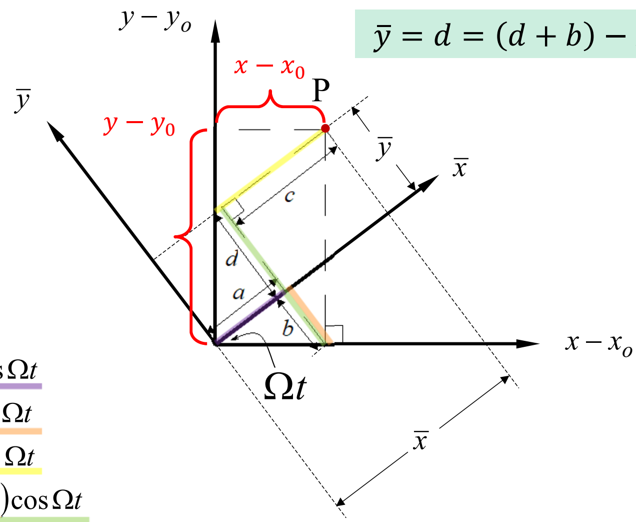


$$a = (x - x_o)\cos\Omega t$$
$$b = (x - x_o)\sin\Omega t$$
$$c = (y - y_o)\sin\Omega t$$
$$d + b = (y - y_o)\cos\Omega t$$

Shear flow in a rotating frame of reference

Location of P:
$x - x_0, y - y_0$

$\bar{y} = d = (d + b) - b$



$a = (x - x_o)\cos\Omega t$

$b = (x - x_o)\sin\Omega t$

$c = (y - y_o)\sin\Omega t$

$d + b = (y - y_o)\cos\Omega t$

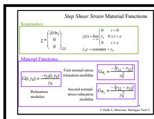
Summary: *Generalized Linear-Viscoelastic (GLVE) Constitutive Equations*

PRO:
- A first constitutive equation with memory
- Can match SAOS, step-strain data very well
- Captures start-up/cessation effects
- Simple to calculate with
- Can be used to calculate the LVE spectrum

CON:
- Fails to predict shear normal stresses
- Fails to predict shear-thinning/thickening
- Only valid at small strains, small rates
- **Not frame-invariant**

GLVE:

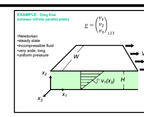
$$\underline{\underline{\tau}}(t) = -\int_{-\infty}^{t} G(t-t')\dot{\underline{\underline{\gamma}}}(t')dt'$$



Predict a material function for a constitutive equation

What kind of rheology calculation am I considering?

Calculate a flow field or stress field for a fluid in a situation



Find the recipe card (has kinematics, definitions of material functions; $\underline{v}(t)$ is known)

↓

For the given kinematics, calc. needed stresses from the constitutive equation

↓

From stresses, calculate the material function(s) from the definition on the recipe card

constitutive equation

$$\underline{\underline{\tau}}(t) = f(\underline{v})$$

Sketch the situation and choose an appropriate coordinate system

↓

Model the flow by asking yourself questions about the velocity field (are some components and derivatives zero? The constitutive equation must be known)

↓

Solve the differential equations (Solve EOM, continuity, and constitutive equation for $\underline{v}$ and p and apply the boundary conditions)

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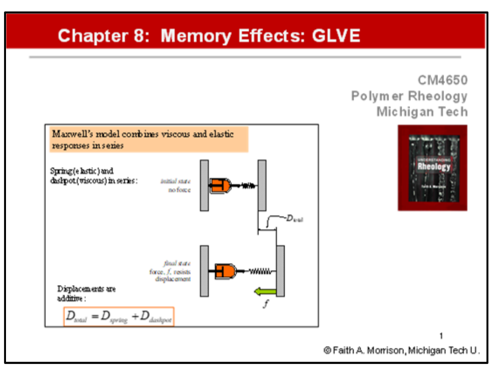
$$\underline{\underline{\tau}}(t) = -\int_{-\infty}^{t} G(t-t')\dot{\underline{\underline{\gamma}}}(t')dt'$$

Done with GLVE.



Let's move on to Advanced Constitutive Equations

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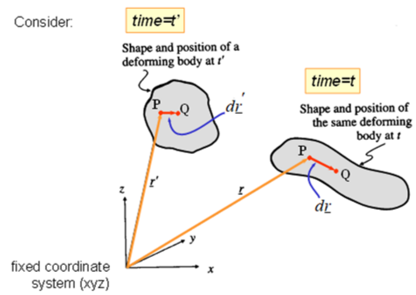
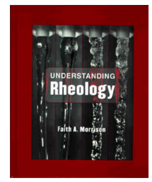
Chapter 9: Advanced Constitutive Models

Michigan Tech
CM4650
Polymer Rheology

We desire a strain tensor that accurately captures large-strain deformation without being affected by rigid-body rotation.





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Michigan Technological University

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