PH3111 Problems - Hamiltonian

1. Prob. 10.28

2. Prob. 10.29

3. Consider two (continuous) functions of generalized coordinates \( q_k \) and corresponding momenta, \( p_k \), for example \( g(q_k, p_k) \) and \( h(q_k, p_k) \). The “Poisson Bracket” is defined by

\[
[g, h] = \sum_k \left( \frac{\partial g}{\partial q_k} \frac{\partial h}{\partial p_k} - \frac{\partial g}{\partial p_k} \frac{\partial h}{\partial q_k} \right)
\]

Show that:

(a) \( \frac{dg}{dt} = [g, H] + \frac{\partial g}{\partial t} \)

(b) \( [p_i, p_j] = [q_i, q_j] = 0 \)

(c) \( [q_i, p_j] = \delta_{ij} \)

where \( H \) is the Hamiltonian and \( \delta_{ij} = 1 \) if \( i = j \) and is zero otherwise. (Hint: if you are spending a lot of time on these, you are doing them wrong!)

Notes:

i. If \( [g, h] = 0 \), then \( [g, h] = [h, g] \) and we say that “\( g \) and \( h \) commute” (for this operation).

ii. If \( [g, h] = 1 \), then \( g \) and \( h \) are said to be “canonically conjugate.”

iii. (a) implies that if the quantity represented by \( g \) is not an explicit function of time, then it will be a constant of the motion as long as \( [g, H] = 0 \), that is, if \( g \) and \( H \) commute. (Notice that \( H \) may be a function of time!)

iv. (a) can be used with \( g = p_k \) and with \( g = q_k \), along with the fact that our coordinates are not explicit functions of time, to show that

\[
\dot{q}_j = [q_j, H], \quad \dot{p}_j = [p_j, H]
\]