AN ELASTIC MICROPOLAR MIXTURE THEORY APPROACH FOR PREDICTING ELASTIC PROPERTIES OF OPEN CELL FOAMS

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ABSTRACT
A constitutive model has been developed to predict the elastic response of two dimensional balsa wood material with a distribution of cell geometries. Two planar triangular grids, each assumed to represent the structural network of an open cell foam material, are superimposed to model an overall cellular structure with a distribution of cell geometries. The elastic mixture theory is applied in conjunction with the micropolar elasticity theory to homogenize the cellular structure and to establish the overall constitutive relationship.

Keywords: Cellular materials, Elasticity

NOMENCLATURE

- \( a^n_{kl} \): Acceleration vector component of \( n^{th} \) constituent
- \( a^n_{kl} \): Material derivative of \( v^n_{kl} \)
- \( A_{strut}^{(n)} \): Cross-sectional area of the struts of the \( n^{th} \) constituent
- \( A_{kl}^{(n)} , A_{kl}^{(n) ,} \): Material moduli of the \( n^{th} \) constituent
- \( b^n_{kl} \): Material derivative of \( \gamma^n_{kl} \)
- \( B_{kl}^{(n)} , B_{kl}^{(n) ,} \): Material moduli of the \( n^{th} \) constituent
- \( C \): Material constant
- \( C_0 \): Constant relating temperature and free energy in the natural state
- \( C_{(n) ,} \): Material moduli of the \( n^{th} \) constituent
- \( D \): Set of all dependent variables
- \( \epsilon^n \): Energy transferred to \( n^{th} \) constituent from the other constituent
- \( E \): Young’s modulus of the mixture
- \( E^n \): Young’s modulus of open cell foam
- \( E_{strut}^{(n)} \): Young’s modulus of strut material
- \( E_{continuum}^{(n)} \): Young’s modulus of the equivalent continuum of the \( n^{th} \) constituent
- \( f^n \): Volume fraction of the \( n^{th} \) constituent
- \( f_{kl}^{(n)} \): Body force vector components of \( n^{th} \) constituent
- \( F_{kl}^{(n)} \): Deformation gradient tensor components of \( n^{th} \) constituent
Open cell foam materials serve important roles as structural materials in sandwich panels which can have high specific bending stiffnesses and strengths and are also important as thermal insulators. This justifies the need for methods to efficiently predict their mechanical behavior.

Open cell foams consist of a complex interconnected framework of material struts that yields a porous material containing open cells as shown in Fig. 1. This microstructural complexity poses significant problems for accurately modeling their mechanical behavior. Two modeling approaches have been used to predict the mechanical response of open cell foam materials: Finite element analysis (FEA) and analytical techniques. Finite Element Analysis (FEA) has been used (Roberts and Garboczi, 2002; Li et al., 2006; Sue and Whitcomb, 2006) to predict the mechanical behavior of open celled foams with randomly-shaped cells.
Li et al. (Li et al., 2006) evaluated the response of an open cell graphite foam with variations in cell size and strut cross sectional area. The Voronoi tessellation technique was used to generate the foam structure and introduce variations in cell size. Roberts and Garboczi (Roberts and Garboczi, 2002) modeled the mechanical response of open cell foams created using a range of structure-building techniques. Sue and Whitcomb (Sue and Whitcomb, 2006) used FEA to simulate a closed-cell foam structure as a regular array of tetrakaidecahedra. While FEA has demonstrated the ability to accurately predict mechanical behavior, the model building and solution procedures can be time-consuming and expensive.

Analytical approaches have been developed that are much more efficient than FEA approaches. For example, Gibson and Ashby (Gibson and Ashby, 1999) proposed the simple relationship

\[
\left( \frac{E'}{E^{\text{strut}}} \right) = C \left( \frac{\rho^*}{\rho^{\text{strut}}} \right)^r
\]

where \(E'\) is the Young's modulus of the foam, \(E^{\text{strut}}\) is the Young's modulus of the solid material composing the struts of the foam, \(\rho^*\) is density of the foam, and \(\rho^{\text{strut}}\) is the density of the solid material of the foam. This model has been very effective in predicting open cell foam properties, however, it requires experimental determination of \(C\) and \(r\) which are functions of microstructure. Therefore, there is a need to establish an efficient and accurate modeling approach to predict the mechanical response of open cell foam materials as a function of the material microstructure.

The objective of this study is to establish a linear-elastic constitutive model for a two dimensional open cell foam material that accounts for distributions of cell size and strut geometry in these materials. Two 2-dimensional triangular grid geometries are individually homogenized using the micropolar elasticity theory and combined into a single homogenized continuum using micropolar mixture theory. The two triangular grid geometries that are selected have different cell sizes and strut geometries and are selected based on the distribution of these parameters in the actual material. The resulting combined constitutive response of the grid mixture results in predicted properties that represent those of the open cell foam. This modeling approach is shown in Fig. 2.

![Triangular grid](image1)

**FIGURE 1 – OPEN CELL POLYURETHANE**

![Triangular grid](image2)

**FIGURE 2 – MODELING APPROACH FOR OPEN CELL FOAM MATERIALS**

First, the kinematics is discussed followed by the general aspects of the micropolar elasticity theory and elastic mixture theory. This is followed by a discussion of the balance laws for a micropolar mixture of 2 solid constituents. Then a brief development of the constitutive theory for a mixture of two micropolar solids is presented. In section 6, the application of the micropolar elasticity theory and the micropolar mixture theory to homogenize the superimposed grid is presented and the expression for the elastic constants of a mixture of two triangular grid geometries is then obtained.

### 2. KINEMATICS

Consider a binary mixture of micropolar elastic solids. A material point of the \(n^{th}\) constituent in the reference configuration is denoted by \(X^n_k\), in Cartesian coordinates.

After a time \(t\), due to a motion denoted by the mapping \(X^n_k\), its position in the spatial configuration is given by the motion equation

\[
x^n_k = X^n_k \cdot t
\]

The corresponding displacement, velocity, and acceleration at time \(t\) are given by, respectively,

\[
u^n = \dot{X}^n - x^n
\]

\[
v^n = \ddot{x}^n = \frac{\partial X^n}{\partial t} \cdot t
\]
\[ \mathbf{a}^n = \mathbf{x}^n = \frac{\partial^2 \mathbf{X}^n}{\partial t^2} \]

where the superposed dot denotes a material time derivative. The deformation gradient tensor of the \( n \)th constituent \( \mathbf{F}^{(n)} \) is

\[ F_{nk}^n = \frac{\partial x_k^n}{\partial X_k^n} \]

The linear strain tensor components and the linear microrotation gradient tensor components of the \( n \)th constituent are given by, respectively,

\[ \varepsilon_{ik}^n = u_{i,k}^n + \varepsilon_{ilm}^n \phi_{lm}^n \]

\[ \gamma_{ik}^n = \phi_{ik}^n \]

where \( u_{i,k}^n \) and \( \phi_{ik}^n \) are the displacement and rotation vector components of a point of the \( n \)th constituent and \( \varepsilon_{ilm}^n \) is the usual permutation symbol. The material derivatives of (6) and (7) are denoted by

\[ \dot{\varepsilon}_{ik}^n = a_{ik}^n \]

\[ \dot{\gamma}_{ik}^n = b_{ik}^n \]

3. BACKGROUND

3.1. MICROPOLAR ELASTICITY

In the theory of micropolar elasticity (Eringen, 1999), the points in a material continuum have six degrees of freedom: three displacement components and three rotations (called microrotations) about the coordinate axes, as shown in Fig. 3. Although the theory of micropolar elasticity has been applied to many fields of engineering, one of its primary uses is to provide a mathematical foundation to describe the mechanical behavior of lattice structures. The analogy between a micropolar continuum and a grid framework has been formulated and substantiated with numerical results (Bazant and Christensen, 1972). The micropolar theory is necessary in the analysis of lattice structures because rotations at the strut joints play a significant role in influencing the overall behavior of the lattice. For example, a micropolar beam model was developed (Noor and Nemeth, 1980b) for four different types of planar lattice grids with rigid joints. In a subsequent study (Noor and Nemeth, 1980a), the model was extended to model three-dimensional lattice structures. An energy approach considering in-plane displacements as well as joint rotations (which leads to micropolarity) was used (Sun and Yang, 1973) to estimate the dynamic characteristic of a square grid.

\[ \lambda \varepsilon_{ij}^{(n)} \delta_{ij} + \mu + \kappa \varepsilon_{ij}^{(n)} + \mu \varepsilon_{kk}^{(n)} \]

\[ n_{ij}^{(n)} = \alpha \phi_{ij}^{(n)} + \beta \phi_{ii}^{(n)} + \gamma \phi_{ij}^{(n)} \]

where \( \lambda \), \( \mu \), \( \kappa \), \( \alpha \), \( \beta \), and \( \gamma \) are the six independent constants necessary to describe the isotropic mechanical behavior. Additional quantities encountered in micropolar elasticity include the microrotation rate \( \gamma_i^{(n)} \), which is the rate of microrotation; the microinertia density \( j^{(n)} \), which is resistance offered to microrotation; and the body couple density \( m_i^{(n)} \), which are similar to rotational body forces. The microinertia density may be different with respect to different axes but in the case where they are the same, the condition is described as microisotropic.

3.2. MIXTURE THEORY

Motivated by Maxwell's kinetic theory of gases, Truesdell and Toupin (Truesdell and Toupin, 1960) presented an axiomatic mixture theory for interacting continua based on the premise that each point of space is simultaneously occupied by all constituents of the mixture. It has been used with profound success to model the mechanical behavior of composite materials (Bedford and Stern, 1972; Stern and Bedford, 1972; Hegemier et al., 1973; Nayfeh and Gurtman, 1974a; McNiven...
and Mengi, 1979a; McNiven and Mengi, 1979b; McNiven and Mengi, 1979c). In this theory it is also assumed that the interactions between the constituents of a mixture are accounted for as interaction forces in the appropriate field equations. The mixture theory concept has been used to model the three-dimensional mechanical behavior of a binary mixture (Bedford and Stern, 1972). In a subsequent study (Stern and Bedford, 1972), the three-dimensional model predicted wave propagation phenomena in a composite material. It was concluded that in order to accurately predict wave propagation properties in a composite material, the material microstructure must be considered, which can be done with considerable ease by using the elastic mixture theory. An alternative version of the mixture theory was developed (Hegemier et al., 1973) for modeling the wave propagation in laminated and unidirectional fibrous composites. The theory was used to determine the distribution of displacements and stresses within individual constituents. This theory was subsequently applied (Nayfeh and Gurtman, 1974b) for both transversely and horizontally polarized shear waves propagating in the plane of the laminate. A mixture theory was developed (McNiven and Mengi, 1979a) for two-phase materials which was implemented (McNiven and Mengi, 1979b; McNiven and Mengi, 1979c) to study the wave propagation in periodically-structured composites.

4. BALANCE LAWS

It is assumed that there are no chemical reactions or mass diffusion processes. It is additionally assumed that interactions between the constituents are purely mechanical. Also, both constituents are considered to be microisotropic, i.e. their moments of inertia are constant and isotropic. The balance laws for the mixture are obtained by summing over the constituents of the mixture. The balance laws are similar to those proposed by Eringen (Eringen, 2003).

4.1 CONSERVATION OF MASS

The balance of mass is

\[ \frac{\partial \rho^n}{\partial t} + \nabla \cdot \rho^n \mathbf{v}^n = 0 \]

where \( \rho^n \) is the mass density of the \( n \)th constituent. Summing over \( n \), the law of conservation of the mass of the mixture is obtained and given by,

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0 \]

where \( \rho = \rho^1 + \rho^2 \) is the density of the mixture and \( \mathbf{v} \) is the velocity of the mixture given by

\[ \mathbf{v} = \frac{1}{\rho} \sum \rho^n \mathbf{v}^n \]

4.2 BALANCE OF LINEAR MOMENTUM

The linear momentum balance is given by

\[ \rho^n \mathbf{x}^n = \nabla \cdot \mathbf{t}^n + \rho^n \mathbf{f}^n + \mathbf{p}^n \]

where \( \mathbf{f}^n \) is the body force vector and \( \mathbf{p}^n \) is the interaction force density vector which represents force exerted on the \( n \)th constituent due to the interaction with the other constituent (terms that represent constituent interactions have the superposed \( \wedge \) symbol). Summing over \( n \) constituents and arguing that the momentum of the mixture is balanced solely by the constituent stresses and body forces reveals

\[ \sum_{n=1}^{2} \mathbf{p}^n = 0 \]

4.3 BALANCE OF ANGULAR MOMENTUM

The balance of angular momentum is given by

\[ \rho^n j^n \mathbf{\omega}^n = m_{t,\wedge}^n + \varepsilon_{\omega \omega} t_{mn}^n + \mathbf{p}^n \omega^n + \hat{m}^n \]

where \( \hat{m}^n \) is the couple exerted on the \( n \)th constituent due to interaction with the other constituent. As with the linear momentum conservation law, summing over both constituents yields

\[ \sum_{n=1}^{2} \hat{m}^n = 0 \]

4.4 CONSERVATION OF ENERGY

The conservation of energy is given by

\[ \rho^n \dot{U}^n = t_{\omega \omega} a_{\omega}^n + m_{t,\wedge}^n b_{\omega}^n + q_{\omega,\wedge}^n + \mathbf{p}^n h^n + \hat{e}^n \]

where \( \dot{U}^n \) is the specific internal energy of the \( n \)th constituent, \( q_{\omega}^n \) is the heat flux vector, \( h^n \) is the internal heat source, and \( \hat{e}^n \) denotes the transfer of energy to the \( n \)th constituent from the other constituent. The Clausius-Duhem inequality is given by

\[ \Psi + \eta \dot{\theta} + \sum_{n} t_{\omega \omega} a_{\omega}^n + m_{t,\wedge}^n b_{\omega}^n - \hat{p}^n \mathbf{v}^{12} - \hat{m}^n \mathbf{\omega}^{12} + \frac{q_{\omega,\wedge}^n \theta}{\theta} \geq 0 \]

where \( \Psi \) is the free energy density, \( \eta \) is the entropy density, \( \theta \) is the absolute temperature (both the constituents and hence the
mixture are assumed to be at a constant temperature $T$, and $v_i^{(12)}$ and $v_i^{(12)}$ are the relative velocity and relative rotation rate vector components, respectively, between the constituents and are given by

$$v_{i}^{12} = v_i^1 - v_i^2 = \ddot{u}_i^1 - \ddot{u}_i^2$$
$$v_{i}^{12} = v_i^1 - v_i^2 = \dot{\phi}_i^1 - \dot{\phi}_i^2$$

(20)

5. CONSTITUTIVE MODELING

The constitutive independent and dependant variables are the sets $I$ and $D$, respectively, given by

$$I = \theta, v_i^1, v_i^2, \gamma_i^1, \gamma_i^2, \psi_i^1, \psi_i^2$$

$$D = \eta, t_i^1, t_i^2, m_i^1, m_i^2, q_m, \hat{\rho}_m, \hat{m}_m$$

(21)

All the independent variables are frame-independent except $v_i^{12}$ and $v_i^{12}$. The admissibility for using these quantities has been established (Eringen, 2003). The free energy is a function of the static independent variables set

$$\Psi = \Psi (0, v_i^1, v_i^2, \gamma_i^1, \gamma_i^2)$$

(22)

where the superposed $\sim$ denotes a response function. Using the chain rule, the material derivative of the free energy is given by

$$\dot{\Psi} = \frac{\partial \Psi}{\partial \theta} \dot{\theta} + \frac{\partial \Psi}{\partial v_i^1} \dot{v_i}^1 + \frac{\partial \Psi}{\partial v_i^2} \dot{v_i}^2 + \frac{\partial \Psi}{\partial \gamma_i^1} \dot{\gamma_i}^1 + \frac{\partial \Psi}{\partial \gamma_i^2} \dot{\gamma_i}^2$$

(23)

Substituting (23) in (19) yields

$$- \left( \frac{\partial \dot{\Psi}}{\partial \theta} + \eta \right) \dot{t_i}^1 - \frac{\partial \dot{\Psi}}{\partial v_i^1} a_i^1 + \left( t_i^2 - \frac{\partial \dot{\Psi}}{\partial v_i^2} \right) a_i^2 + \left( m_i^1 - \frac{\partial \dot{\Psi}}{\partial \gamma_i^1} \right) b_i^1 + \left( m_i^2 - \frac{\partial \dot{\Psi}}{\partial \gamma_i^2} \right) b_i^2$$

(24)

$$- \hat{\rho}_m v_i^{12} - \hat{m}_m v_i^{12} + \frac{1}{\theta} q_m \theta_m \geq 0$$

If (24) is to be satisfied for arbitrary values of $\dot{\theta}$, $a_i^+$, and $b_i^+$,

$$\eta = - \frac{\partial \dot{\Psi}}{\partial \theta}$$
$$t_i^1 = \frac{\partial \dot{\Psi}}{\partial v_i^1}$$
$$t_i^2 = \frac{\partial \dot{\Psi}}{\partial v_i^2}$$
$$m_i^1 = \frac{\partial \dot{\Psi}}{\partial \gamma_i^1}$$
$$m_i^2 = \frac{\partial \dot{\Psi}}{\partial \gamma_i^2}$$

(25)

Substituting (25) back into (24)

$$- \hat{\rho}_m v_i^{12} - m_i^2 v_i^{12} + \frac{1}{\theta} q_m \theta_m \geq 0$$

(26)

The thermodynamic forces $Y$ and fluxes $J$ (Eringen, 1999) are given by

$$Y = \left( v_i^{12}, v_i^{12}, \frac{\theta}{\theta} \right)$$

$$J = - \hat{\rho}_m, - \hat{m}_m$$

(27)

The associated constitutive equations are

$$\hat{\rho}_m = - \frac{\partial \Phi}{\partial \theta}$$
$$\hat{m}_m = - \frac{\partial \Phi}{\partial \theta}$$

(28)

where $\Phi$ is the dissipation potential (Eringen, 1999).

The free energy function is given by

$$\Psi = \Psi^0 + \Psi^2$$

$$\Psi^1 = S_0 - \frac{1}{2} \eta \dot{v}_i^1 T - \frac{C_g}{4 T_0} T^2 - A_i^1 T \dot{v}_i^1 - B_i^1 T \gamma_i^1$$

$$+ \frac{1}{2} A_{klm}^1 e_i^{1^2} e_m^1 + B_{klm}^1 \dot{\gamma}_i^1 \gamma_i^1 + 2 C_{klm} \gamma_i^1 \gamma_i^1$$

$$\Psi^2 = S_0 - \frac{1}{2} \eta \dot{v}_i^2 T - \frac{C_g}{4 T_0} T^2 - A_i^2 T \dot{v}_i^2 - B_i^2 T \gamma_i^2$$

$$+ \frac{1}{2} A_{klm}^2 e_i^{2^2} e_m^2 + B_{klm}^2 \dot{\gamma}_i^2 \gamma_i^2 + 2 C_{klm} \gamma_i^2 \gamma_i^2$$

(29)

where $\Psi^{(n)}$ is the free energy density of constituent $n$. The subscript 0 denotes constants or variables in their natural state, that is a state that is free of stress and couple stress. $S_0$ is the free energy in the natural state, $C_0$ is a constant relating temperature and free energy in the natural state, $T_0$ is the ambient temperature, $T$ is the change in ambient temperature resulting in current temperature $\theta = T_0 + T$, where $T = T_0, T_0 > 0$, and $A_{klm}^{(n)}$, $B_{klm}^{(n)}$, $A_{klm}^{(n)}$, $B_{klm}^{(n)}$, $C_{klm}^{(n)}$ are material moduli.

The dissipation potential is given by

$$2 \Phi = \xi v_i^{12} v_i^{12} + 2 \zeta v_i^{12} \frac{\theta}{\theta} + K \theta \frac{\theta}{\theta} + m v_i^{12} v_i^{12}$$

(30)
where $\zeta$ is the momentum generation coefficient due to velocity difference, $\zeta$ is the heat generation for unit change in velocity, $K$ is the classical Fourier constant, and $\tau$ is the momentum generation due to difference in gyrations. The linear constitutive equations are obtained by substituting (29) and (30) into (25) and (28).

$$\eta = \eta_0 + \frac{C_0}{T_0} + A_{ij} \dot{e}_{ij} + B_{ij} \ddot{e}_{ij} + A_{ij} \dot{v}_{ij} + B_{ij} \ddot{v}_{ij}$$

$$t_{ij}^1 = -A_{ij} T + A_{klmn} \dot{e}_{mn} + C_{klmn} \dot{\gamma}_{mn}$$

$$t_{ij}^2 = -A_{ij} T + A_{klmn} \ddot{e}_{mn} + C_{klmn} \ddot{\gamma}_{mn}$$

$$m_{ij}^1 = -B_{ij} T + B_{klmn} \dot{e}_{mn} + C_{klmn} \dot{\gamma}_{mn}$$

$$m_{ij}^2 = -B_{ij} T + B_{klmn} \ddot{e}_{mn} + C_{klmn} \ddot{\gamma}_{mn}$$

$$\dot{p}_k = -\xi v_k \ddot{\gamma}_{kk} - \zeta \ddot{T}_k$$

$$\dot{m}_k = -\gamma v_k$$

$$q_k = \zeta v_k + K \ddot{T}_k$$

For isotropic constituents (Eringen, 1999)

$$B_{ij}^1 = B_{ij}^2 = C_{klmn} = C_{2klmn} = 0$$

$$A_{ij}^1 = B_{ij} \dot{\delta}_{ij}$$

$$A_{ij}^2 = B_{ij} \ddot{\delta}_{ij}$$

$$A_{klmn}^1 = \lambda \dot{\delta}_{kl} \dot{\delta}_{mn} + \mu \dot{\delta}_{ms} \dot{\delta}_{lm} + \mu \dot{\delta}_{lm} \dot{\delta}_{ms}$$

$$A_{klmn}^2 = \lambda \ddot{\delta}_{kl} \ddot{\delta}_{mn} + \mu \ddot{\delta}_{ms} \ddot{\delta}_{lm} + \mu \ddot{\delta}_{lm} \ddot{\delta}_{ms}$$

$$B_{klmn}^1 = \alpha \dot{\delta}_{kl} \dot{\delta}_{mn} + \beta \dot{\delta}_{ms} \dot{\delta}_{lm} + \gamma \dot{\delta}_{lm} \dot{\delta}_{ms}$$

$$B_{klmn}^2 = \alpha \ddot{\delta}_{kl} \ddot{\delta}_{mn} + \beta \ddot{\delta}_{ms} \ddot{\delta}_{lm} + \gamma \ddot{\delta}_{lm} \ddot{\delta}_{ms}$$

(31)

where $\alpha$, $\beta$, $\gamma$, $\lambda$, $\mu$, $\kappa$ are the six micropolar elastic constants and $\beta_0$ is a thermal expansion constant for determining micropolar isotropic behavior. Substitution of (32) into (31) yields

$$\eta = \eta_0 + \frac{C_0}{T_0} + \beta_{ij} \dot{v}_{ij} + \dot{\beta}_{ij} \ddot{v}_{ij}$$

$$t_{ij}^1 = -\beta_{ij} T + \lambda \dot{e}_{ij} + \mu \dot{v}_{ij} + \dot{\mu} \ddot{v}_{ij}$$

$$t_{ij}^2 = -\beta_{ij} T + \lambda \ddot{e}_{ij} + \mu \ddot{v}_{ij} + \dot{\mu} \ddot{v}_{ij}$$

$$m_{ij}^1 = \alpha \dot{e}_{ij} + \beta \dot{v}_{ij} + \gamma \dot{\gamma}_{ij}$$

$$m_{ij}^2 = \alpha \ddot{e}_{ij} + \beta \ddot{v}_{ij} + \gamma \ddot{\gamma}_{ij}$$

(33)

$$\dot{p}_k = \zeta v_k \frac{\ddot{T}_k}{\beta_0}$$

$$\dot{m}_k = -\beta_0$$

$$q_k = \zeta v_k + K \frac{\ddot{T}_k}{\beta_0}$$

6. THEORETICAL DEVELOPMENT

6.1 HOMOGENIZATION OF A TRIANGULAR GRID

The equivalent micropolar continuum for a triangular grid composed of Timoshenko beams has been previously established (Ostoja-Starzewski, 2002). In this formulation, given the length of the strut, cross-sectional dimensions of the strut, and the material properties of the grid material, the elastic constants of the $n$th equivalent micropolar continuum are given by

$$\lambda^n = \frac{3}{8} R_{strut} - \hat{R}_{strut}$$

$$\mu^n = \frac{3}{8} R_{strut} - \hat{R}_{strut}$$

$$\kappa^n = \frac{3}{2} \hat{R}_{strut}$$

$$\gamma^n = \frac{3}{2} \hat{R}_{strut}$$

$$\alpha^n = \beta^n = 0$$

where

$$R_{strut} = \frac{E_{strut} A_{strut}^n}{l_{strut}}$$

$$\hat{R}_{strut} = \frac{24 E_{strut} I_{strut}^n}{l_{strut}^3 \sqrt{3} (1 + \delta_{strut}^n)}$$

$$S_{strut}^n = \frac{2 E_{strut} I_{strut}^n}{l_{strut}^3 \sqrt{3}}$$

(34)

$$\delta_{strut}^n = \left( \frac{G_{strut}^n}{\mu_{strut}} \right)^2$$
where \( E_{\text{strut}} \) is the Young’s modulus of the strut material, \( A_{\text{strut}} \) is the cross-sectional area of the strut, \( l_{\text{strut}} \) is the length of the strut, \( w_{\text{strut}} \) is width of the cross section, \( G_{\text{strut}} \) is the shear modulus of the strut material, and \( I_{\text{strut}} \) is the moment of inertia of the cross section of the strut. The \( \delta_{\text{strut}} \) term depicts the ratio of the bending stiffness to shearing stiffness of the beam. The inclusion of the term \( \delta_{\text{strut}} \) helps account for excessively large bending rigidities that result when the strut length become small compared to the strut width. The homogenized triangular grid results in an equivalent micropolar continuum, as shown in Fig. 2.

6.2 HOMOGENIZATION OF THE MIXTURE

Once the equivalent micropolar properties are determined for each of the two triangular grids, the mixture theory is used to combine the micropolar constituents into one homogeneous micropolar continuum, as shown in Fig. 2.

The stresses and couple stresses in the individual continua are as denoted by (31) and when the stresses in the individual micropolar continua \( t^{(1)}_{ij} \) and \( t^{(2)}_{ij} \) are known, the stress in the micropolar mixture continuum is given by

\[
t_{ij} = f^{(1)} t^{(1)}_{ij} + f^{(2)} t^{(2)}_{ij}
\]

(36)

where \( f^{(1)} \) and \( f^{(2)} \) are the volume fractions of constituents 1 and 2, respectively, in the mixture. A similar equation holds for the couple stress.

\[
m_{ij} = f^{(1)} m^{(1)}_{ij} + f^{(2)} m^{(2)}_{ij}
\]

(37)

For the binary mixture considered in this study \( f^{(1)} + f^{(2)} = 1 \). The stresses \( t_{ij} \) and \( m_{ij} \) represent the constitutive response of the micropolar mixture continuum, the equivalent continuum of the superimposed grid. For simplicity, it is assumed that \( u^{(1)}_i = u^{(2)}_i \) and \( \phi^{(1)}_i = \phi^{(2)}_i \) during the deformation in the mixture.

6.3 SUPERIMPOSED GRID SUBJECTED TO LOADING

Consider the plane-stress uniaxial deformation of the superimposed grid where the applied deformation is parallel to the plane of the superposed lattice structures, as shown in Fig. 4. The deformation is carried out in an isothermal manner. Under this deformation condition, the in-plane Young’s modulus of a micropolar continuum that represents a single grid is (Nowacki, 1974; Gauthier and Jahsman, 1975; Eringen, 1999)

\[
E_{\text{continuum}}^{(s)} = \frac{t^{(s)}_{11}}{\varepsilon^{(s)}_{11}} = \frac{2\mu^{(s)} + \kappa^{(s)} + 3\lambda^{(s)}}{2\lambda^{(s)} + 2\mu^{(s)} + \kappa^{(s)}}
\]

(38)

Substitution of (36) into (39) results in

\[
E = f^{(1)} E_{\text{continuum}}^{(1)} + f^{(2)} E_{\text{continuum}}^{(2)}
\]

(40)

where \( E_{\text{continuum}}^{(s)} \) is the Young’s modulus of the equivalent continuum of the \( n^{th} \) grid. The Young’s modulus of the mixture is

\[
E = \frac{f^{(1)} t^{(1)}_{11} + f^{(2)} t^{(2)}_{11}}{\varepsilon^{(1)}_{11}}
\]

(39)

where \( \varepsilon^{(1)}_{11} = \varepsilon^{(2)}_{11} \) because it is assumed that \( u^{(1)} = u^{(2)} \) and \( \phi^{(1)} = \phi^{(2)} \) (discussed above). Substituting (38) into (40) reveals the Young’s modulus of the mixture in terms of the micropolar moduli of the individual constituents

\[
E = f^{(1)} \left[ \frac{2\mu^{(1)} + \kappa^{(1)} + 3\lambda^{(1)}}{2\lambda^{(1)} + 2\mu^{(1)} + \kappa^{(1)}} \right] \]

\[
+ f^{(2)} \left[ \frac{2\mu^{(2)} + \kappa^{(2)} + 3\lambda^{(2)}}{2\lambda^{(2)} + 2\mu^{(2)} + \kappa^{(2)}} \right]
\]

(41)

The Poisson’s ratio of the equivalent micropolar continuum of the \( n^{th} \) grid is (Gauthier and Jahsman, 1975)

\[
\nu^{(n)}_{\text{continuum}} = -\frac{E_{\text{continuum}}^{(n)}}{E_{\text{continuum}}^{(n)}} = -\frac{\lambda^{(n)}}{2\lambda^{(n)} + 2\mu^{(n)} + \kappa^{(n)}}
\]

(42)
The transverse strains \( e_{22} \) of the individual grids subjected to individual loadings are given by

\[
e_{22}^1 = -e_{11}^1 \nu_{\text{continuum}} = -e_{11}^1 \frac{\lambda^1}{2\lambda^1 + 2\mu^1 + \kappa^1}
\]

\[
e_{22}^2 = -e_{11}^2 \nu_{\text{continuum}} = -e_{11}^2 \frac{\lambda^2}{2\lambda^2 + 2\mu^2 + \kappa^2}
\]

(43)

It is assumed that the transverse strain of the mixture is related to the transverse strains of the individual grids by

\[
e_{22} = f^1 e_{22}^1 + f^2 e_{22}^2
\]

(44)

The mixture Poisson’s ratio in terms of the individual grid Poisson’s ratios is

\[
\nu = \frac{e_{22}}{e_{11}} = \frac{1}{f^1} \frac{\nu_{\text{continuum}}}{e_{11}}
\]

\[
= f^1 \nu_{\text{continuum}}^1 + f^2 \nu_{\text{continuum}}^2
\]

(45)

Substituting (42) into (45) yields

\[
\nu = -f^1 \frac{\lambda^1}{2\lambda^1 + 2\mu^1 + \kappa^1} - f^2 \frac{\lambda^2}{2\lambda^2 + 2\mu^2 + \kappa^2}
\]

(46)

Consider the plane-stress simple shear deformation of the homogenous equivalent continuum where the applied deformations are parallel to the plane of the superposed lattice structures, as shown in the Fig. 5. The solid is deformed in an isothermal manner.

Under this deformation conditions, the in-plane shear modulus of the mixture is (Gauthier and Jahsman, 1976; Eringen, 1999)

\[
G_{\text{continuum}} = \mu^n + \frac{K^n}{2} = \frac{E_{\text{continuum}}^n}{2(1 + \nu_{\text{continuum}}^n)}
\]

(47)

The mixture shear modulus is given by

\[
G = \frac{E}{2(1 + \nu)} = \frac{f^1 E^1 + f^2 E^2}{2(1 + f^1 \nu^1 + f^2 \nu^2)}
\]

(48)

Therefore, the mixture properties (Young’s modulus, Poisson’s ratio, and shear modulus) have been established in terms of the mechanical properties of the constituent materials.

6.3 APPLICATION OF MODEL TO BALSA WOOD

A cross sectional slice of balsa wood closely approximates a two-dimensional cellular structure with a distribution of cell sizes. Fig. 6, which is an image of an axial cross-section of balsa wood, shows three types of cells. Most of the volume is occupied by nearly hexagonal normal cells, with parallel bands of rectangular ray cells. The larger sap cells occupy a much smaller volume than the normal and ray cells over the entire cross section (Fig. 6 is focused on an area crowded with sap cells), thus their relative volume fraction is insignificant compared to those of the normal and ray cells.

**FIGURE 6 – SEM IMAGE OF A LONGITUDINAL CROSS SECTION OF BALSA WOOD** (Image copyright Dennis Kunkel Microscopy, Inc., used with permission)
A binary mixture model was constructed in which the 1\textsuperscript{st} and 2\textsuperscript{nd} constituents were the equivalent continua of the normal and ray cells, respectively. The structural and mechanical parameters for the two lattices are shown in Table 1. The values of $s^{(n)}$, $E^{(n)}$, and $f^{(n)}$ were previously determined by Easterling \textit{et al.} (Easterling \textit{et al}., 1982). The values of $f^{(n)}$ were determined by equating the average cell areas reported by Easterling \textit{et al.} (Easterling \textit{et al}., 1982) with triangular cell areas for the triangular lattice. The values of $G^{(n)}$ were calculated assuming a cell wall Poisson’s ratio of 0.33.

<table>
<thead>
<tr>
<th>Normal cells</th>
<th>Ray cells</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>1</td>
</tr>
<tr>
<td>$s^{(n)}$</td>
<td>1.5 μm</td>
</tr>
<tr>
<td>$f^{(n)}$</td>
<td>44 μm</td>
</tr>
<tr>
<td>$E^{(n)}$</td>
<td>10 GPa</td>
</tr>
<tr>
<td>$G^{(n)}$</td>
<td>3.8 GPa</td>
</tr>
</tbody>
</table>

Table 1 – Properties of the equivalent lattices of balsa wood

Using Equations (34), (35), (38), (42), (47), (41), (46) and (48), the in-plane Young’s modulus, Poisson’s ratio, and shear modulus of balsa wood were predicted to be 376 MPa, 0.25, and 150 MPa, respectively. Experimental measurements of in-plane Young’s modulus of balsa wood range from 10 – 300 MPa (Easterling \textit{et al}., 1982). Therefore, the predicted Young’s modulus is in reasonable agreement with the empirical value. The model did not take into consideration the presence of sap channels and imperfections in the material, which may explain slight overestimate of the predicted properties. Of course, more constituents can be added to the model to account for these structural effects, if a more accurate prediction is needed.

8. SUMMARY

An analytical modeling approach has been developed to predict the elastic properties of two dimensional open cell materials without the need for complex and inefficient FEA modeling. The modeling approach directly accounts for the distribution of cell geometries that are present in most cellular materials. The approach combines mixture theory and micropolar elasticity theory to predict elastic response of cellular materials to a wide range of loading conditions. The modeling approach was applied to the two-dimensional balsa wood material. Predicted properties were in good agreement with experimentally-determined properties. This agreement demonstrates that the model has the potential to predict the elastic response of other cellular solids, such as open cell and closed cell foams.

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