Remarks.

(A) Closed and bounded centrally symmetric sets S in $\mathbb{E}^n$ can also be characterized by the property that for each n-dimensional simplex T with vertices in S there is a translate of $-T$ also having its vertices in S. This, of course, is a consequence of Theorem 1, since the three-point sets belonging to S are subsets of the $(n+1)$-point sets with vertices in S.

(B) Theorem 1 does not generalize to closed sets without some other condition imposed on them. By way of example, consider the closed half-plane $S = \{(x, y) : y \geq 0\}$ in $\mathbb{E}^2$. Note that for each three-point subset T of S, there exists a translate of $-T$ that is a subset of S, yet S is not centrally symmetric. A closed half-space provides a similar example in $\mathbb{E}^n$ when $n > 2$.

(C) The theorem also fails to generalize to bounded sets without further restrictions. To see this, let

$$S = \left\{(x, y) \in \mathbb{E}^2 : x^2 + y^2 < 1\right\} \cup \left\{(1, 0)\right\}.$$

Then S is not centrally symmetric, yet S satisfies the three-point property. To check this, suppose first $T = \{x_1, x_2, x_3\} \subset S$ with $x_i \neq (1, 0)$ for $i = 1, 2, 3$. Then $-T$ itself is a subset of S. On the other hand if, say, $x_1 = (1, 0)$, with x_2 and x_3 contained in the open disk $\{(x, y) : x^2 + y^2 < 1\}$, then we have $-T = \{-x_1, -x_2, -x_3\}$, where $-x_2$ and $-x_3$ are contained in the open disk and $-x_1 = (-1, 0)$. If we take $p = (\varepsilon, 0)$, with $\varepsilon > 0$ and ε sufficiently small, then $p + (-T)$ is a subset of the open disk and a fortiori a subset of S. An open ball with a boundary point adjoined provides a suitable counterexample in $\mathbb{E}^n$ with $n > 2$.

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L'Hospital Rules for Monotonicity and the Wilker-Anglesio Inequality

Iosif Pinelis

1. L'HOSPITAL RULES FOR MONOTONICITY. Let $-\infty \leq a < b \leq \infty$, let f and g be continuously differentiable functions defined on the interval (a, b), and let $r = f/g$. Suppose one wants to prove the inequality $f > g$. Then it would suffice to show, for example, that $g > 0$, r is increasing, and $r(a+) = \lim_{x \to a^+} r(x) = 1$.

In [**11**], general "rules" for monotonicity patterns, resembling the usual l'Hospital rules for limits, were given. In particular, according to Proposition 1.9 in [**11**], one has the following dependence of the monotonicity pattern of r (on (a, b)) on that of $\rho := f'/g'$ (and also on the sign of gg', assuming that gg' does not vanish anywhere on (a, b)):

Table 1.

ρ	gg'	r
$\nearrow$	> 0	$\nearrow$ or $\searrow$ or $\searrow\nearrow$
$\searrow$	> 0	$\nearrow$ or $\searrow$ or $\nearrow\searrow$
$\nearrow$	< 0	$\nearrow$ or $\searrow$ or $\nearrow\searrow$
$\searrow$	< 0	$\nearrow$ or $\searrow$ or $\searrow\nearrow$

Here, for instance, $r \searrow\nearrow$ means that there is some c in (a, b) such that r is decreasing ($\searrow$) on (a, c) and increasing ($\nearrow$) on (c, b). Now suppose that one also knows whether r is increasing or decreasing in a right neighborhood of a and in a left neighborhood of b. Then Table 1 uniquely determines the monotonicity pattern of r.

Clearly, these l'Hospital-type rules for monotonicity patterns are helpful wherever the l'Hospital rules for limits are. Moreover, the monotonicity rules apply even outside such contexts, because they do not require that both f and g (or either of them) tend to 0 or ∞ at any point. In the special case when both f and g vanish at an endpoint of the interval (a, b), l'Hospital-type rules for monotonicity can be found, in different forms and with different proofs, in [**1**]–[**7**] and [**9**]–[**13**]. In view of what has been said here, it should not be surprising that a very wide variety of applications of these l'Hospital-type rules for monotonicity patterns were given in these papers.

Given Table 1, one can generally infer the monotonicity pattern of r given that of ρ, however complicated the latter might be. This is illustrated in Table 2.

Table 2.

ρ	gg'	r
$\nearrow\searrow$	> 0	$\nearrow$ or $\searrow$ or $\nearrow\searrow$ or $\searrow\nearrow$ or $\searrow\nearrow\searrow$
$\searrow\nearrow$	> 0	$\nearrow$ or $\searrow$ or $\nearrow\searrow$ or $\searrow\nearrow$ or $\nearrow\searrow\nearrow$
$\nearrow\searrow$	< 0	$\nearrow$ or $\searrow$ or $\nearrow\searrow$ or $\searrow\nearrow$ or $\nearrow\searrow\nearrow$
$\searrow\nearrow$	< 0	$\nearrow$ or $\searrow$ or $\nearrow\searrow$ or $\searrow\nearrow$ or $\searrow\nearrow\searrow$

2. THE WILKER-ANGLESIO INEQUALITY.

This inequality provides another good opportunity to illustrate the foregoing "rules." Let

$$A(x) = \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2$$

and $B(x) = x^3 \tan x$. In [**15**], Wilker asked to (a) prove that $A > 0$ on $(0, \pi/2)$ and (b) find the largest constant C such that $A > CB$ on $(0, \pi/2)$. In [**14**], Anglesio showed that the ratio A/B is decreasing on $(0, \pi/2)$, so

$$C = \lim_{x \to (\pi/2)^-} \frac{A(x)}{B(x)} = \frac{16}{\pi^4}.$$

In particular, this implies Wilker's inequality (a).

Anglesio's proof is elementary but nontrivial. In fact, of all this MONTHLY's readers at the time, only Anglesio and the Lamar University Problem Solving Group were able to solve part (b) of Wilker's problem. A shorter proof of Anglesio's result was given just recently in [**8**]; however, the latter proof uses certain properties of the Bernoulli numbers, which apparently cannot be established without complex analysis. Here we show that Anglesio's result can be proved in an almost algorithmic manner, based on the foregoing l'Hospital-type rules for monotonicity.

Let us consider the inverse ratio $B(x)/A(x)$, in which we make the substitution $x = \operatorname{arccot} c$. This is done because the derivative of the arccotangent is a rational function. Essentially, this substitution is the only ingredient in our proof that could count as any sort of "idea." Once this is done, the rest of the proof is rather algorithmic. Thus, the monotonicity of A/B is seen to be equivalent to the assertion that the ratio $r := f/g$ is decreasing on $(0, \infty)$, where f and g are given by

$$f(c) = \operatorname{arccot}^5 c, \quad g(c) = \frac{c}{1+c^2} + \operatorname{arccot} c - 2c \operatorname{arccot}^2 c.$$

Let $f_0 = f$. For any natural number n, define f_n recursively on $(0, \infty)$ by $f_n(c) = f'_{n-1}(c)$ if n is odd and $f_n(c) = f'_{n-1}(c)(1+c^2)^2$ if n is even. Functions g_n are defined in the analogous manner, with $g_0 = g$. We then let $r_n = f_n/g_n$ and $\rho_n = r_{n+1}$. Note that $\rho_n = f'_n/g'_n$ for $n = 0, 1, 2 \dots$.

Calculations (best done with Mathematica or similar software) show that

$$g_3(c) = -\frac{16}{(1+c^2)^2}, \quad g_4(c) = \frac{64c}{1+c^2},$$

$$g_{2m}(c) = (-1)^{m-1} \frac{2^{3m-2} c(c^2-3)}{1+c^2},$$

$$g_{2m+1}(c) = (-1)^{m-1} \frac{2^{3m-2}(c^4+6c^2-3)}{(1+c^2)^2}$$

for $m = 3, 4, \dots$. Hence, for each $n \geq 3$, the function g_n does not change sign on any of the intervals

$$\left(0, \sqrt{2\sqrt{3}-3}\right), \quad \left(\sqrt{2\sqrt{3}-3}, 1\right), \quad \left(1, \sqrt{3}\right), \quad \left(\sqrt{3}, \infty\right). \tag{1}$$

Moreover, for $n = 0$, 1, or 2, it is the case that $(-1)^n g_n > 0$ on the entire interval $(0, \infty)$, which follows because $g_3 < 0$ and $g_n(c) \to 0$ as $c \to \infty$ for $n = 0$, 1, or 2.

Next,

$$r_{10}(c) = \frac{15(1+c^2)}{32(c^2-3)},$$

so r_{10} is decreasing on each of the intervals in (1). This puts us in a position to go back from r_{10} to $r_0 (= r)$ on each of these intervals.

First consider $(a, b) = (0, \sqrt{2\sqrt{3}-3})$. Then, using Tables 1 and 2, one successively obtains the monotonicity patterns of $r_9, \dots, r_0 = r$ on (a, b):

n	ρ_n	$g_n g_n'$	r_n in a r.n. of a	r_n in a l.n. of b	r_n (on (a, b))
9	$\searrow$	< 0		$\searrow$	$\searrow$
8, 7, 6, 5	$\searrow$		$\searrow$	$\searrow$	$\searrow$
4	$\searrow$		$\nearrow$	$\searrow$	$\nearrow\searrow$
3, 2	$\nearrow\searrow$		$\nearrow$	$\searrow$	$\nearrow\searrow$
1	$\nearrow\searrow$	< 0	$\searrow$	$\searrow$	$\searrow$
0	$\searrow$		$\searrow$	$\searrow$	$\searrow$

Here, the conclusions regarding the pattern of r_n on (a, b) do not depend on the content of the five empty cells in this table. The local monotonicity patterns of r_n in a right neighborhood of a and in a left neighborhood of b are determined according to the signs of the nonzero limits $r_n'(a+)$ and $r_n'(b-)$. The only exception is the zero value of $r_1'(a+)$; however, $r_1''(a+) < 0$.

On each of the intervals $(\sqrt{2\sqrt{3}-3}, 1)$ and $(1, \sqrt{3})$ the matter is much simpler. Namely, letting (a, b) signify either of these two intervals, one obtains successively for $n = 9, \dots, 0$ that ρ_n is decreasing on (a, b) and r_n is decreasing in a right neighborhood of a and in a left neighborhood of b, whence r_n is decreasing on (a, b). (Here, $r_n'(a+)$ and $r_n'(b-)$ are negative for all $n = 9, \dots, 0$.)

It remains to consider the interval $(a, b) = (\sqrt{3}, \infty)$. For this interval one obtains the following table:

n	ρ_n	$g_n g_n'$	r_n in a r.n. of a	r_n in a l.n. of b	r_n (on (a, b))
9	$\searrow$		$\searrow$	$\nearrow$	$\searrow\nearrow$
8	$\searrow\nearrow$	> 0	$\searrow$	$\searrow$	$\searrow$
7	$\searrow$		$\searrow$	$\nearrow$	$\searrow\nearrow$
6	$\searrow\nearrow$	> 0	$\searrow$	$\searrow$	$\searrow$
$5, \dots, 0$	$\searrow$		$\searrow$	$\searrow$	$\searrow$

(Here, $r_n'(a+)$ is nonzero for all $n = 9, \dots, 0$, while $\lim_{c\to\infty} c^3 r_n'(c)$ is positive for $n = 7$ or 9 and negative for n in $\{0, \dots, 6, 8\}$.) Thus, r is decreasing on the entire interval $(0, \infty)$ from $r(0+) = \pi^4/16 = 6.08\dots$ to $r(\infty-) = 45/8 = 5.62\dots$.

3. CONCLUSION. The proofs in [**8**] and [**14**] require a good deal of ingenuity that cleverly exploits specific features of the problem. In contrast, the argument just presented is straightforward and rather mechanical.

This is exactly the point that we wish to make in this paper. Now a wide class of inequalities become almost trivial in that ad hoc creativity is no longer needed for many such problems. But then is there any excitement left? Yes, what is exciting now is to have such general rules for monotonicity!

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Infinitely Many Insolvable Diophantine Equations

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Let $f(x_1, \dots, x_n)$ be a quadratic form in n variables $x_1, \dots, x_n$ with integral coefficients, let p be a prime, and let k be a positive integer. The congruence $f(x_1, \dots, x_n) \equiv 0 \pmod{p^k}$ is said to be *solvable nontrivially* if there exist integers $x_1, \dots, x_n$ such that $f(x_1, \dots, x_n) \equiv 0 \pmod{p^k}$ with at least one of $x_1, \dots, x_n$ not divisible by p. Thus the congruence $x_1^2 + x_2^2 \equiv 0 \pmod{3^k}$ is solvable (with $x_1 = x_2 = 0$) but is not solvable nontrivially as any solution x_1, x_2 satisfies $x_1 \equiv x_2 \equiv 0 \pmod 3$. Let m be a positive integer larger than 1. The congruence $f(x_1, \dots, x_n) \equiv 0 \pmod m$ is said to be solvable nontrivially if $f(x_1, \dots, x_n) \equiv 0 \pmod{p^k}$ is solvable nontrivially for each prime