On the Bennett-Hoeffding inequality

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$^2$Supported by NSF grant DMS-0805946


October 14, 2009
1. Introduction

2. Main results

3. Sketch of proof

4. Computation of bounds

5. Comparison of bounds
$X_1, \ldots, X_n$: indep. r.v.'s s.t.
$X_i \leq y$ a.s. for some $y > 0$;
$\mathbb{E}X_i \leq 0$;
$\sum_i \mathbb{E}X_i^2 \leq \sigma^2$.

$S := X_1 + \cdots + X_n$. 
$X_1, \ldots, X_n$: indep. r.v.’s s.t. $X_i \leq y$ a.s. for some $y > 0$; $E X_i \leq 0$; $\sum_i E X_i^2 \leq \sigma^2$. $S := X_1 + \cdots + X_n$. 
Classes of (generalized moment) functions $f : \mathbb{R} \to \mathbb{R}$

\[
f \in \mathcal{E} \iff \exists \lambda > 0 \ \forall u \in \mathbb{R} \ f(u) = e^{\lambda u};
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\[
f \in \mathcal{H}_+^\alpha \iff \exists \mu \geq 0 \ \forall u \in \mathbb{R} \ f(u) = \int_{-\infty}^{\infty} (u - t)^\alpha \mu(dt).
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\[
0 < \beta < \alpha \implies \mathcal{H}_+^\alpha \subseteq \mathcal{H}_+^\beta.
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For $\alpha = 1, 2, \ldots$:

- $f \in \mathcal{H}_+^\alpha$ iff $f^{(\alpha - 1)}$ is convex and $f^{(j)}(-\infty) = 0$ for $j = 0, \ldots, \alpha - 1$.

- \[
\bigcap_{\alpha > 0} \mathcal{H}_+^\alpha = \{ f : f(x) = \int_{(0,\infty)} e^{tx} \mu(dt) \ \forall x \in \mathbb{R} \}.
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Let $\Gamma_{a^2}$ and $\Pi_\theta$ be any indep. r.v.’s s.t.

$$\Gamma_{a^2} \sim \mathcal{N}(0, a^2) \text{ and } \Pi_\theta \sim \text{Pois}(\theta).$$

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Take any $\beta > 0$ s.t. $\epsilon := \beta \sigma^2 y \in (0, 1)$. Suppose that $\epsilon \in \mathcal{E}(X_i^\beta)$.

Then

$\mathbb{E}[S] \geq \mathbb{E}[f(\Gamma^{1-\epsilon} \sigma^2 + \epsilon \tilde{\Pi} \epsilon \sigma^2 / y^2)] \quad \forall f \in \mathcal{H}^3 +$.
Theorem (Main)

Take any $\beta > 0$ s.t.

$$\varepsilon := \frac{\beta}{\sigma^2 y} \in (0, 1).$$

Suppose that

$$\sum_i E(X_i)^3 \leq \beta.$$

Then

$$E f(S) \leq E f\left(\Gamma(1-\varepsilon)\sigma^2 + y\tilde{\Pi}_{\varepsilon\sigma^2/y^2}\right) \quad \forall f \in \mathcal{H}_+^3.$$
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$$\mathbb{E} f(S) \leq \mathbb{E} f \left( \Gamma (1 - \varepsilon) \sigma^2 + y \tilde{\Pi} \varepsilon \sigma^2 / y^2 \right) \quad \forall f \in \mathcal{H}_+^3.$$
Proposition (Exactness for each $f$)

For each triple $(\sigma, y, \beta)$ as in Theorem (Main) and each $f \in \mathcal{H}_+^3$, the upper bound $E f \left( \Gamma (1 - \varepsilon) \sigma^2 + y \tilde{\Pi}_\varepsilon \sigma^2 / y^2 \right)$ on $E f(S)$ is exact.

Proposition (Exactness in $p$)

For any given $p \in (0, 3)$, one cannot replace $\mathcal{H}_+^3$ in Theorem (Main) by the larger class $\mathcal{H}_+^p$. 

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Related preceding results – all of the form:

\[ \forall f \in \mathcal{F} \quad \sup E f(S) = E f(\eta), \]

sup over all indep. \( X_i \)'s as before, with the cond.
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imposed or not, and where the class \( \mathcal{F} \) of functions and the r.v. \( \eta \) are as in the following table:

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<th>Bound</th>
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**Corollary (Upper bound on the tail)**

*Under the conditions of Theorem (Main), $\forall x \in \mathbb{R}$*

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P(S \geq x) \leq \text{Pin}(x) := P_{H^3_+} (\Gamma (1-\varepsilon) \sigma^2 + y \tilde{\eta} \varepsilon \sigma^2 / y^2; x)
\leq c_{3,0} \ P^{\text{LC}} (\Gamma (1-\varepsilon) \sigma^2 + y \tilde{\eta} \varepsilon \sigma^2 / y^2 \geq x),
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$c_{3,0} = 2e^3 / 9 = 4.46 \ldots$.

Here, $P^{\text{LC}} (\eta \geq \cdot)$ is the least log-concave majorant of $P(\eta \geq \cdot)$, and

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P_F (\eta; x) = \inf \left\{ \frac{E f(\eta)}{f(x)} : f \in \mathcal{F}, \ f(x) > 0 \right\},
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the best upper bound on $P(S \geq x)$ based on comparison $E f(S) \leq E f(\eta)$ for all $f \in \mathcal{F}$. 
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Remark

Since the class \( \mathcal{H}_+^3 \) of generalized moment functs. is shift-invariant, it is enough to prove Theorem (Main) just for \( n = 1 \).

Fix any \( \sigma > 0 \) and \( y > 0 \).

For any \( a \geq 0 \) and \( b > 0 \), let \( X_{a,b} \) denote any r.v. with the unique zero-mean distr. on the two-point set \( \{-a, b\} \).
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Lemma (Possible values of $E X_3^+$)

(i) For any r.v. $X$ s.t. $X \leq y$ a.s., $EX \leq 0$, and $EX^2 \leq \sigma^2$,

$$EX_+^3 \leq \frac{y^3 \sigma^2}{y^2 + \sigma^2}.$$

(ii) For any

$$\beta \in \left(0, \frac{y^3 \sigma^2}{y^2 + \sigma^2}\right)$$

$$\exists! (a, b) \in (0, \infty) \times (0, \infty) \text{ s.t. } X_{a,b} \leq y \text{ a.s., } EX_{a,b}^2 = \sigma^2, \text{ and } E(X_{a,b})_+^3 = \beta.$$

In particular, the ineq. in part (i) is exact.
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In particular, the ineq. in part (i) is exact.
Lemma (2-point zero-mean distr. are extremal)

Fix any $w \in \mathbb{R}$, $y > 0$, $\sigma > 0$, and $\beta$ s.t. $\beta \in \left(0, \frac{y^3 \sigma^2}{y^2 + \sigma^2}\right]$, and let $(a, b)$ be the unique pair as in the previous lemma. Then

$$\max \{E(X - w)^3_+ : X \leq y \text{ a.s., } E X \leq 0, E X^2 \leq \sigma^2, E X^3_+ \leq \beta\}$$

$$= \begin{cases} 
E(X_{a,b} - w)^3_+ & \text{if } w \leq 0, \\
E(X_{\tilde{a}, \tilde{b}} - w)^3_+ & \text{if } w \geq 0,
\end{cases}$$

where $\tilde{b} := y$ and $\tilde{a} := \frac{\beta y}{y^3 - \beta}$. At that, $\tilde{a} > 0$, $X_{\tilde{a}, \tilde{b}} \leq y$ a.s., $E X_{\tilde{a}, \tilde{b}} = 0$, and $E(X_{\tilde{a}, \tilde{b}})^3_+ = \beta$, but one can only say that $E X^2_{\tilde{a}, \tilde{b}} \leq \sigma^2$, and the latter inequality is strict if $\beta \neq \frac{y^3 \sigma^2}{y^2 + \sigma^2}$. 
Lemma (2-point zero-mean distrs. are extremal)

Fix any $w \in \mathbb{R}$, $y > 0$, $\sigma > 0$, and $\beta$ s.t. $\beta \in \left(0, \frac{y^3\sigma^2}{y^2+\sigma^2}\right]$, and let $(a, b)$ be the unique pair as in the previous lemma. Then

$$\max\{E(X - w)^3_+ : X \leq y \text{ a.s., } EX \leq 0, EX^2 \leq \sigma^2, EX^3_+ \leq \beta\}$$

$$= \begin{cases} E(X_{a, b} - w)^3_+ & \text{if } w \leq 0, \\ E(X_{\tilde{a}, \tilde{b}} - w)^3_+ & \text{if } w \geq 0, \end{cases}$$

where $\tilde{b} := y$ and $\tilde{a} := \frac{\beta y}{y^3 - \beta}$. At that, $\tilde{a} > 0$, $X_{\tilde{a}, \tilde{b}} \leq y$ a.s., $EX_{\tilde{a}, \tilde{b}} = 0$, and $E(X_{\tilde{a}, \tilde{b}})^3_+ = \beta$, but one can only say that $EX_{\tilde{a}, \tilde{b}}^2 \leq \sigma^2$, and the latter inequality is strict if $\beta \neq \frac{y^3\sigma^2}{y^2+\sigma^2}$. 


Lemma (2-point zero-mean distrs. are extremal)

Fix any \( w \in \mathbb{R}, \ y > 0, \ \sigma > 0, \) and \( \beta \) s.t. \( \beta \in \left( 0, \frac{y^3\sigma^2}{y^2+\sigma^2} \right) \), and let \((a, b)\) be the unique pair as in the previous lemma. Then

\[
\max\{E(X - w)^3_+ : X \leq y \ a.s., \ E X \leq 0, \ E X^2 \leq \sigma^2, \ E X^3_+ \leq \beta\} = \begin{cases} 
E(X_{a,b} - w)^3_+ & \text{if } w \leq 0, \\
E(X_{\tilde{a},\tilde{b}} - w)^3_+ & \text{if } w \geq 0,
\end{cases}
\]

where \( \tilde{b} := y \) and \( \tilde{a} := \frac{\beta y}{y^3 - \beta} \). At that, \( \tilde{a} > 0, \ X_{\tilde{a},\tilde{b}} \leq y \ a.s., \ E X_{\tilde{a},\tilde{b}} = 0, \) and \( E(X_{\tilde{a},\tilde{b}})^3_+ = \beta \), but one can only say that \( E X^2_{\tilde{a},\tilde{b}} \leq \sigma^2 \), and the latter inequality is strict if \( \beta \neq \frac{y^3\sigma^2}{y^2+\sigma^2} \).
2-point zero-mean distrs. are extremal

**Lemma (2-point zero-mean distrs. are extremal)**

Fix any \( w \in \mathbb{R}, y > 0, \sigma > 0, \) and \( \beta \) s.t. \( \beta \in \left( 0, \frac{y^3 \sigma^2}{y^2 + \sigma^2} \right) \), and let \((a, b)\) be the unique pair as in the previous lemma. Then

\[
\max \{ E(X - w)^3_+ : X \leq y \text{ a.s.}, E\, X \leq 0, E\, X^2 \leq \sigma^2, E\, X^3_+ \leq \beta \}
= \begin{cases} 
E(X_{a,b} - w)^3_+ & \text{if } w \leq 0, \\
E(X_{\tilde{a},\tilde{b}} - w)^3_+ & \text{if } w \geq 0,
\end{cases}
\]

where \( \tilde{b} := y \) and \( \tilde{a} := \frac{\beta y}{y^3 - \beta} \). At that, \( \tilde{a} > 0, X_{\tilde{a},\tilde{b}} \leq y \text{ a.s.}, E\, X_{\tilde{a},\tilde{b}} = 0, \) and \( E(X_{\tilde{a},\tilde{b}})^3_+ = \beta \), but one can only say that \( E\, X^2_{\tilde{a},\tilde{b}} \leq \sigma^2 \), and the latter inequality is strict if \( \beta \neq \frac{y^3 \sigma^2}{y^2 + \sigma^2} \).
Lemma (2-point zero-mean distr. are extremal)

Fix any $w \in \mathbb{R}$, $y > 0$, $\sigma > 0$, and $\beta$ s.t. $\beta \in \left(0, \frac{y^3 \sigma^2}{y^2 + \sigma^2}\right]$, and let $(a, b)$ be the unique pair as in the previous lemma. Then

$$\max\left\{E(X - w)^3_{+} : X \leq y \text{ a.s., } E X \leq 0, E X^2 \leq \sigma^2, E X^3_{+} \leq \beta\right\}$$

$$= \begin{cases} E(X_{a,b} - w)^3_{+} & \text{if } w \leq 0, \\ E(X_{\tilde{a},\tilde{b}} - w)^3_{+} & \text{if } w \geq 0, \end{cases}$$

where $\tilde{b} := y$ and $\tilde{a} := \frac{\beta y}{y^3 - \beta}$. At that, $\tilde{a} > 0$, $X_{\tilde{a},\tilde{b}} \leq y$ a.s., $E X_{\tilde{a},\tilde{b}} = 0$, and $E(X_{\tilde{a},\tilde{b}})^3_{+} = \beta$, but one can only say that $E X^2_{\tilde{a},\tilde{b}} \leq \sigma^2$, and the latter inequality is strict if $\beta \neq \frac{y^3 \sigma^2}{y^2 + \sigma^2}$. 
Lemma (2-point zero-mean distr. are extremal)

Fix any $w \in \mathbb{R}$, $y > 0$, $\sigma > 0$, and $\beta$ s.t. $\beta \in \left(0, \frac{y^3 \sigma^2}{y^2 + \sigma^2}\right]$, and let $(a, b)$ be the unique pair as in the previous lemma. Then

$$
\max\{E(X - w)^3_+ : X \leq y \ a.s., \ E \ X \leq 0, \ E \ X^2 \leq \sigma^2, \ E \ X^3_+ \leq \beta\}
$$

$$
= \begin{cases} 
E(X_{a,b} - w)^3_+ & \text{if } w \leq 0,
\end{cases}
$$

$$
E(X_{\tilde{a},\tilde{b}} - w)^3_+ & \text{if } w \geq 0,
\end{cases}
$$

where $\tilde{b} := y$ and $\tilde{a} := \frac{\beta y}{y^3 - \beta}$. At that, $\tilde{a} > 0$, $X_{\tilde{a},\tilde{b}} \leq y$ a.s., $E X_{\tilde{a},\tilde{b}} = 0$, and $E(X_{\tilde{a},\tilde{b}})^3_+ = \beta$, but one can only say that $E X^2_{\tilde{a},\tilde{b}} \leq \sigma^2$, and the latter inequality is strict if $\beta \neq \frac{y^3 \sigma^2}{y^2 + \sigma^2}$.
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\[
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E(X_{a,b} - w)^3_+ & \text{if } w \leq 0, \\
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\end{cases}
\]

where \( \tilde{b} := y \) and \( \tilde{a} := \frac{\beta y}{y^3 - \beta} \). At that, \( \tilde{a} > 0, X_{\tilde{a},\tilde{b}} \leq y \text{ a.s.}, E X_{\tilde{a},\tilde{b}} = 0, \) and \( E(X_{\tilde{a},\tilde{b}})^3_+ = \beta \), but one can only say that \( E X^2_{\tilde{a},\tilde{b}} \leq \sigma^2 \), and the latter inequality is strict if \( \beta \neq \frac{y^3\sigma^2}{y^2+\sigma^2} \).
Lemma (2-point zero-mean distrs. are extremal)

Fix any \( w \in \mathbb{R}, y > 0, \sigma > 0, \text{ and } \beta \text{ s.t. } \beta \in \left(0, \frac{y^3 \sigma^2}{y^2 + \sigma^2}\right), \) and let \((a, b)\) be the unique pair as in the previous lemma. Then

\[
\max \{E(X - w)^3_+: X \leq y \text{ a.s., } E X \leq 0, E X^2 \leq \sigma^2, E X^3_+ \leq \beta\} = \begin{cases} E(X_{a,b} - w)_+^3 & \text{if } w \leq 0, \\ E(X_{\tilde{a},\tilde{b}} - w)_+^3 & \text{if } w \geq 0, \end{cases}
\]

where \( \tilde{b} := y \) and \( \tilde{a} := \frac{\beta y}{y^3 - \beta} \). At that, \( \tilde{a} > 0, X_{\tilde{a},\tilde{b}} \leq y \text{ a.s., } E X_{\tilde{a},\tilde{b}} = 0, \text{ and } E(X_{\tilde{a},\tilde{b}})_+^3 = \beta, \) but one can only say that

\[
E X_{\tilde{a},\tilde{b}}^2 \leq \sigma^2, \text{ and the latter inequality is strict if } \beta \neq \frac{y^3 \sigma^2}{y^2 + \sigma^2}.
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where $\tilde{b} := y$ and $\tilde{a} := \frac{\beta y}{y^3-\beta}$. At that, $\tilde{a} > 0$, $X_{\tilde{a},\tilde{b}} \leq y$ a.s., $E X_{\tilde{a},\tilde{b}} = 0$, and $E(X_{\tilde{a},\tilde{b}})_+^3 = \beta$, but one can only say that $E X_{\tilde{a},\tilde{b}}^2 \leq \sigma^2$, and the latter inequality is strict if $\beta \neq \frac{y^3\sigma^2}{y^2+\sigma^2}$. 
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Fix any \( w \in \mathbb{R}, \ y > 0, \ \sigma > 0, \ \text{and} \ \beta \ \text{s.t.} \ \beta \in \left(0, \frac{y^3 \sigma^2}{y^2 + \sigma^2}\right), \ \text{and let} \ (a, b) \ \text{be the unique pair as in the previous lemma. Then}

\[
\max\{E(X - w)^3 : X \leq y \ a.s., \ E X \leq 0, \ E X^2 \leq \sigma^2, \ E X^3_+ \leq \beta\} \\
= \begin{cases} \\
E(X_{a,b} - w)^3_+ & \text{if} \ w \leq 0, \\
E(X_{\tilde{a},\tilde{b}} - w)^3_+ & \text{if} \ w \geq 0,
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\]

where \( \tilde{b} := y \) and \( \tilde{a} := \frac{\beta y}{y^3 - \beta} \). At that, \( \tilde{a} > 0, \ X_{\tilde{a},\tilde{b}} \leq y \ a.s., \ E X_{\tilde{a},\tilde{b}} = 0, \ \text{and} \ E(X_{\tilde{a},\tilde{b}})^3_+ = \beta, \ \text{but one can only say that} \ E X^2_{\tilde{a},\tilde{b}} \leq \sigma^2, \ \text{and the latter inequality is strict if} \ \beta \neq \frac{y^3 \sigma^2}{y^2 + \sigma^2}. \)
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$$
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$$

$$
= \begin{cases} 
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where $\tilde{b} := y$ and $\tilde{a} := \frac{\beta y}{y^3 - \beta}$. At that, $\tilde{a} > 0$, $X_{\tilde{a},\tilde{b}} \leq y$ a.s., $E X_{\tilde{a},\tilde{b}} = 0$, and $E(X_{\tilde{a},\tilde{b}})^3_+ = \beta$, but one can only say that

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\[
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where $\tilde{b} := y$ and $\tilde{a} := \frac{\beta y}{y^3 - \beta}$. At that, $\tilde{a} > 0$, $X_{\tilde{a},\tilde{b}} \leq y$ a.s., $E X_{\tilde{a},\tilde{b}} = 0$, and $E(X_{\tilde{a},\tilde{b}})^3_+ = \beta$, but one can only say that

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$$
\max\{E(X - w)^3_+ : X \leq y \text{ a.s., } E X \leq 0, E X^2 \leq \sigma^2, E X^3_+ \leq \beta\} = \begin{cases} 
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where $\tilde{b} := y$ and $\tilde{a} := \frac{\beta y}{y^3 - \beta}$. At that, $\tilde{a} > 0$, $X_{\tilde{a},\tilde{b}} \leq y$ a.s., $E X_{\tilde{a},\tilde{b}} = 0$, and $E(X_{\tilde{a},\tilde{b}})^3_+ = \beta$, but one can only say that $E X_{\tilde{a},\tilde{b}}^2 \leq \sigma^2$, and the latter inequality is strict if $\beta \neq \frac{y^3 \sigma^2}{y^2 + \sigma^2}$. 
Lemma (Monotonicity in $\sigma$ and $\beta$)

Take any $\sigma_0, \beta_0, \sigma, \beta$ s.t.

$0 \leq \sigma_0 \leq \sigma$, $0 \leq \beta_0 \leq \beta$,

$\beta_0 \leq \sigma_0^2 y$, and $\beta \leq \sigma^2 y$. Then

$$E f(\Gamma_{\sigma_0^2 - \beta_0/y + y \tilde{\Pi}_{\beta_0/y^3}}) \leq E f(\Gamma_{\sigma^2 - \beta/y + y \tilde{\Pi}_{\beta/y^3}}) \quad (1)$$

$\forall f \in \mathcal{H}^2_+$, and hence $\forall f \in \mathcal{H}^3_+$. 
Lemma (Monotonicity in $\sigma$ and $\beta$)

Take any $\sigma_0, \beta_0, \sigma, \beta$ s.t.
$0 \leq \sigma_0 \leq \sigma$, $0 \leq \beta_0 \leq \beta$,
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$$
E f(\Gamma_{\sigma_0^2 - \beta_0/y} + y \tilde{\Pi}_{\beta_0/y^3}) \leq E f(\Gamma_{\sigma^2 - \beta/y} + y \tilde{\Pi}_{\beta/y^3}) \quad (1)
$$

$\forall f \in \mathcal{H}_+^2$, and hence $\forall f \in \mathcal{H}_+^3$. 
**Monotonicity in $\sigma$ and $\beta$**

**Lemma (Monotonicity in $\sigma$ and $\beta$)**

*Take any $\sigma_0, \beta_0, \sigma, \beta$ s.t.*

$0 \leq \sigma_0 \leq \sigma$, $0 \leq \beta_0 \leq \beta$,

$\beta_0 \leq \sigma_0^2 y$, and $\beta \leq \sigma^2 y$. Then

$$
E f(\Gamma_{\sigma_0^2 - \beta_0/y} + y \tilde{\Pi}_{\beta_0/y^3}) \leq E f(\Gamma_{\sigma^2 - \beta/y} + y \tilde{\Pi}_{\beta/y^3}) \quad (1)
$$

$\forall f \in \mathcal{H}_+^2$, and hence $\forall f \in \mathcal{H}_+^3$. 


Lemma (Monotonicity in \( \sigma \) and \( \beta \))

Take any \( \sigma_0, \beta_0, \sigma, \beta \) s.t.

\[
0 \leq \sigma_0 \leq \sigma, \ 0 \leq \beta_0 \leq \beta, \\
\beta_0 \leq \sigma_0^2 y, \text{ and } \beta \leq \sigma^2 y. \text{ Then}
\]

\[
\mathbb{E} f\left(\Gamma_{\sigma_0^2 - \beta_0/y} + y \tilde{\Pi}_{\beta_0/y^3}\right) \leq \mathbb{E} f\left(\Gamma_{\sigma^2 - \beta/y} + y \tilde{\Pi}_{\beta/y^3}\right) \tag{1}
\]

\( \forall f \in \mathcal{H}_+^2, \text{ and hence } \forall f \in \mathcal{H}_+^3. \)
Lemma (Monotonicity in $\sigma$ and $\beta$)

Take any $\sigma_0, \beta_0, \sigma, \beta$ s.t.

$0 \leq \sigma_0 \leq \sigma$, $0 \leq \beta_0 \leq \beta$,

$\beta_0 \leq \sigma_0^2 y$, and $\beta \leq \sigma^2 y$. Then

$$E f(\Gamma_{\sigma_0^2-\beta_0/y} + y \tilde{\Pi}_{\beta_0/y^3}) \leq E f(\Gamma_{\sigma^2-\beta/y} + y \tilde{\Pi}_{\beta/y^3})$$

(1)

$\forall f \in \mathcal{H}^2_+, \text{ and hence } \forall f \in \mathcal{H}^3_+$. 
Lemma (Monotonicity in $\sigma$ and $\beta$)

Take any $\sigma_0, \beta_0, \sigma, \beta$ s.t.

$0 \leq \sigma_0 \leq \sigma$, $0 \leq \beta_0 \leq \beta,$

$\beta_0 \leq \sigma_0^2 y$, and $\beta \leq \sigma^2 y$. Then

$$
E f(\Gamma_{\sigma_0^2-\beta_0/y} + y \tilde{\Pi}_{\beta_0/y^3}) \leq E f(\Gamma_{\sigma^2-\beta/y} + y \tilde{\Pi}_{\beta/y^3}) \quad (1)
$$

$\forall f \in \mathcal{H}^2_+, \text{ and hence } \forall f \in \mathcal{H}^3_+.$
Monotonicity in $\sigma$ and $\beta$

Lemma (Monotonicity in $\sigma$ and $\beta$)

Take any $\sigma_0, \beta_0, \sigma, \beta$ s.t.

$0 \leq \sigma_0 \leq \sigma$, $0 \leq \beta_0 \leq \beta$,

$\beta_0 \leq \sigma_0^2 y$, and $\beta \leq \sigma^2 y$. Then

$$
E f(\Gamma \sigma_0^2 - \beta_0 / y + y \tilde{\Pi} \beta_0 / y^3) \leq E f(\Gamma \sigma^2 - \beta / y + y \tilde{\Pi} \beta / y^3) \quad (1)
$$

$\forall f \in \mathcal{H}_+^2$, and hence $\forall f \in \mathcal{H}_+^3$. 
**Lemma (Main)**

Let $X$ be any r.v such that $X \leq y$ a.s., $E X \leq 0$, $E X^2 \leq \sigma^2$, and $E X^3_+ \leq \beta$, where $\beta \in \left(0, \frac{y^3\sigma^2}{y^2+\sigma^2}\right]$. Then

$$E f(X) \leq E f(\Gamma_{\sigma^2-\beta/y} + y \tilde{\Pi}_{\beta/y^3}) \quad \forall f \in \mathcal{H}^3_+.$$  

**Sketch of proof**  By the “2-point zero-mean distrs. are extremal” lemma and the “monotonicity in $\sigma$ and $\beta$” lemma, w.l.o.g. $X = X_{a_0,b_0}$ for some $a_0 > 0$ and $b_0 > 0$. Also, w.l.o.g. $f(x) \equiv (x - w)_+^3$. Also, by rescaling, w.l.o.g. $y = 1$.  

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**Outline**

- Introduction
- Main results
- Sketch of proof
- Computation of bounds
- Comparison of bounds
Lemma (Main)

Let $X$ be any r.v such that $X \leq y$ a.s., $E X \leq 0$, $E X^2 \leq \sigma^2$, and $E X_+^3 \leq \beta$, where $\beta \in \left(0, \frac{y^3 \sigma^2}{y^2 + \sigma^2}\right]$. Then

$$E f(X) \leq E f\left(\Gamma \sigma^2 - \frac{\beta}{y} + y \tilde{\Pi} \frac{\beta}{y^3}\right) \quad \forall f \in \mathcal{H}_+^3.$$  

Sketch of proof  By the “2-point zero-mean distrs. are extremal” lemma and the “monotonicity in $\sigma$ and $\beta$” lemma, w.l.o.g. $X = X_{a_0, b_0}$ for some $a_0 > 0$ and $b_0 > 0$. Also, w.l.o.g. $f(x) \equiv (x - w)_+^3$. Also, by rescaling, w.l.o.g. $y = 1$. 

Lemma (Main)

Let $X$ be any r.v such that $X \leq y$ a.s., $E X \leq 0$, $E X^2 \leq \sigma^2$, and $E X_+^3 \leq \beta$, where $\beta \in \left(0, \frac{y^3 \sigma^2}{y^2 + \sigma^2}\right]$. Then

$$E f(X) \leq E f\left(\Gamma_{\sigma^2-\beta/y} + y \tilde{\Pi}_{\beta/y^3}\right) \quad \forall f \in \mathcal{H}_+^3.$$ 

Sketch of proof  By the “2-point zero-mean distrs. are extremal” lemma and the “monotonicity in $\sigma$ and $\beta$” lemma, w.l.o.g. $X = X_{a_0, b_0}$ for some $a_0 > 0$ and $b_0 > 0$. Also, w.l.o.g. $f(x) \equiv (x - w)^3_+$. Also, by rescaling, w.l.o.g. $y = 1$. 
Lemma (Main)

Let $X$ be any r.v such that $X \leq y$ a.s., $E X \leq 0$, $E X^2 \leq \sigma^2$, and $E X^3_+ \leq \beta$, where $\beta \in \left(0, \frac{y^3\sigma^2}{y^2+\sigma^2}\right]$. Then

$$E f(X) \leq E f\left(\Gamma_{\sigma^2-\beta/y} + y \tilde{\Pi}_{\beta/y^3}\right) \quad \forall f \in \mathcal{H}_3^+.$$ 

Sketch of proof

By the “2-point zero-mean distr.s. are extremal” lemma and the “monotonicity in $\sigma$ and $\beta$” lemma, w.l.o.g. $X = X_{a_0,b_0}$ for some $a_0 > 0$ and $b_0 > 0$. Also, w.l.o.g. $f(x) \equiv (x - w)_+^3$. Also, by rescaling, w.l.o.g. $y = 1$. 

Lemma (Main)

Let $X$ be any r.v such that $X \leq y$ a.s., $E X \leq 0$, $E X^2 \leq \sigma^2$, and $E X_+^3 \leq \beta$, where $\beta \in \left(0, \frac{y^3 \sigma^2}{y^2 + \sigma^2}\right]$. Then

$$E f(X) \leq E f(\Gamma_{\sigma^2 - \beta/y} + y \tilde{\Pi}_{\beta/y^3}) \quad \forall f \in \mathcal{H}_+^3.$$

Sketch of proof By the “2-point zero-mean distrs. are extremal” lemma and the “monotonicity in $\sigma$ and $\beta$” lemma, w.l.o.g. $X = X_{a_0, b_0}$ for some $a_0 > 0$ and $b_0 > 0$. Also, w.l.o.g. $f(x) \equiv (x - w)_+^3$. Also, by rescaling, w.l.o.g. $y = 1$. 
Lemma (Main)

Let $X$ be any r.v such that $X \leq y$ a.s., $E X \leq 0$, $E X^2 \leq \sigma^2$, and $E X^3_+ \leq \beta$, where $\beta \in \left(0, \frac{y^3 \sigma^2}{y^2 + \sigma^2}\right]$. Then

$$E f(X) \leq E f\left(\Gamma_\sigma^2 - \frac{\beta}{y} + y \tilde{\Pi}_{\beta/y^3}\right) \quad \forall f \in \mathcal{H}^3_+.$$ 

Sketch of proof  By the “2-point zero-mean distrs. are extremal” lemma and the “monotonicity in $\sigma$ and $\beta$” lemma, w.l.o.g. $X = X_{a_0, b_0}$ for some $a_0 > 0$ and $b_0 > 0$. Also, w.l.o.g. $f(x) \equiv (x - w)_+^3$. Also, by rescaling, w.l.o.g. $y = 1$. 

14/32  Main lemma
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Let $X$ be any r.v such that $X \leq y$ a.s., $E X \leq 0$, $E X^2 \leq \sigma^2$, and $E X_+^3 \leq \beta$, where $\beta \in \left(0, \frac{y^3 \sigma^2}{y^2 + \sigma^2}\right]$. Then

$$E f(X) \leq E f\left(\sigma^2 - \beta/y + y \tilde{\Pi}_{\beta/y^3}\right) \quad \forall f \in \mathcal{H}_+^3.$$

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Let $X$ be any r.v such that $X \leq y$ a.s., $E X \leq 0$, $E X^2 \leq \sigma^2$, and $E X_+^3 \leq \beta$, where $\beta \in \left(0, \frac{y^3 \sigma^2}{y^2 + \sigma^2}\right)$. Then

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Lemma (Main)

Let $X$ be any r.v such that $X \leq y$ a.s., $E X \leq 0$, $E X^2 \leq \sigma^2$, and $E X_+^3 \leq \beta$, where $\beta \in \left(0, \frac{y^3 \sigma^2}{y^2 + \sigma^2}\right)$. Then

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The initial infinitesimal step:
Start with the r.v. $X_{a_0,b_0}$. Decrease $a_0$ and $b_0$ simultaneously by infinitesimal amounts $\Delta a > 0$ and $\Delta b > 0$ so that
\[
E(X_{a_0,b_0} - w)_+^3 \leq E(X_{a,b} + X_{\Delta_1,\Delta_1} + X_{\Delta_2,1} - w)_+^3 \quad \forall w \in \mathbb{R},
\]
where $X_{a,b}$, $X_{\Delta_1,\Delta_1}$, $X_{\Delta_2,1}$ are independent, $a = a_0 - \Delta a$ and $b = b_0 - \Delta b$, and $0 < \Delta_1 \approx 0$ and $0 < \Delta_2 \approx 0$ are chosen, together with $\Delta a$ and $\Delta b$, so that to keep the balance of the total variance and that of the positive-part third moments closely enough:
\[
E X_{a,b}^2 + E X_{\Delta_1,\Delta_1}^2 + E X_{\Delta_2,1}^2 \approx E X_{a_0,b_0}^2
\]
and
\[
E(X_{a,b})_+^3 + E(X_{\Delta_1,\Delta_1})_+^3 + E(X_{\Delta_2,1})_+^3 \approx E(X_{a_0,b_0})_+^3.
\]
Refer to $X_{\Delta_1,\Delta_1}$ and $X_{\Delta_2,1}$ as the symm. and highly asymm. infinitesimal spin-offs, resp.
The initial infinitesimal step:
Start with the r.v. $X_{a_0,b_0}$. Decrease $a_0$ and $b_0$ simultaneously by infinitesimal amounts $\Delta a > 0$ and $\Delta b > 0$ so that
$$E(X_{a_0,b_0} - w)_+^3 \leq E(X_{a,b} + X_{\Delta_1,\Delta_1} + X_{\Delta_2,1} - w)_+^3 \quad \forall w \in \mathbb{R},$$
where $X_{a,b}, X_{\Delta_1,\Delta_1}, X_{\Delta_2,1}$ are indep., $a = a_0 - \Delta a$ and $b = b_0 - \Delta b$, and $0 < \Delta_1 \approx 0$ and $0 < \Delta_2 \approx 0$ are chosen, together with $\Delta a$ and $\Delta b$, so that to keep the balance of the total variance and that of the positive-part third moments closely enough:
$$E X_{a,b}^2 + E X_{\Delta_1,\Delta_1}^2 + E X_{\Delta_2,1}^2 \approx E X_{a_0,b_0}^2$$
and
$$E(X_{a,b})_+^3 + E(X_{\Delta_1,\Delta_1})_+^3 + E(X_{\Delta_2,1})_+^3 \approx E(X_{a_0,b_0})_+^3.$$
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Start with the r.v. $X_{a_0,b_0}$. Decrease $a_0$ and $b_0$ simultaneously by infinitesimal amounts $\Delta a > 0$ and $\Delta b > 0$ so that
\[ E(X_{a_0,b_0} - w)_+^3 \leq E(X_{a,b} + X_{\Delta_1,\Delta_1} + X_{\Delta_2,1} - w)_+^3 \quad \forall w \in \mathbb{R}, \]
where $X_{a,b}$, $X_{\Delta_1,\Delta_1}$, $X_{\Delta_2,1}$ are indep., $a = a_0 - \Delta a$ and $b = b_0 - \Delta b$, and $0 < \Delta_1 \approx 0$ and $0 < \Delta_2 \approx 0$ are chosen, together with $\Delta a$ and $\Delta b$, so that to keep the balance of the total variance and that of the positive-part third moments closely enough:
\[ E X_{a,b}^2 + E X_{\Delta_1,\Delta_1}^2 + E X_{\Delta_2,1}^2 \approx E X_{a_0,b_0}^2 \]
and
\[ E(X_{a,b})_+^3 + E(X_{\Delta_1,\Delta_1})_+^3 + E(X_{\Delta_2,1})_+^3 \approx E(X_{a_0,b_0})_+^3. \]
Refer to $X_{\Delta_1,\Delta_1}$ and $X_{\Delta_2,1}$ as the symm. and highly asymm. infinitesimal spin-offs, resp.
The initial infinitesimal step:
Start with the r.v. $X_{a_0,b_0}$. Decrease $a_0$ and $b_0$ simultaneously by infinitesimal amounts $\Delta a > 0$ and $\Delta b > 0$ so that
$E(X_{a_0,b_0} - w)_+^3 \leq E(X_{a,b} + X_{\Delta_1,1} + X_{\Delta_2,1} - w)_+^3 \quad \forall w \in \mathbb{R}$,
where $X_{a,b}, X_{\Delta_1,1}, X_{\Delta_2,1}$ are indep., $a = a_0 - \Delta a$ and $b = b_0 - \Delta b$, and $0 < \Delta_1 \approx 0$ and $0 < \Delta_2 \approx 0$ are chosen, together with $\Delta a$ and $\Delta b$, so that to keep the balance of the total variance and that of the positive-part third moments closely enough:
$E X_{a,b}^2 + E X_{\Delta_1,1}^2 + E X_{\Delta_2,1}^2 \approx E X_{a_0,b_0}^2$
and $E(X_{a,b})_+^3 + E(X_{\Delta_1,1})_+^3 + E(X_{\Delta_2,1})_+^3 \approx E(X_{a_0,b_0})_+^3$.
Refer to $X_{\Delta_1,1}$ and $X_{\Delta_2,1}$ as the symm. and highly asymm. infinitesimal spin-offs, resp.
The initial infinitesimal step:
Start with the r.v. $X_{a_0,b_0}$. Decrease $a_0$ and $b_0$ simultaneously by infinitesimal amounts $\Delta a > 0$ and $\Delta b > 0$ so that
\[
E(X_{a_0,b_0} - w)_+^3 \leq E(X_a + X_{\Delta_1,\Delta_1} + X_{\Delta_2,1} - w)_+^3 \quad \forall w \in \mathbb{R},
\]
where $X_{a,b}$, $X_{\Delta_1,\Delta_1}$, $X_{\Delta_2,1}$ are indep., $a = a_0 - \Delta a$ and $b = b_0 - \Delta b$, and $0 < \Delta_1 \approx 0$ and $0 < \Delta_2 \approx 0$ are chosen, together with $\Delta a$ and $\Delta b$, so that to keep the balance of the total variance and that of the positive-part third moments closely enough:
\[
E X_{a,b}^2 + E X_{\Delta_1,\Delta_1}^2 + E X_{\Delta_2,1}^2 \approx E X_{a_0,b_0}^2
\]
and
\[
E(X_{a,b})_+^3 + E(X_{\Delta_1,\Delta_1})_+^3 + E(X_{\Delta_2,1})_+^3 \approx E(X_{a_0,b_0})_+^3.
\]
Refer to $X_{\Delta_1,\Delta_1}$ and $X_{\Delta_2,1}$ as the symm. and highly asymm. infinitesimal spin-offs, resp.
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Start with the r.v. $X_{a_0,b_0}$. Decrease $a_0$ and $b_0$ simultaneously by infinitesimal amounts $\Delta a > 0$ and $\Delta b > 0$ so that
\[
E(X_{a_0,b_0} - w)_+^3 \leq E(X_{a,b} + X_{\Delta_1,\Delta_1} + X_{\Delta_2,1} - w)_+^3 \quad \forall w \in \mathbb{R},
\]
where $X_{a,b}, X_{\Delta_1,\Delta_1}, X_{\Delta_2,1}$ are indep., $a = a_0 - \Delta a$ and $b = b_0 - \Delta b$, and $0 < \Delta_1 \approx 0$ and $0 < \Delta_2 \approx 0$ are chosen, together with $\Delta a$ and $\Delta b$, so that to keep the balance of the total variance and that of the positive-part third moments closely enough:
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E X_{a,b}^2 + E X_{\Delta_1,\Delta_1}^2 + E X_{\Delta_2,1}^2 \approx E X_{a_0,b_0}^2
\]
and
\[
E(X_{a,b})_+^3 + E(X_{\Delta_1,\Delta_1})_+^3 + E(X_{\Delta_2,1})_+^3 \approx E(X_{a_0,b_0})_+^3.
\]
Refer to $X_{\Delta_1,\Delta_1}$ and $X_{\Delta_2,1}$ as the symm. and highly asymm. infinitesimal spin-offs, resp.
Main idea of the proof of Lemma (Main): \textit{infinitesimal spin-off}

The initial infinitesimal step:
Start with the r.v. $X_{a_0,b_0}$. Decrease $a_0$ and $b_0$ simultaneously by infinitesimal amounts $\Delta a > 0$ and $\Delta b > 0$ so that
\[
E(X_{a_0,b_0} - w)^3_+ \leq E(X_{a,b} + X_{\Delta_1,\Delta_1} + X_{\Delta_2,1} - w)^3_+ \quad \forall w \in \mathbb{R},
\]
where $X_{a,b}, X_{\Delta_1,\Delta_1}, X_{\Delta_2,1}$ are indep., $a = a_0 - \Delta a$ and $b = b_0 - \Delta b$, and $0 < \Delta_1 \approx 0$ and $0 < \Delta_2 \approx 0$ are chosen, together with $\Delta a$ and $\Delta b$, so that to keep the balance of the total variance and that of the positive-part third moments closely enough:
\[
E X_{a,b}^2 + E X_{\Delta_1,\Delta_1}^2 + E X_{\Delta_2,1}^2 \approx E X_{a_0,b_0}^2
\]
and $E(X_{a,b})^3_+ + E(X_{\Delta_1,\Delta_1})^3_+ + E(X_{\Delta_2,1})^3_+ \approx E(X_{a_0,b_0})^3_+$.
Refer to $X_{\Delta_1,\Delta_1}$ and $X_{\Delta_2,1}$ as the symm. and highly asymm. infinitesimal spin-offs, resp.
The initial infinitesimal step:

Start with the r.v. $X_{a_0,b_0}$. Decrease $a_0$ and $b_0$ simultaneously by infinitesimal amounts $\Delta a > 0$ and $\Delta b > 0$ so that

$$E(X_{a_0,b_0} - w)_+^3 \leq E(X_{a,b} + X_{\Delta_1,\Delta_1} + X_{\Delta_2,1} - w)_+^3 \quad \forall w \in \mathbb{R},$$

where $X_{a,b}$, $X_{\Delta_1,\Delta_1}$, $X_{\Delta_2,1}$ are independent, $a = a_0 - \Delta a$ and $b = b_0 - \Delta b$, and $0 < \Delta_1 \approx 0$ and $0 < \Delta_2 \approx 0$ are chosen, together with $\Delta a$ and $\Delta b$, so that to keep the balance of the total variance and that of the positive-part third moments closely enough:

$$E X_{a,b}^2 + E X_{\Delta_1,\Delta_1}^2 + E X_{\Delta_2,1}^2 \approx E X_{a_0,b_0}^2$$

and

$$E(X_{a,b})_+^3 + E(X_{\Delta_1,\Delta_1})_+^3 + E(X_{\Delta_2,1})_+^3 \approx E(X_{a_0,b_0})_+^3.$$

Refer to $X_{\Delta_1,\Delta_1}$ and $X_{\Delta_2,1}$ as the symm. and highly asymm. infinitesimal spin-offs, resp.
The initial infinitesimal step:
Start with the r.v. $X_{a_0,b_0}$. Decrease $a_0$ and $b_0$ simultaneously by infinitesimal amounts $\Delta a > 0$ and $\Delta b > 0$ so that
\[ E(X_{a_0,b_0} - w)^3_+ \leq E(X_{a,b} + X_{\Delta_1,\Delta_1} + X_{\Delta_2,1} - w)^3_+ \quad \forall w \in \mathbb{R}, \]
where $X_{a,b}$, $X_{\Delta_1,\Delta_1}$, $X_{\Delta_2,1}$ are indep., $a = a_0 - \Delta a$ and $b = b_0 - \Delta b$, and $0 < \Delta_1 \approx 0$ and $0 < \Delta_2 \approx 0$ are chosen, together with $\Delta a$ and $\Delta b$, so that to keep the balance of the total variance and that of the positive-part third moments closely enough:
\[ E X_{a,b}^2 + E X_{\Delta_1,\Delta_1}^2 + E X_{\Delta_2,1}^2 \approx E X_{a_0,b_0}^2 \]
and
\[ E(X_{a,b})^3_+ + E(X_{\Delta_1,\Delta_1})^3_+ + E(X_{\Delta_2,1})^3_+ \approx E(X_{a_0,b_0})^3_+. \]
Refer to $X_{\Delta_1,\Delta_1}$ and $X_{\Delta_2,1}$ as the symm. and highly asymm. infinitesimal spin-offs, resp.
The initial infinitesimal step:
Start with the r.v. $X_{a_0,b_0}$. Decrease $a_0$ and $b_0$ simultaneously by infinitesimal amounts $\Delta a > 0$ and $\Delta b > 0$ so that

$$E(X_{a_0,b_0} - w)^3_+ \leq E(X_{a,b} + X_{\Delta_1,\Delta_1} + X_{\Delta_2,1} - w)^3_+ \quad \forall w \in \mathbb{R},$$

where $X_{a,b}, X_{\Delta_1,\Delta_1}, X_{\Delta_2,1}$ are indep., $a = a_0 - \Delta a$ and $b = b_0 - \Delta b$, and $0 < \Delta_1 \approx 0$ and $0 < \Delta_2 \approx 0$ are chosen, together with $\Delta a$ and $\Delta b$, so that to keep the balance of the total variance and that of the positive-part third moments closely enough:

$$EX_{a,b}^2 + EX_{\Delta_1,\Delta_1}^2 + EX_{\Delta_2,1}^2 \approx EX_{a_0,b_0}^2$$

and

$$E(X_{a,b})^3_+ + E(X_{\Delta_1,\Delta_1})^3_+ + E(X_{\Delta_2,1})^3_+ \approx E(X_{a_0,b_0})^3_+.$$

Refer to $X_{\Delta_1,\Delta_1}$ and $X_{\Delta_2,1}$ as the symm. and highly asymm. infinitesimal spin-offs, resp.
The initial infinitesimal step:
Start with the r.v. $X_{a_0,b_0}$. Decrease $a_0$ and $b_0$ simultaneously by infinitesimal amounts $\Delta a > 0$ and $\Delta b > 0$ so that
\[
E(X_{a_0,b_0} - w)^3_+ \leq E(X_{a,b} + X_{\Delta_1,\Delta_1} + X_{\Delta_2,1} - w)^3_+ \quad \forall w \in \mathbb{R},
\]
where $X_{a,b}$, $X_{\Delta_1,\Delta_1}$, $X_{\Delta_2,1}$ are indep., $a = a_0 - \Delta a$ and $b = b_0 - \Delta b$, and $0 < \Delta_1 \approx 0$ and $0 < \Delta_2 \approx 0$ are chosen, together with $\Delta a$ and $\Delta b$, so that to keep the balance of the total variance and that of the positive-part third moments closely enough:
\[
E X_{a,b}^2 + E X_{\Delta_1,\Delta_1}^2 + E X_{\Delta_2,1}^2 \approx E X_{a_0,b_0}^2
\]
and
\[
E(X_{a,b})_+^3 + E(X_{\Delta_1,\Delta_1})_+^3 + E(X_{\Delta_2,1})_+^3 \approx E(X_{a_0,b_0})_+^3.
\]
Refer to $X_{\Delta_1,\Delta_1}$ and $X_{\Delta_2,1}$ as the symm. and highly asymm. infinitesimal spin-offs, resp.
Main idea of the proof of Lemma (Main): *infinitesimal spin-off*

The initial infinitesimal step:
Start with the r.v. $X_{a_0,b_0}$. Decrease $a_0$ and $b_0$ simultaneously by infinitesimal amounts $\Delta a > 0$ and $\Delta b > 0$ so that
\[
E(X_{a_0,b_0} - w)_+^3 \leq E(X_{a,b} + X_{\Delta_1,\Delta_1} + X_{\Delta_2,1} - w)_+^3 \quad \forall w \in \mathbb{R},
\]
where $X_{a,b}$, $X_{\Delta_1,\Delta_1}$, $X_{\Delta_2,1}$ are indep., $a = a_0 - \Delta a$ and $b = b_0 - \Delta b$, and $0 < \Delta_1 \approx 0$ and $0 < \Delta_2 \approx 0$ are chosen, together with $\Delta a$ and $\Delta b$, so that to keep the balance of the total variance and that of the positive-part third moments closely enough:
\[
E X_{a,b}^2 + E X_{\Delta_1,\Delta_1}^2 + E X_{\Delta_2,1}^2 \approx E X_{a_0,b_0}^2
\]
and
\[
E(X_{a,b})_+^3 + E(X_{\Delta_1,\Delta_1})_+^3 + E(X_{\Delta_2,1})_+^3 \approx E(X_{a_0,b_0})_+^3.
\]
Refer to $X_{\Delta_1,\Delta_1}$ and $X_{\Delta_2,1}$ as the symm. and highly asymm. infinitesimal spin-offs, resp.
The initial infinitesimal step:
Start with the r.v. $X_{a_0, b_0}$. Decrease $a_0$ and $b_0$ simultaneously by infinitesimal amounts $\Delta a > 0$ and $\Delta b > 0$ so that
\[
\mathbb{E}(X_{a_0, b_0} - w)^3_+ \leq \mathbb{E}(X_{a, b} + X_{\Delta_1, \Delta_1} + X_{\Delta_2, 1} - w)^3_+ \quad \forall w \in \mathbb{R},
\]
where $X_{a, b}, X_{\Delta_1, \Delta_1}, X_{\Delta_2, 1}$ are indep., $a = a_0 - \Delta a$ and $b = b_0 - \Delta b$, and $0 < \Delta_1 \approx 0$ and $0 < \Delta_2 \approx 0$ are chosen, together with $\Delta a$ and $\Delta b$, so that to keep the balance of the total variance and that of the positive-part third moments closely enough:
\[
\mathbb{E}X_{a, b}^2 + \mathbb{E}X_{\Delta_1, \Delta_1}^2 + \mathbb{E}X_{\Delta_2, 1}^2 \approx \mathbb{E}X_{a_0, b_0}^2
\]
and $\mathbb{E}(X_{a, b})^3_+ + \mathbb{E}(X_{\Delta_1, \Delta_1})^3_+ + \mathbb{E}(X_{\Delta_2, 1})^3_+ \approx \mathbb{E}(X_{a_0, b_0})^3_+.$
Refer to $X_{\Delta_1, \Delta_1}$ and $X_{\Delta_2, 1}$ as the symm. and highly asymm. infinitesimal spin-offs, resp.
Continue decreasing $a$ and $b$ while “spinning off” the indep. pairs of indep. infinitesimal spin-offs $X_{\Delta_1,\Delta_1}$ and $X_{\Delta_2,1}$, at that keeping the balance of the total variance and that of the positive-part third moments, as described. Stop when $X_{a,b} = 0$ a.s., i.e., when $a$ or $b$ is decreased to 0 (if ever); such a termination point is indeed attainable. Then the sum of all the symm. indep. infinitesimal spin-offs $X_{\Delta_1,\Delta_1}$ will have a centered Gaussian distr., while the sum of the highly asymmetric spin-offs $X_{\Delta_2,1}$’s will give a centered Poisson component. At that, the balances of the variances and positive-part third moments will each be kept (the infinitesimal $X_{\Delta_1,\Delta_1}$’s will provide in the limit a total zero contribution to the balance of the positive-part third moments).
Continue decreasing $a$ and $b$ while “spinning off” the indep. pairs of indep. infinitesimal spin-offs $X_{\Delta_1,\Delta_1}$ and $X_{\Delta_2,1}$, at that keeping the balance of the total variance and that of the positive-part third moments, as described. Stop when $X_{a,b} = 0$ a.s., i.e., when $a$ or $b$ is decreased to 0 (if ever); such a termination point is indeed attainable. Then the sum of all the symm. indep. infinitesimal spin-offs $X_{\Delta_1,\Delta_1}$ will have a centered Gaussian distr., while the sum of the highly asymmetric spin-offs $X_{\Delta_2,1}$’s will give a centered Poisson component. At that, the balances of the variances and positive-part third moments will each be kept (the infinitesimal $X_{\Delta_1,\Delta_1}$’s will provide in the limit a total zero contribution to the balance of the positive-part third moments).
Continue decreasing \( a \) and \( b \) while “spinning off” the indep. pairs of indep. infinitesimal spin-offs \( X_{\Delta_1, \Delta_1} \) and \( X_{\Delta_2, 1} \), at that keeping the balance of the total variance and that of the positive-part third moments, as described. Stop when \( X_{a, b} = 0 \) a.s., i.e., when \( a \) or \( b \) is decreased to 0 (if ever); such a termination point is indeed attainable. Then the sum of all the symm. indep. infinitesimal spin-offs \( X_{\Delta_1, \Delta_1} \) will have a centered Gaussian distr., while the sum of the highly asymmetric spin-offs \( X_{\Delta_2, 1} \)'s will give a centered Poisson component. At that, the balances of the variances and positive-part third moments will each be kept (the infinitesimal \( X_{\Delta_1, \Delta_1} \)'s will provide in the limit a total zero contribution to the balance of the positive-part third moments).
Continue decreasing $a$ and $b$ while “spinning off” the indep. pairs of indep. infinitesimal spin-offs $X_{\Delta_1,\Delta_1}$ and $X_{\Delta_2,1}$, at that keeping the balance of the total variance and that of the positive-part third moments, as described. Stop when $X_{a,b} = 0$ a.s., i.e., when $a$ or $b$ is decreased to 0 (if ever); such a termination point is indeed attainable. Then the sum of all the symm. indep. infinitesimal spin-offs $X_{\Delta_1,\Delta_1}$ will have a centered Gaussian distr., while the sum of the highly asymmetric spin-offs $X_{\Delta_2,1}$’s will give a centered Poisson component. At that, the balances of the variances and positive-part third moments will each be kept (the infinitesimal $X_{\Delta_1,\Delta_1}$’s will provide in the limit a total zero contribution to the balance of the positive-part third moments).
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Formalizing the spin-off idea, with a time-changed Lévy process

Introduce a family of r.v.’s of the form
\[ \eta_b := X_{a(b), b} + \xi_{\tau(b)} \] for \( b \in [\varepsilon, b_0] \), where

\[ \varepsilon := \beta / \sigma^2 = b_0^2 / (b_0 + a_0) < b_0, \]
\[ a(b) := (b/\varepsilon - 1)b, \quad \tau(b) := a_0 b_0 - a(b)b, \quad \text{(balances)} \]
\[ \xi_t := W_{(1-\varepsilon)t} + \tilde{\Pi}_{\varepsilon t}, \]

\( W \) and \( \tilde{\Pi} \) are indep. standard Wiener and centered standard Poisson processes, indep. of \( X_{a(b), b} \) for each \( b \in [\varepsilon, b_0] \). Note: \( a(b_0) = a_0 \) and \( a(\varepsilon) = 0 \), \( \tau(b_0) = 0 \) and \( \tau(\varepsilon) = a_0 b_0 = \sigma^2 \), so that
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Formalizing the spin-off idea, with a time-changed Lévy process

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Proposition (PU(x) computation)

For all $\sigma > 0$, $y > 0$, $\varepsilon \in (0, 1)$, and $x \geq 0$

$$PU(x) = e^{-\lambda_x x} PU_{\text{exp}}(\lambda_x)$$

$$= \exp \frac{(1 - \varepsilon)^2 (w_x + 1)^2 - (\varepsilon + xy/\sigma^2)^2 - (1 - \varepsilon^2)}{2(1 - \varepsilon)y^2/\sigma^2},$$

$$\lambda_x := \frac{1}{y} \left( \frac{\varepsilon + xy/\sigma^2}{1 - \varepsilon} - w_x \right), \quad w_x := L\left( \frac{\varepsilon}{1 - \varepsilon} \exp \frac{\varepsilon + xy/\sigma^2}{1 - \varepsilon} \right),$$

and $L$ is the Lambert product-log funct.: $\forall z \geq 0$, $w = L(z)$ is the only real root of the equation $we^w = z$.

Moreover, $\lambda_x$ incr. in $x$ from 0 to $\infty$ as $x$ does so.

So, PU(x) is about as easy to compute as BH(x).
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and $L$ is the Lambert product-log funct.: $\forall z \geq 0$, $w = L(z)$ is the only real root of the equation $we^w = z$.
Moreover, $\lambda_x$ incr. in $x$ from 0 to $\infty$ as $x$ does so.

So, PU(x) is about as easy to compute as BH(x).
Proposition (PU(x) computation)

For all $\sigma > 0$, $y > 0$, $\varepsilon \in (0, 1)$, and $x \geq 0$

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PU(x) = e^{-\lambda_x x} \cdot PU_{\exp}(\lambda_x) = \exp \left( \frac{(1 - \varepsilon)^2(w_x + 1)^2 - (\varepsilon + xy/\sigma^2)^2 - (1 - \varepsilon^2)}{2(1 - \varepsilon)y^2/\sigma^2} \right),
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Recall:
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P_\alpha(\eta; x) := \inf_{t \in (-\infty, x)} \frac{E(\eta - t)^\alpha}{(x - t)^\alpha}.
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An efficient procedure to compute \( P_\alpha(\eta; x) \) in general was given in Pinelis '98.

In the case of \( \text{Be}(x) = P_2(y\tilde{\eta}\sigma^2/y^2; x) \), this general procedure can be much simplified. Indeed, if \( \alpha \) is natural and \( \cdots < d_k < d_{k+1} < \cdots \) are the atoms of the distr. of \( \eta \), then \( E(\eta - t)^\alpha_+ \) can be easily expressed for \( t \in [d_k, d_{k+1}) \) in terms of the truncated moments \( E(\eta - d_k)^j_+ \) with \( j = 0, \ldots, \alpha \).
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A good way to compute $\text{Pin}(x)$ turns out to be to express the positive-part moments $E(\eta - t)_+^\alpha$ for $\eta = \Gamma(1-\varepsilon)\sigma^2 + y\tilde{\Pi}_\varepsilon\sigma^2/y^2$ in terms of the Fourier or Fourier-Laplace transform of the distribution of $\eta$. Such expressions were developed in Pinelis ’09 (with this specific motivation in mind). A reason for this approach to work is that the Fourier-Laplace transform of the distribution of the r.v. $\Gamma(1-\varepsilon)\sigma^2 + y\tilde{\Pi}_\varepsilon\sigma^2/y^2$ has a simple expression.
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\[ E X_+^p = \frac{\Gamma(p + 1)}{\pi} \int_0^\infty \Re \frac{E e_j((s + it)X)}{(s + it)^{p+1}} \, dt, \]

where \( p \in (0, \infty), \, s \in (0, \infty), \) \( \Gamma \) is the Gamma function, \( \Re z := \) the real part of \( z, \) \( i = \sqrt{-1}, \) \( j = -1, 0, \ldots, \ell, \)
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Also,

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Comparison

Compare the bounds BH, PU, Be, and Pin, and also the Cantelli bound
\[ Ca(x) := Ca_{\sigma^2}(x) := \frac{\sigma^2}{\sigma^2 + x^2} \]
and the best exp. bound
\[ EN(x) := EN_{\sigma^2}(x) \inf_{\lambda > 0} e^{-\lambda x} E e^{\lambda \Gamma \sigma^2} = \exp \left\{ -\frac{x^2}{2\sigma^2} \right\} \]
on the tail of \( N(0, \sigma^2) \); of course, in general \( EN(x) \) is not an upper bound on \( P(S \geq x) \).
The bound \( Ca(x) \) is optimal in its own terms.

Proposition

Take any \( x \in [0, \infty) \), \( \sigma \in (0, \infty) \), and r.v.’s \( \xi \) and \( \eta \) s.t. \( E \xi \leq 0 = E \eta \) and \( E \xi^2 \leq E \eta^2 = \sigma^2 \). Then
\[ P(\xi \geq x) \leq Ca(x) = \inf_{t \in (-\infty, x)} \frac{E(\eta - t)^2}{(x - t)^2}. \]
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**Proposition**

Take any $x \in [0, \infty)$, $\sigma \in (0, \infty)$, and r.v.'s $\xi$ and $\eta$ s.t. $E \xi \leq 0 = E \eta$ and $E \xi^2 \leq E \eta^2 = \sigma^2$. Then

$$P(\xi \geq x) \leq Ca(x) = \inf_{t \in (-\infty, x)} \frac{E(\eta - t)^2}{(x - t)^2}.$$
Comparison

Compare the bounds BH, PU, Be, and Pin, and also the Cantelli bound

\[ Ca(x) := Ca_\sigma^2(x) := \frac{\sigma^2}{\sigma^2 + x^2} \]

and the best exp. bound

\[ EN(x) := EN_\sigma^2(x) \inf_{\lambda > 0} e^{-\lambda x} E e^{\lambda \Gamma \sigma^2} = \exp \left\{ - \frac{x^2}{2\sigma^2} \right\} \]

on the tail of \( N(0,\sigma^2) \); of course, in general \( EN(x) \) is not an upper bound on \( P(S \geq x) \).

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Proposition

For all $x > 0$, $\sigma > 0$, $y > 0$, and $\varepsilon \in (0, 1)$,

1. $\text{Pin}(x) \leq \text{PU}(x) \leq \text{BH}(x)$ and $\text{Be}(x) \leq \text{Ca}(x) \land \text{BH}(x)$;
2. $\text{Be}(x) = \text{Ca}(x)$ for all $x \in [0, y]$;
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4. $\exists u_{y/\sigma} \in (0, \infty)$ s.t. $\text{Ca}(x) < \text{BH}(x)$ if $x \in (0, \sigma u_{y/\sigma})$ and $\text{Ca}(x) > \text{BH}(x)$ if $x \in (\sigma u_{y/\sigma}, \infty)$; moreover, $u_{y/\sigma}$ incr. from $u_{0+} = 1.585 \ldots$ to $\infty$ as $y/\sigma$ incr. from 0 to $\infty$; in particular, $\text{Ca}(x) < \text{EN}(x)$ if $x/\sigma \in (0, 1.585)$ and $\text{Ca}(x) > \text{EN}(x)$ for $x/\sigma \in (1.586, \infty)$.
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For all \( x > 0, \sigma > 0, y > 0, \) and \( \varepsilon \in (0, 1), \)

(I) \( \text{Pin}(x) \leq \text{PU}(x) \leq \text{BH}(x) \) and \( \text{Be}(x) \leq \text{Ca}(x) \land \text{BH}(x); \)

(II) \( \text{Be}(x) = \text{Ca}(x) \) for all \( x \in [0, y]; \)

(III) \( \text{BH}(x) \) increases from \( \text{EN}(x) \) to 1 as \( y \) increases from 0 to \( \infty; \)

(IV) \( \exists u_{y/\sigma} \in (0, \infty) \) s.t. \( \text{Ca}(x) < \text{BH}(x) \) if \( x \in (0, \sigma u_{y/\sigma}) \) and \( \text{Ca}(x) > \text{BH}(x) \) if \( x \in (\sigma u_{y/\sigma}, \infty); \) moreover, \( u_{y/\sigma} \) incr. from \( u_{0+} = 1.585 \ldots \) to \( \infty \) as \( y/\sigma \) incr. from 0 to \( \infty; \) in particular, \( \text{Ca}(x) < \text{EN}(x) \) if \( x/\sigma \in (0, 1.585) \) and \( \text{Ca}(x) > \text{EN}(x) \) for \( x/\sigma \in (1.586, \infty). \)

(V) \( \text{PU}(x) \) incr. from \( \text{EN}(x) \) to \( \text{BH}(x) \) as \( \varepsilon \) incr. from 0 to 1.
Proposition

For all \( x > 0, \sigma > 0, y > 0, \) and \( \varepsilon \in (0, 1) \),

(I) \( \text{Pin}(x) \leq \text{PU}(x) \leq \text{BH}(x) \) and \( \text{Be}(x) \leq \text{Ca}(x) \land \text{BH}(x) \);

(II) \( \text{Be}(x) = \text{Ca}(x) \) for all \( x \in [0, y] \);

(III) \( \text{BH}(x) \) increases from \( \text{EN}(x) \) to 1 as \( y \) increases from 0 to \( \infty \);

(IV) \( \exists u_{y/\sigma} \in (0, \infty) \) s.t. \( \text{Ca}(x) < \text{BH}(x) \) if \( x \in (0, \sigma u_{y/\sigma}) \) and \( \text{Ca}(x) > \text{BH}(x) \) if \( x \in (\sigma u_{y/\sigma}, \infty) \); moreover, \( u_{y/\sigma} \) incr. from \( u_{0+} = 1.585 \ldots \) to \( \infty \) as \( y/\sigma \) incr. from 0 to \( \infty \); in particular, \( \text{Ca}(x) < \text{EN}(x) \) if \( x/\sigma \in (0, 1.585) \) and \( \text{Ca}(x) > \text{EN}(x) \) for \( x/\sigma \in (1.586, \infty) \).

(V) \( \text{PU}(x) \) incr. from \( \text{EN}(x) \) to \( \text{BH}(x) \) as \( \varepsilon \) incr. from 0 to 1.
Proposition

For all $\sigma > 0$, $y > 0$, $\varepsilon \in (0, 1)$, and $x > 0$

$$PU(x) = \max_{\alpha \in (0, 1)} EN_{(1-\varepsilon)\sigma^2}((1 - \alpha)x) BH_{\varepsilon\sigma^2, y}(\alpha x)$$

$$= EN_{(1-\varepsilon)\sigma^2}((1 - \alpha_x)x) BH_{\varepsilon\sigma^2, y}(\alpha_x x),$$

where $\alpha_x$ is the only root in $(0, 1)$ of the equation

$$\frac{(1-\alpha)x^2}{(1-\varepsilon)\sigma^2} - \frac{x}{y} \ln \left(1 + \frac{\alpha xy}{\varepsilon\sigma^2}\right) = 0.$$ 

Moreover, $\alpha_x$ incr. from $\varepsilon$ to 1 as $x$ incr. from 0 to $\infty$.

So, the bound $PU(x)$ is the product of the best exp. upper bounds on the tails $P(\Gamma_{(1-\varepsilon)\sigma^2} \geq (1 - \alpha)x)$ and $P(\Pi_{\varepsilon\sigma^2} \geq \alpha x) —$ for some $\alpha \in (0, 1)$ (in fact, the $\alpha \in (\varepsilon, 1)$).

This proposition is useful in establishing asymptotics of $PU(x)$. 
Proposition

For all $\sigma > 0$, $y > 0$, $\varepsilon \in (0, 1)$, and $x > 0$

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PU(x) = \max_{\alpha \in (0,1)} EN_{(1-\varepsilon)\sigma^2}((1 - \alpha)x) \ BH_{\varepsilon\sigma^2,y}(\alpha x)
\]

\[
= EN_{(1-\varepsilon)\sigma^2}((1 - \alpha_x)x) \ BH_{\varepsilon\sigma^2,y}(\alpha_x x),
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So, the bound $PU(x)$ is the product of the best exp. upper bounds on the tails $P \left( \Gamma_{(1-\varepsilon)\sigma^2} \geq (1 - \alpha)x \right)$ and $P \left( \tilde{\Pi}_{\varepsilon\sigma^2} \geq \alpha x \right)$ — for some $\alpha \in (0, 1)$ (in fact, the $\alpha \in (\varepsilon, 1)$). This proposition is useful in establishing asymptotics of $PU(x)$. 
## Proposition

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So, the bound $PU(x)$ is the product of the best exp. upper bounds on the tails $P\left(\Gamma_{(1-\varepsilon)\sigma^2} \geq (1 - \alpha)x\right)$ and $P\left(\tilde{\Pi}_{\varepsilon\sigma^2} \geq \alpha x\right)$ — for some $\alpha \in (0, 1)$ (in fact, the $\alpha \in (\varepsilon, 1)$).

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For all $\sigma > 0$, $y > 0$, $\varepsilon \in (0, 1)$, and $x > 0$

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For all $\sigma > 0$, $y > 0$, $\varepsilon \in (0, 1)$, and $x > 0$

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PU(x) = \max_{\alpha \in (0,1)} EN_{(1-\varepsilon)\sigma^2}((1 - \alpha)x) BH_{\varepsilon\sigma^2,y}(\alpha x) \\
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## Proposition

For all $\sigma > 0$, $y > 0$, $\varepsilon \in (0, 1)$, and $x > 0$

$$PU(x) = \max_{\alpha \in (0,1)} EN(1-\varepsilon)^{\sigma^2}((1-\alpha)x) BH_{\varepsilon\sigma^2,y}(\alpha x)$$

$$= EN(1-\varepsilon)^{\sigma^2}((1-\alpha_x)x) BH_{\varepsilon\sigma^2,y}(\alpha_x x),$$

where $\alpha_x$ is the only root in $(0, 1)$ of the equation

$$\frac{(1-\alpha)x^2}{(1-\varepsilon)^{\sigma^2}} - \frac{x}{y} \ln \left(1 + \frac{\alpha xy}{\varepsilon \sigma^2}\right) = 0.$$

Moreover, $\alpha_x$ incr. from $\varepsilon$ to 1 as $x$ incr. from 0 to $\infty$.

So, the bound $PU(x)$ is the product of the best exp. upper bounds on the tails $P(\Gamma_{(1-\varepsilon)^{\sigma^2}} \geq (1-\alpha)x)$ and $P(\tilde{\Pi}_{\varepsilon\sigma^2} \geq \alpha x)$ — for some $\alpha \in (0, 1)$ (in fact, the $\alpha \in (\varepsilon, 1)$). This proposition is useful in establishing asymptotics of $PU(x)$. 
**Proposition**

For all $\sigma > 0$, $y > 0$, $\varepsilon \in (0, 1)$, and $x > 0$

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PU(x) = \max_{\alpha \in (0, 1)} EN_{(1-\varepsilon)\sigma^2}((1 - \alpha)x) BH_{\varepsilon\sigma^2, y}(\alpha x)
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So, the bound $PU(x)$ is the product of the best exp. upper bounds on the tails $P\left(\Gamma_{(1-\varepsilon)\sigma^2} \geq (1 - \alpha)x\right)$ and $P\left(\tilde{\Pi}_{\varepsilon\sigma^2} \geq \alpha x\right)$ — for some $\alpha \in (0, 1)$ (in fact, the $\alpha \in (\varepsilon, 1)$).

This proposition is useful in establishing asymptotics of $PU(x)$. 
Proposition

For all \( \sigma > 0, \ y > 0, \ \varepsilon \in (0, 1), \) and \( x > 0 \)

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PU(x) = \max_{\alpha \in (0,1)} EN_{(1-\varepsilon)\sigma^2}((1 - \alpha)x) BH_{\varepsilon\sigma^2,y}(\alpha x)
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\]

Moreover, \( \alpha_x \) incr. from \( \varepsilon \) to 1 as \( x \) incr. from 0 to \( \infty \).

So, the bound \( PU(x) \) is the product of the best exp. upper bounds on the tails \( P \left( (1-\varepsilon)\sigma^2 \geq (1 - \alpha)x \right) \) and
\( P \left( \tilde{\Pi}_{\varepsilon\sigma^2} \geq \alpha x \right) \) — for some \( \alpha \in (0, 1) \) (in fact, the \( \alpha \in (\varepsilon, 1) \)).

This proposition is useful in establishing asymptotics of \( PU(x) \).
Proposition

For all $\sigma > 0$, $y > 0$, $\varepsilon \in (0, 1)$, and $x > 0$

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PU(x) = \max_{\alpha \in (0,1)} EN(1-\varepsilon)\sigma^2((1 - \alpha)x) BH_{\varepsilon\sigma^2,y}(\alpha x)
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Comparison: asymptotics for large $x > 0$

**Proposition**

For any fixed $\sigma > 0$, $y > 0$, and $\varepsilon \in (0, 1)$, and all $x \geq 0$

\[
\text{Pin}(x) \leq \text{PU}(x) = (\varepsilon + o(1))^{x/y} \text{Be}(x) \leq (\varepsilon + o(1))^{x/y} \text{BH}(x)
\]

as $x \to \infty$.

That is, for large $x$, the bound $\text{PU}(x)$ and, hence, the better bound $\text{Pin}(x)$ are each exponentially better than $\text{Be}(x)$ and hence than $\text{BH}(x)$ — especially when $\varepsilon \ll 1$. 
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That is, for large $x$, the bound $\text{PU}(x)$ and, hence, the better bound $\text{Pin}(x)$ are each exponentially better than $\text{Be}(x)$ and hence than $\text{BH}(x)$ — especially when $\epsilon \ll 1$. 
Here, $\sigma$ is normalized to be 1. In the next 4 frames, the graphs $G(P) := \{(x, \log_{10} \frac{P(x)}{BH(x)}) : 0 < x \leq x_{\text{max}}\}$ for $P = \text{Ca, PU, Be, Pin}$, with the benchmark BH, will be shown, for $\varepsilon \in \{0.1, 0.9\}$, $y \in \{0.1, 1\}$, and $x_{\text{max}} = 3$ or $4$, depending on whether $y = 0.1$ (little skewed-to-the-right $X_i$'s) or $y = 1$ (much skewed-to-the-right $X_i$'s).

¶ For such choices of $x_{\text{max}}$, the values of $BH(x_{\text{max}}) \approx 0.016$ or $0.017$, whether $y = 0.1$ or $y = 1$.

¶ $G(\text{Ca})$ is shown only on the interval $(0, u_y)$, on which $\text{Ca} < BH$, i.e., $\log_{10} \frac{\text{Ca}}{BH} < 0$.

¶ For $y = 1$, $\text{Ca}(x) < BH(x)$ for all $x \in (0, 2.66)$.

¶ For Pin, actually two approx. graphs are shown: the dashed and thin solid lines – produced using the Fourier-Laplace and Fourier formulas.
Here, $\sigma$ is normalized to be 1. In the next 4 frames, the graphs $G(P) := \{(x, \log_{10} \frac{P(x)}{BH(x)}) : 0 < x \leq x_{\text{max}}\}$ for $P = \text{Ca}, \text{PU}, \text{Be}, \text{Pin}$, with the benchmark BH, will be shown, for $\varepsilon \in \{0.1, 0.9\}$, $y \in \{0.1, 1\}$, and $x_{\text{max}} = 3$ or $4$, depending on whether $y = 0.1$ (little skewed-to-the-right $X_i$'s) or $y = 1$ (much skewed-to-the-right $X_i$'s).

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For Pin, actually two approx. graphs are shown: the dashed and thin solid lines – produced using the Fourier-Laplace and Fourier formulas.
Graphic comparison for moderate deviations: $x \in [0, 3]$ or $x \in [0, 4]$

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On the BH ineq.

Outline

Introduction

Main results

Sketch of proof

Computation of bounds

Comparison of bounds

Graphic comparison for moderate deviations: $x \in [0, 3]$ or $x \in [0, 4]$

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¶ For Pin, actually two approx. graphs are shown: the dashed and thin solid lines – produced using the Fourier-Laplace and Fourier formulas.
Comparison: $x \in [0, 4], \varepsilon = 0.1, y = 1$

If the weight of the Poisson component is small ($\varepsilon = 0.1$) and the Poisson component is quite distinct from the Gaussian component ($y = 1$), then $\text{Be}(x)$ is about 9.93 times worse (i.e., greater) than $\text{Pin}(x)$ at $x = 4$. Moreover, for these values of $\varepsilon$ and $y$, even the bound $\text{PU}(x)$ is better than $\text{Be}(x)$ already at about $x = 2.5$. 

$\text{BH} \sim 0$

$(\text{BH}(4) \approx 0.017)$

$\text{Ca}$

$\text{Be}$

$\text{PU}$

$\text{Pin}$
Comparison: \( x \in [0, 4], \varepsilon = 0.1, y = 1 \)

If the weight of the Poisson component is small (\( \varepsilon = 0.1 \)) and the Poisson component is quite distinct from the Gaussian component (\( y = 1 \)), then \( \text{Be}(x) \) is about 9.93 times worse (i.e., greater) than \( \text{Pin}(x) \) at \( x = 4 \). Moreover, for these values of \( \varepsilon \) and \( y \), even the bound \( \text{PU}(x) \) is better than \( \text{Be}(x) \) already at about \( x = 2.5 \).
Comparison: $x \in [0, 4]$, $\varepsilon = 0.1$, $y = 1$

If the weight of the Poisson component is small ($\varepsilon = 0.1$) and the Poisson component is quite distinct from the Gaussian component ($y = 1$), then $\text{Be}(x)$ is about 9.93 times worse (i.e., greater) than $\text{Pin}(x)$ at $x = 4$. Moreover, for these values of $\varepsilon$ and $y$, even the bound $\text{PU}(x)$ is better than $\text{Be}(x)$ already at about $x = 2.5$. 

$\text{BH} \sim 0$

$(\text{BH}(4) \approx 0.017)$

$\text{Be}$

$\text{PU}$

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Comparison: $x \in [0, 4]$, $\varepsilon = 0.1$, $y = 1$

If the weight of the Poisson component is small ($\varepsilon = 0.1$) and the Poisson component is quite distinct from the Gaussian component ($y = 1$), then $\text{Be}(x)$ is about 9.93 times worse (i.e., greater) than $\text{Pin}(x)$ at $x = 4$. Moreover, for these values of $\varepsilon$ and $y$, even the bound $\text{PU}(x)$ is better than $\text{Be}(x)$ already at about $x = 2.5$. 

$\text{BH} \sim 0$

$(\text{BH}(4) \approx 0.017)$

$\text{Pu}$

$\text{Be}$

$\text{Pu}$

$\text{Pin}$
Comparison: $x \in [0, 4], \varepsilon = 0.1, y = 1$

If the weight of the Poisson component is small ($\varepsilon = 0.1$) and the Poisson component is quite distinct from the Gaussian component ($y = 1$), then $\text{Be}(x)$ is about 9.93 times worse (i.e., greater) than $\text{Pin}(x)$ at $x = 4$. Moreover, for these values of $\varepsilon$ and $y$, even the bound $\text{PU}(x)$ is better than $\text{Be}(x)$ already at about $x = 2.5$. 

$\text{BH} \sim 0$

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$\text{Ca}$

$\text{Be}$

$\text{PU}$

$\text{Pin}$
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$\text{BH} \sim 0$

$(\text{BH}(4) \approx 0.017)$

$\text{Ca}$

$\text{Be}$

$\text{PU}$

$\text{Pin}$
Comparison: $x \in [0, 3]$, $\varepsilon = 0.1$, $y = 0.1$

If the weight of the Poisson component is small ($\varepsilon = 0.1$) and the Poisson component is close to the Gaussian component ($y = 0.1$), then $Be(x)$ is still about 20% greater than $Pin(x)$ at $x = 3$. 

\begin{align*}
BH & \sim 0 \\
(BH(4) & \approx 0.016) \\
Ca & \\
Be & \\
PU & \\
Pin & 
\end{align*}
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Comparison: $x \in [0, 3]$, $\varepsilon = 0.1$, $y = 0.1$
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$\text{BH} \sim 0$

($\text{BH}(4) \approx 0.016$)

$\text{Ca}$

$\text{Be}$

$\text{PU}$

$\text{Pin}$
Comparison: $x \in [0, 3]$, $\varepsilon = 0.1$, $y = 0.1$

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$\text{BH} \sim 0$

$\text{BH}(4) \approx 0.016$
Comparison: \( x \in [0, 4], \varepsilon = 0.9, y = 1 \)

If the weight of the Poisson component is large \((\varepsilon = 0.9)\) and the Poisson component is quite distinct from the Gaussian component \((y = 1)\), then Be\((x)\) is about 8\% better than Pin\((x)\) at \(x = 4\). For \(x \in [0, 4]\), Pin\((x)\) and Be\((x)\) are close to each other and both are significantly better than either BH\((x)\) or PU\((x)\) (which latter are also close to each other).
Comparison: $x \in [0, 4]$, $\varepsilon = 0.9$, $y = 1$

If the weight of the Poisson component is large ($\varepsilon = 0.9$) and the Poisson component is quite distinct from the Gaussian component ($y = 1$), then $Be(x)$ is about 8% better than $Pin(x)$ at $x = 4$. For $x \in [0, 4]$, $Pin(x)$ and $Be(x)$ are close to each other and both are significantly better than either $BH(x)$ or $PU(x)$ (which latter are also close to each other).
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Comparison: $x \in [0, 4], \varepsilon = 0.9, y = 0.1$

If the weight of the Poisson component is large ($\varepsilon = 0.9$) and the Poisson component is close to the Gaussian component ($y = 0.1$), then $\text{Be}(x)$ is about 12% better than $\text{Pin}(x)$ at $x = 3$. For $x \in [0, 3]$, $\text{Pin}(x)$ and $\text{Be}(x)$ are close to each other and both are significantly better than either $\text{BH}(x)$ or $\text{PU}(x)$ (which latter are very close to each other).
Comparison: $x \in [0, 4], \varepsilon = 0.9, y = 0.1$

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Comparison: $x \in [0, 4]$, $\varepsilon = 0.9$, $y = 0.1$

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Comparison: $x \in [0, 4], \varepsilon = 0.9, y = 0.1$

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Comparison: \( x \in [0, 4], \varepsilon = 0.9, y = 0.1 \)

If the weight of the Poisson component is large (\( \varepsilon = 0.9 \)) and the Poisson component is close to the Gaussian component (\( y = 0.1 \)), then \( \text{Be}(x) \) is about 12% better than \( \text{Pin}(x) \) at \( x = 3 \). For \( x \in [0, 3] \), \( \text{Pin}(x) \) and \( \text{Be}(x) \) are close to each other and both are significantly better than either \( \text{BH}(x) \) or \( \text{PU}(x) \) (which latter are very close to each other).
30/32  Comparison: $x \in [0, 4], \varepsilon = 0.9, y = 0.1$

If the weight of the Poisson component is large ($\varepsilon = 0.9$) and the Poisson component is close to the Gaussian component ($y = 0.1$), then $\text{Be}(x)$ is about 12% better than $\text{Pin}(x)$ at $x = 3$. For $x \in [0, 3]$, $\text{Pin}(x)$ and $\text{Be}(x)$ are close to each other and both are significantly better than either $\text{BH}(x)$ or $\text{PU}(x)$ (which latter are very close to each other).
Row 1: \( \varepsilon = 0.1 \): heavy-tail Poisson component of little weight
Row 2: \( \varepsilon = 0.9 \): heavy-tail Poisson component of large weight
Column 1: \( y = 1 \): distrs. of the \( X_i \)'s may be much skewed to the right
Column 2: \( y = 0.1 \): distrs. of the \( X_i \)'s may be only a little skewed to the right.
Row 1: $\varepsilon = 0.1$: heavy-tail Poisson component of little weight
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Comparison: Graphics grid

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Thank you!