A coupled BEM and FEM for the interior transmission problem

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Abstract

The interior transmission problem (ITP) is a boundary value problem arising in inverse scattering theory which has important applications in qualitative methods. In this paper, we propose a coupled boundary element and finite element method for the ITP. The coupling procedure is realized by applying the direct boundary integral equation method to define the so-called Dirichlet-to-Neumann (DtN) mappings. We show the existence of the solution to the ITP for anisotropic medium. Numerical examples are provided to illustrate the practicability of the coupling procedure.

1 Introduction

The interior transmission problem is a boundary value problem introduced in the inverse scattering theory for the study of the far field patterns for transmission problems [4, 8]. It arises in the scattering of time-harmonic waves by an inhomogeneous medium of compact support. The interior transmission problem and the associated transmission eigenvalue problem have attracted a lot of attention recently [6, 9, 14]. This is mainly due to their importance in the qualitative methods, such as the linear sampling method. In addition, the transmission eigenvalues can be determined from the far field pattern and be used to obtain estimates of the physical properties, such as the index of refraction, of the scattering object.

The ITP is a non-standard partial differential equation (PDE). The problem seems new and has not been covered by the classical PDE theory. In this paper, we consider the ITP for acoustic wave scattering by the anisotropic medium. In particular, we propose a coupled boundary element and finite element method for the ITP. In spite of many papers devoted to the theory of the ITP and the associated transmission eigenvalues, the study of numerical methods for them are quite limited. In this regard, some finite element methods to compute the transmission eigenvalues are available in [5, 15]. While we focus on the interior transmission problem for the acoustic waves, we refer the reader to

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[2, 10] and references therein for the interior transmission problem in vector case, i.e., Maxwell's equations.

The paper is organized as follows. In Section 2 we present the ITP for an anisotropic medium and point out its importance in the qualitative methods in inverse scattering theory. In Section 3 we introduce the DtN mappings and give a boundary integral formulation for the ITP. The existence of the solution is shown in Section 4 using the Fredholm alternative. In Section 5, we present a coupled boundary element and finite element method for the ITP and present some preliminary numerical experiments to demonstrate the practicability of the coupling procedure.

2 The ITP for an anisotropic medium

Let $D \subset \mathbb{R}^2$ be an open bounded domain with a C^2 boundary $\Gamma = \partial D$. Let A be a symmetric matrix value function in $\bar{D}$ such that $\bar{\xi} \cdot \mathrm{Im}(A)\xi \leq 0$ and $\bar{\xi} \cdot \mathrm{Re}(A)\xi \geq \gamma |x|^2$ for all $\xi \in \mathbb{C}^2$ and $x \in \bar{D}$ where $\gamma > 0$. For a function $u \in C^1(\bar{D})$ the conormal derivative is defined by

$$\frac{\partial u}{\partial \nu_A}(x) := v(x) \cdot A(x) \nabla u(x), \quad x \in \Gamma$$

where ν is the unit outward normal to Γ. The direct scattering problem for an anisotropic medium is to find functions $w \in H^1(D)$, $u \in H^1_{loc}(\mathbb{R}^2 \setminus \bar{D})$ such that

$$\nabla \cdot A \nabla w + k^2 n w = 0 \quad \text{in } D, \tag{2.1a}$$

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^2 \setminus \bar{D}, \tag{2.1b}$$

$$w - u = f \quad \text{on } \Gamma, \tag{2.1c}$$

$$\frac{\partial w}{\partial \nu_A} - \frac{\partial u}{\partial \nu} = g \quad \text{on } \Gamma, \tag{2.1d}$$

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u}{\partial r} - iku \right) = 0, \tag{2.1e}$$

where $k > 0$ is the wavenumber, $r = |x|$, $f := e^{ikx \cdot d}$ and $g := (\partial / \partial \nu) e^{ikx \cdot d}$, $d \in \Omega = \{x \in \mathbb{R}^2 : |x| = 1\}$. Here $n = n(x)$ is the index of refraction of the medium in D. We assume that $n - 1$ has compact support $\bar{D}$, $n(x) > 0, x \in \bar{D}$, and $n(x) \in L^\infty(D)$. The Sommerfeld radiation condition (2.1e) is assumed to hold uniformly in all direction by the scattered field u.

It is well known that the field u has the asymptotic behavior

$$u(x) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\hat{x}, d) + O(r^{-3/2}) \quad r \to \infty$$

where $\hat{x} = x/|x|$ and u_∞ is called the far field pattern of u. Assuming $g \in L^2(\Omega)$, the far field operator $F : L^2(\Omega) \to L^2(\Omega)$ is defined by

$$(Fg)(\hat{x}) := \int_\Omega u_\infty(\hat{x}, d) g(d) \, \mathrm{d}s(d). \tag{2.2}$$

It can be shown that $(Fg)(\hat{x})$ is the far field pattern of u in (2.1) due to the incident field given by the Herglotz wave function v_g:

$$v_g(x) := \int_{\Omega} e^{ikx\cdot d} g(d)\,\mathrm{d}s(d). \tag{2.3}$$

The associated interior transmission problem can be stated as follows. Given $f \in H^{1/2}(\Gamma)$ and $g \in H^{-1/2}(\Gamma)$, find two functions $v \in H^1(D)$ and $w \in H^1(D)$ satisfying

$$\nabla \cdot A\nabla w + k^2 n(x) w = 0 \qquad \text{in } D, \tag{2.4a}$$

$$\Delta v + k^2 v = 0 \qquad \text{in } D, \tag{2.4b}$$

$$w - v = f \qquad \text{on } \Gamma, \tag{2.4c}$$

$$\frac{\partial w}{\partial \nu_A} - \frac{\partial v}{\partial \nu} = g \qquad \text{on } \Gamma. \tag{2.4d}$$

As in [1], we assume that the ellipticity constant $\gamma > 1$ for $x \in D$. The importance of the interior transmission problem lies in the fact that it serves as an important tool for studying the kernel of the far field operator [6]. Although we shall not discuss here known as the interior transmission eigenvalues, we would like to state its definition here.

Definition 2.1. *If $k > 0$ is such that the homogeneous interior transmission problem has a nontrivial solution, than k is called a transmission eigenvalue.*

3 A coupled BIE formulation

In this section, we will reduce the interior transmission problem (2.4a)–(2.4d) to two non-local boundary value problems in D in terms of two different forms of the Dirichlet to Neumann (DtN) operator.

3.1 Dirichlet-to-Neumann mappings

To construct the DtN mappings on Γ, we first consider the following interior Dirichlet problem: Given $\phi \in H^{1/2}(\Gamma)$, find $v \in H^1(D)$ satisfying

$$\Delta v + k^2 v = 0 \qquad \text{in } D, \tag{3.5a}$$

$$v = \phi, \qquad \text{on } \Gamma. \tag{3.5b}$$

We state without proof of the following uniqueness result [3].

Theorem 3.1. *Let* $\mathrm{Im}(k) > 0$*. Then the interior Dirichlet problem* (3.5) *has at most one solution.*

As is well-known the solution v of the boundary value problem (3.5) can be represented by the Green's representation formula in terms of the fundamental solution $E(x,y)$ for the two-dimensional Helmholtz equation,

$$E(x,y) = \frac{i}{4} H_0^{(1)}(k|x-y|), \tag{3.6}$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order zero. This representation assumes the form

$$v(x) := \int_\Gamma E(x,y)\sigma(y)\,ds_y - \int_{\partial D} \frac{\partial E(x,y)}{\partial \nu_y}\mu(y)\,ds_y, \quad \forall\, x \in D, \tag{3.7}$$

where

$$\mu(y) = v^-, \quad \sigma(y) = \frac{\partial v}{\partial \nu_y}^{-}$$

denote the Cauchy data on Γ for the solution v. Here and in the sequel, we write w^- for the boundary limit of any function or distribution w defined in D. Letting x in (3.7) approach to the boundary Γ, and employing the jump conditions, we obtain the boundary integral equation (BIE), which relates the Cauchy data σ and μ,

$$V\sigma(x) = (\frac{1}{2}I + K)\mu(x), \quad \forall\, x \in \Gamma. \tag{3.8}$$

Here, I stands for the identity operator, and V and K are basic simple- and double-layer boundary integral operators defined by

$$V\sigma(x) \quad = \quad \int_\Gamma E(x,y)\sigma(y)ds_y, \quad \forall\, x \in \Gamma, \tag{3.9}$$

$$K\mu(x) \quad = \quad \int_\Gamma \frac{\partial E(x,y)}{\partial \nu_y}\mu(y)ds_y, \quad \forall\, x \in \Gamma, \tag{3.10}$$

respectively. Prior to defining the first DtN mapping (also known as the *Steklov–Poincaré* operator) in terms of boundary integral operators, we state some of the mapping properties for the boundary integral operators V and K on *Sobolev* spaces $H^s(\Gamma)$, for $s = -1/2, 1/2$ (see [11]). For sufficiently smooth boundary Γ, we have

1. The simple-layer boundary integral operator V is an isomorphism from $H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$, provided k^2 is not an eigenvalue for the interior Dirichlet problem for the negative Laplacian $-\Delta$ in D.

2. The double-layer boundary integral operator K is a continuous mapping from $H^{1/2}(\Gamma) \to H^{3/2}(\Gamma)$. In particular, due to the embedding theorem, the operator K is compact from $H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$.

Applying the properties of boundary integral operators V and K, we now define the first DtN mapping $T : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ by

$$T : \varphi \to T\varphi := V^{-1}(\frac{1}{2}I + K)\varphi(x)\ , \quad \forall\, \varphi \in H^{1/2}(\Gamma). \tag{3.11}$$

It is worthy mentioning that the reduction of the boundary value problem (3.5) to an equivalent boundary integral equation is generally by no means a unique process. Alternatively, we define another DtN mapping by employing the symmetric coupling procedure (Costabel [7]). Note that other formulations are also possible. For instance, Marin et al. [13] employed the indirect method in terms of layer ansatz to define a DtN mapping on Γ. By computing the normal derivative for both sides of representation formula (3.7) and taking the limits as $x \to \Gamma$, we arrive at the second boundary integral equation

$$(3.12) \qquad \sigma(x) = (\frac{1}{2}I + K')\sigma(x) + W\mu(x), \quad \forall\, x \in \Gamma,$$

where K' is the transpose of K in (3.10), and W is the hypersingular integral operator defined respectively by

$$(3.13) \qquad K^{'}\sigma(x) = \int_\Gamma \frac{\partial E(x,y)}{\partial \nu_x}\sigma(y)ds_y, \quad \forall\, x \in \Gamma,$$

$$(3.14) \qquad W\mu(x) = -\frac{\partial}{\partial \nu_x}\int_\Gamma \frac{\partial E(x,y)}{\partial \nu_y}\mu(y)ds_y, \quad \forall\, x \in \Gamma.$$

Similarly, the mapping properties of K' and W are well-known (see e.g., [11, 12]),

1. The boundary integral operator $K^{'}$ is a continuous mapping from $H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$.

2. The hypersingular boundary integral operator W is a continuous mapping from $H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$.

In terms of K' and W, in addition to V and K, we may now define an alternative DtN mapping $T : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ as

$$(3.15) \quad T : \varphi \to T\varphi := (\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K)\varphi(x) + W\varphi(x), \quad \forall\, \varphi \in H^{1/2}(\Gamma),$$

provided V is invertible.

We summarize the results in the following theorem.

Theorem 3.2. *The DtN mapping T in* (3.11) *or* (3.15) *is a bounded linear operator from $H^{1/2}(\Gamma)$ to $H^{-1/2}(\Gamma)$, provided k^2 is not an eigenvalue of the interior Dirichlet problem for the negative Laplacian $-\Delta$ in D.*

3.2 Non-local boundary value problem for w

Now with the DtN mapping T available for the interior Dirichlet problem, from the transmission conditions (2.4c) and (2.4d), we may replace ϕ in (3.5b) by $w^- - f$ to eliminate v. The ITP can be reduced to a non-local boundary value problem for w consisting of (2.4a), namely

$$(3.16) \qquad \nabla \cdot A\nabla w + k^2 n(x) w = 0 \quad \text{in } D$$

together with the non-local boundary condition:

$$(3.17) \qquad \frac{\partial w}{\partial \nu_A} = \frac{\partial v}{\partial \nu} + g = Tw + (g - Tf).$$

The existence of the solution of ITP then reduces to that of the non-local boundary value problem defined by (3.16) and (3.17).

4 Existence of the solution

For the existence proof, we first consider the following modified ITP (MITP)

(4.18a) $\nabla \cdot A\nabla w - mw = 0$ in D,

(4.18b) $\Delta v - k^2 v = 0$ in D,

(4.18c) $w - v = f$ on Γ,

(4.18d) $\dfrac{\partial w}{\partial \nu_A} - \dfrac{\partial v}{\partial \nu} = g$ on Γ,

where, without loss of generality, we also require that $m(x) > \gamma$ for $\gamma > 1$. First we reduce the MITP to a non-local problem for w as in the previous section for the original ITP by introducing a corresponding DtN mapping T_0. Next we show that the solution of the corresponding non-local problem for the MITP exists. Finally we show that the non-local boundary value problem (3.16) and (3.17) is a compact perturbation of the corresponding boundary value problem for the MITP. Hence uniqueness implies existence, which establishes the existence of the solution of the ITP.

For the solution of the MITP we define the DtN mapping

$$T_0\phi := V_0^{-1}\left(\frac{1}{2}I + K_0\right)\phi, \quad \text{for } \phi \in H^{1/2}(\Gamma)$$

where V_0, K_0 corresponding to the simple- and double- layer boundary integral operators in terms of the fundamental solution $\Phi_0(k|x-y|)$ of

$$-\triangle u + k^2 u = 0.$$

Then the MITP (4.18a) and (4.18b) reduces to the non-local boundary value problem consisting of (4.18a) and the non-local boundary condition

$$\frac{\partial w}{\partial \nu_A} = T_0 w + (g - T_0 f) \quad \text{on } \Gamma. \tag{4.19}$$

Now we consider the weak formulation of (4.18a) and (4.19). The corresponding sesquilinear form is

$$a(w,\bar{\tilde{w}}) := \int_D A\nabla w \cdot \nabla \bar{\tilde{w}}\, dx + \int_D m w \bar{\tilde{w}}\, dx - \langle T_0 w, \bar{\tilde{w}}\rangle_\Gamma.$$

Thus

$$\mathrm{Re}\{a(w,w)\} \geq \gamma \|\nabla w\|^2 + \mathrm{Re}\int_D m w \bar{w}\, dx - \mathrm{Re}\langle T_0 w, \bar{w}\rangle_\Gamma \quad \forall w \in H^1(D).$$

But since $m(x) \geq \gamma$ as required, and by construction, we see that

$$\int_D (\nabla w \cdot \nabla \bar{w} + w\bar{w})\, dx = \langle T_0 w, \bar{w}\rangle_\Gamma,$$

i.e.,

$$\langle T_0 w, \bar{w}\rangle_\Gamma = \|w\|^2_{H^1(D)}$$

from the Green's identity. Consequently we have

$$\begin{aligned}\mathrm{Re}\{a(w,w)\} &\geq \gamma\left(\|\nabla w\|_0^2+\|w\|_0^2\right)-\|w\|_{H^1(D)}^2\\ &= (\gamma-1)\|w\|_{H^1(D)}^2 \quad \text{for } \gamma>1.\end{aligned}$$

This implies the existence of solution of (4.18a) and (4.19). Thus we have proved the following theorem.

Theorem 4.1. *Assume that $\bar{\xi}\cdot Re(A)\xi \geq \gamma|x|^2$ for all $\xi \in \mathbb{C}^2$ and $x \in \bar{D}$ and $m(x) > \gamma$ for some $\gamma > 1$. Then there exists a unique solution to the modified interior transmission problem* (4.18).

Now we return to the non-local boundary value problem for the original ITP. The weak form for the non-local problem (4.18a) and (4.19) reads: Given $f \in H^{1/2}(\Gamma)$ and $g \in H^{-1/2}(\Gamma)$, find $w \in H^1(D)$ such that

$$a(w,\hat{w}) = \int_D \left(A\nabla w\cdot\overline{\nabla\hat{w}} - nw\bar{\hat{w}}\right)dx - \langle Tw,\bar{\hat{w}}\rangle_\Gamma = \langle g-Tf,\bar{\hat{w}}\rangle_\Gamma \tag{4.20}$$

for all $\hat{w} \in H^1(D)$. The sesquilinear form can be rewritten in the form:

$$\begin{aligned}\mathrm{Re}\{a(w,w)\} &= \int_D A\nabla w\cdot\nabla\bar{\hat{w}}\,dx + \int_D mw\bar{\hat{w}}\,dx - \langle T_0 w,\bar{\hat{w}}\rangle_\Gamma\\ &\quad -\left\{\int_D (n-m)w\hat{w}\,dx + \langle (T-T_0)w,\hat{w}\rangle_\Gamma\right\}\\ &\geq (\gamma-1)\|w\|_{H^1(D)}^2 - \mathrm{Re}(Cw,\hat{w})_{H^1(D)}.\end{aligned}$$

Here the operator $C : H^1(D) \to H^1(D)$ is defined by

$$(Cw,\hat{w})_{H^1(D)} := \int_D (n-m)w\bar{\hat{w}}\,dx + \langle (T-T_0)w,\bar{\hat{w}}\rangle_\Gamma$$

is a compact, since both terms on the right hand side are compact. This means $a(w,\hat{w})$ satisfies the Gårding's inequality in the form

$$\mathrm{Re}\left\{a(w,w)+(Cw,w)_{H^1(D)}\right\} \geq c_0\|w\|_{H^1(D)}^2 \quad \forall w \in H^1(D) \tag{4.21}$$

where $c_0 = \gamma - 1 > 0$ is a constant independent of $w \in H^1(D)$.

As a consequence, the existence follows from the standard Fredholm alternative argument, i.e., uniqueness implies existence. Thus we have established the following:

Theorem 4.2. *Assume that $\bar{\xi}\cdot Re(A)\xi \geq \gamma|x|^2$ for all $\xi \in \mathbb{C}^2$ and $x \in \bar{D}$ for some $\gamma > 1$. Let $f \in H^{1/2}(\Gamma)$ and $g \in H^{-1/2}(\Gamma)$. Then there exists a unique solution to the interior transmission problem* (2.4).

5 Numerical scheme

Finite element method is employed for the numerical solution of variational equation (4.20). To this end, we need to replace the function space $H^1(D)$ with some finite element subspace $S_h \subset H^1(D)$. The standard finite element implementation for (4.20) can be found in many books. In the following, we only briefly introduce how the boundary element method is applied.

Table 1: Relative error

h	Relative error	h	Relative error	h	Relative error
0.01	0.0103	0.001	0.0026	0.0001	0.0006

5.1 Computation of $\langle Tw, \hat{w}\rangle_\Gamma$

To find the finite element solution of (4.20), we must be able to numerically evaluate the sesquilinear form $\langle Tw, \hat{w}\rangle_\Gamma$. In the discrete formulation, this amounts to computing the integrals

$$-\int_\Gamma (T\varphi_j)\varphi_i ds, \tag{5.22}$$

where φ_i, $i = 1, 2, \cdots, N$, are basis functions of the finite element space S_h. Here, N is the degree of freedom. According to the definition of T in (3.11), we need to solve the N boundary integral equations

$$\frac{\partial \varphi_j}{\partial \nu} = T\varphi_j = V^{-1}(\frac{1}{2}I + K)\varphi_j, \quad j = 1, 2, \cdots, N, \tag{5.23}$$

and then compute the integral (5.22) using appropriate quadrature rule. In particular, we employ the Galerkin boundary element method for the numerical solution of (5.23). The computational task is formidable in the general case since one has to solve N boundary integral equations. Therefore, fast and accurate evaluation of the boundary integral equation (5.23) is of great significance for the improvement of the coupling methods. In our simulations, the finite element space consists of piecewise linear functions, and most of them vanish on the boundary Γ correspondingly eliminating the complexity of the above procedure. Finally, it is worth mentioning that the mesh size for the boundary element method may not be the same as that for the finite element method, and is usually dependent on many numerical parameters such as basis functions for the boundary element space and quadrature rules, etc. In our study, based on the piecewise linear and constant basis functions for function spaces $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$, respectively, and midpoint quadrature rules, the global optimal accuracy is obtained as the half mesh size of finite element methods is employed during the boundary element discretization.

5.2 Numerical Examples

In this section, we compute two numerical examples to illustrate the practicability of the coupling procedure. The first example provides a model for which the analytic solution is available for validating the performance of the methods. Let D be the unit disk and $k = 1$. Let $A = \text{diag}(2, 2)$ and $n(x) = 8$ on D. The Bessel's functions $w = J_0(2r)$ and $v = J_0(r)$ are the solutions of (2.4a) and (2.4b) respectively on D. Therefore $v = J_0(r)$ and $w = J_0(2r)$ solve the interior transmission problem (2.4) with $f = J_0(2) - J_0(1)$, $g = -4J_1(2) + J_1(1)$.

The relative numerical error is computed as the relative discrete L_2-error on the unit disk via

$$\text{Relative error} = \frac{\sqrt{\sum_{i=1}^{NP}(w_i - w_i^{ex})^2}}{\sqrt{\sum_{i=1}^{NP}(w_i^{ex})^2}}\ , \tag{5.24}$$

where NP is the number of points on the unit disk, w_i and w_i^{ex} are the finite element approximations and exact solutions, respectively. In Table 1 we present the relative error of the finite element approximation as a function of mesh size h. It can be observed that the increased accuracy follows the mesh refinement.

The second example is the ITP considered in Colton and Monk [4] to determine the refractive index n. Let D be the unit disk, $A = I$, $n = 4$ and f and g be given by

$$f = v - w = \frac{1}{r}e^{-ikr} \quad \text{on} \quad \Gamma,$$
$$g = \frac{\partial v}{\partial \nu} - \frac{\partial w}{\partial v} = \frac{\partial}{\partial \nu}\left(\frac{1}{r}e^{-ikr}\right) \quad \text{on} \quad \Gamma.$$

Thus

$$f = e^{-i}, \quad g = -e^{-i} + ie^{-i} \quad \text{on} \quad \Gamma.$$

Note that this case is not covered by the result in the previous section on the existence of the ITP. Nevertheless, we preform the computation and the numerical result is convergent.

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