C^0IPG for a non-standard fourth order eigenvalue problem

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Abstract. This paper concerns numerical computation of a non-standard fourth order eigenvalue problem. For high order problems, Discontinuous Galerkin methods are competitive since they avoid some difficulties arising from other approaches. We show the well-posedness of the source problem. An interior penalty discontinuous Galerkin method using Lagrange elements (C^0IPG) is proposed and its convergence is studied. The method is then used to compute the eigenvalues. We show that the method is spectrally correct and prove the optimal convergence. Numerical examples are presented to validate the theory.

AMS subject classifications: 65N25, 65N30, 65N22

Key words: fourth order eigenvalue problem, discontinuous Galerkin method, spectrum approximation.

1 Introduction

In this paper, we consider a non-standard fourth order eigenvalue problem arising in the study of transmission eigenvalues, which have important applications in inverse scattering theory [8, 20]. There exist quite a few finite element methods in the literature including the conforming elements, partition of unity finite element methods, non-conforming elements, and mixed methods. Construction of high regularity conforming elements is difficult in general [2, 10]. Moreover, they usually involve a large number of degrees of freedom. Partition of unity finite element methods [9, 21], are difficult to implement and the resulting linear systems can be severely ill-conditioned. Non-conforming methods such as Morley method do not have a good hierarchy [19] and thus

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cannot capture smooth solutions efficiently. Mixed methods may produce spurious solutions for non-convex domains [7].

As competitive alternatives, discontinuous Galerkin method become popular for high order elliptic problems [6,13,14]. In particular, in [6,13], the authors employ an interior penalty Galerkin method using Lagrange elements (C^0IPG) for the biharmonic equation. The method has a good hierarchy and there is no need to enforce the jump of the solution since the finite element space is H^1 conforming. Recently, Brenner et.al. employ the C^0IPG to compute biharmonic eigenvalues [7]. Using the classical theory of Babuška and Osborn, they prove the converge of C^0IPG for biharmonic eigenvalue problems and compare it with the Argyris element, the mixed method, and the Morley element.

In this paper, we use C^0IPG to compute a non-standard fourth order problem. Due to lower order terms, we choose to adopt the method by Antonietti et.al. [1] which in turn follows the abstract theory by Descloux, Nassif and Rappaz [11,12]. Convergence theory of the finite element methods for eigenvalue problem has been studied by many researchers since 1970s. We refer the book chapter by Babuška and Osborn [3] and the references therein for studies before 1991. For recent developments, we refer the readers to the survey paper [4].

The rest of the paper is arranged as follows. In section 2, we present a fourth order eigenvalue problem with low order terms and show the well-posedness. Section 3 describes the C^0IPG and shows convergence of the discrete problem. We develop the convergence theory in section 4. Numerical examples are given in section 5.

2 A non-standard fourth order eigenvalue problem

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$ with unit outward normal n. Let $m(x)$ be a bounded smooth function such that $m(x)>\gamma>0$ and γ,τ be positive constants. In addition, let $\|\cdot\|$ denote the L^2 norm and C,C_1,C_2 denote generic constants. We consider a non-standard fourth order eigenvalue problem of finding μ and u such that

$$(\triangle+\tau)m(x)(\triangle+\tau)u+\tau^2 u=\mu\triangle u \quad \text{in } \Omega, \tag{2.1}$$

with boundary conditions

$$u=0, \quad \frac{\partial u}{\partial n}=0, \quad \text{on } \partial\Omega. \tag{2.2}$$

The corresponding source problem can be stated as to find u such that

$$(\triangle+\tau)m(x)(\triangle+\tau)u+\tau^2 u=\triangle f, \tag{2.3}$$

with the same boundary conditions (2.2).

Let $\mathcal{A}: H_0^2(\Omega)\times H_0^2(\Omega)\to\mathbb{C}$ and $\mathcal{B}: H_0^1(\Omega)\times H_0^1(\Omega)\to\mathbb{C}$ be defined as

$$\mathcal{A}(u,v)=(m(\triangle u+\tau u),(\triangle v+\tau v))+\tau^2(u,v), \tag{2.4a}$$

$$\mathcal{B}(u,v)=(\nabla u,\nabla v). \tag{2.4b}$$

The variational formulation for the eigenvalue problem (2.1) is to find $\mu \in \mathbb{R}$ and $u \in H_0^2(\Omega)$, such that

$$\mathcal{A}(u,v) - \mu\mathcal{B}(u,v) = 0 \quad \text{for all } v \in H_0^2(\Omega). \tag{2.5}$$

The associated source problem is to find $u \in H_0^2(\Omega)$ for $f \in H_0^1(\Omega)$ such that

$$\mathcal{A}(u,v) = \mathcal{B}(f,v) \quad \text{for all } v \in H_0^2(\Omega). \tag{2.6}$$

It can be shown that $\mathcal{A}$ is a coercive sesquilinear form on $H_0^2(\Omega)\times H_0^2(\Omega)$. In fact, we have that

$$\begin{aligned}
\mathcal{A}(u,u) &\geq \gamma\|\triangle u+\tau u\|^2+\tau^2\|u\|^2 \\
&\geq \gamma\|\triangle u\|^2-2\gamma\tau\|\triangle u\|\|u\|+(\gamma+1)\tau^2\|u\|^2 \\
&= \epsilon(\tau\|u\|-\gamma/\epsilon\|\triangle u\|)^2+\gamma(1-\gamma/\epsilon)\|\triangle u\|^2+(1+\gamma-\epsilon)\tau^2\|u\|^2 \\
&\geq \gamma(1-\gamma/\epsilon)\|\triangle u\|^2+(1+\gamma-\epsilon)\tau^2\|u\|^2,
\end{aligned} \tag{2.7}$$

for any ϵ such that $\gamma < \epsilon < \gamma+1$. Moreover, since $u \in H_0^2(\Omega)$, using the Poincaré inequality, we have that

$$\|\nabla u\|^2 \leq C\|\triangle u\|^2.$$

Thus we obtain that

$$\mathcal{A}(u,u) \geq C\|u\|_{H^2}^2,$$

for some positive constant C.

For boundedness, employing the Cauchy-Schwatz inequality, we obtain that

$$\begin{aligned}
|\mathcal{A}(u,v)| &\leq C\|\triangle u+\tau u\|\|\triangle v+\tau v\|+\tau^2\|u\|\|v\| \\
&\leq C(\|\triangle u\|+\tau\|u\|)(\|\triangle v\|+\tau\|v\|)+\tau^2\|u\|\|v\| \\
&\leq C\|u\|_{H^2}\|v\|_{H^2},
\end{aligned}$$

for some constant C. We have shown the well-posedness of the source problem.

Theorem 2.1. *Let $f \in H_0^1(\Omega)$. There exists a unique solution $u \in H_0^2$ to* (2.6) *such that*

$$\|u\|_{H^2} \leq C|f|_{H^1},$$

for some constant C *independent of* u *and* f.

Due to the fact that (2.1) is a fourth order problem with lower order perturbations, there exists an $\alpha > 0$, called the index of elliptic regularity [15] such that

$$\|u\|_{H^{2+\alpha}(\Omega)} \leq C|f|_{H^1(\Omega)}, \tag{2.8}$$

where C is a constant. The elliptic index α depends on the corner of Ω. Furthermore, $\alpha \in (\frac{1}{2},1]$ for a polygonal domain and $\alpha = 1$ if Ω is convex.

3 C^0 interior penalty method (C^0IPG)

In this section, we describe the C^{0}IPG method for the source problem (2.3). Note that our formulation is different than that in [6] since we need to incooperate lower order terms. Let $\mathcal{T}_h$ be a regular triangulation for Ω and $V_h \subset H_0^1(\Omega)$ be the P_k $(k \geq 2)$ Lagrange finite element space associated with $\mathcal{T}_h$ with zero boundary condition on the boundary $\partial\Omega$.

Assuming the solution u is smooth enough, we start with the following integration by parts formula

$$\int_T \triangle(m\triangle u)v dx = \int_{\partial T} \left(\frac{\partial(m\triangle u)}{\partial n} v - m\triangle u \frac{\partial v}{\partial n} \right) ds + \int_T m\triangle u \triangle v dx. \tag{3.1}$$

Summing up (3.1) over all the triangles in $\mathcal{T}_h$, with cancelations we have

$$\sum_{T\in\mathcal{T}_h} \int_T \triangle(m\triangle u)v dx = -\sum_{T\in\mathcal{T}_h} \int_{\partial T} m\triangle u \frac{\partial v}{\partial n} ds + \sum_{T\in\mathcal{T}_h} \int_T m\triangle u \triangle v dx. \tag{3.2}$$

For an interior edge e shared by two triangles $T_\pm$ where n_e points from T_- to T_+, we define

$$\left[\!\left[\frac{\partial u}{\partial n_e} \right]\!\right] = n_e \cdot (\nabla u_+ - \nabla u_-), \quad \{\!\{ m\triangle u \}\!\} = \frac{1}{2}(m_-\triangle u_- + m_+\triangle u_+), \tag{3.3}$$

where $u_\pm = u|_{T_\pm}$.

For a boundary edge e, we take n_e to be the unit normal pointing towards the outside of Ω and define

$$[[\frac{\partial u}{\partial n_e}]] = -n_e \cdot \nabla u, \quad \{\!\{ m\triangle u \}\!\} = m\triangle u. \tag{3.4}$$

We rewrite the first term on the right-hand side of (3.2) as a sum over the edges

$$-\sum_{T\in\mathcal{T}_h} \int_{\partial T} m\triangle u \frac{\partial v}{\partial n} ds = \sum_{e\in\mathcal{E}_h} \int_e m\triangle u [[\frac{\partial v}{\partial n_e}]] ds, \tag{3.5}$$

where $\mathcal{E}_h$ is the set of all the edges of $\mathcal{T}_h$.

Replacing $m\triangle u$ in the above equation by $\{\!\{ m\triangle u \}\!\}$, introducing the symmetric term $\int_e \{\!\{ m\triangle v \}\!\} [[\frac{\partial w}{\partial n_e}]] \mathrm{d}s$, and adding the penalty term $\frac{1}{|e|}\int_e [[\frac{\partial w}{\partial n_e}]][[\frac{\partial v}{\partial n_e}]] \mathrm{d}s$, we obtain the following discrete problem: for $f \in H_0^1(\Omega)$, find $u_h \in V_h$ such that

$$\mathcal{A}_h(u_h, v) = \mathcal{B}_h(u, v) \quad \text{for all } v \in V_h, \tag{3.6}$$

where

$$\mathcal{A}_h(w,v) = a_h(w,v) + b_h(w,v) + \sigma c_h(w,v), \tag{3.7}$$

$$\mathcal{B}_h(u,v) = \sum_{T\in\mathcal{T}_h} \int_T \nabla u \cdot \nabla v dx, \tag{3.8}$$

and

$$
\begin{aligned}
a_h(w,v) &= \sum_{T\in\mathcal{T}_h}\int_T m(\triangle+\tau)w(\triangle+\tau)v+\tau^2 wv\,\mathrm{d}x,\\
b_h(w,v) &= \sum_{e\in\mathcal{E}_h}\int_e \{\{m\triangle w\}\}[[\frac{\partial v}{\partial n_e}]]+\{\{m\triangle v\}\}[[\frac{\partial w}{\partial n_e}]]\,\mathrm{d}s,\\
c_h(w,v) &= \sum_{e\in\mathcal{E}_h}\frac{1}{|e|}\int_e [[\frac{\partial w}{\partial n_e}]][[\frac{\partial v}{\partial n_e}]]\,\mathrm{d}s.
\end{aligned}
$$

Here $\sigma>0$ is the penalty parameter.

Let $V(h)=H_0^2(\Omega)+V_h$. We define the mesh dependent norm $\|\cdot\|_h$ on $V(h)$ as

$$
\|v\|_h^2=\sum_{T\in\mathcal{T}_h}\|\triangle v\|_{L_2(T)}^2+\sigma\sum_{e\in\mathcal{E}_h}\frac{1}{|e|}\left\|[[\frac{\partial v}{\partial n_e}]]\right\|_{L^2(e)}^2. \tag{3.9}
$$

It is easy to see that we have the following Poincaré inequality.

Lemma 3.1. *(Poincaré inequality) For every $v\in V(h)$, $\|v\|\le C\|v\|_h$.*

The bilinear form $\mathcal{A}_h(\cdot,\cdot)$ is bounded, i.e.,

$$
|\mathcal{A}_h(w,v)|\le C\|w\|_h\|v\|_h \quad \text{for all } w,v\in V_h. \tag{3.10}
$$

This is the result of Lemma 3.1, standard inverse estimates and the Cauchy-Schwarz inequality since

$$
\begin{aligned}
\sum_{e\in\mathcal{E}_h}\left|\int_e \{\{m\triangle w\}\}[[\frac{\partial v}{\partial n_e}]]ds\right| &\le \left(\sum_{e\in\mathcal{E}_h}|e|\,\|\{\{m\triangle w\}\}\|_{L_2(e)}^2\right)^{\frac12}\left(\sum_{e\in\mathcal{E}_h}|e|^{-1}\left\|[[\frac{\partial v}{\partial n_e}]]\right\|_{L_2(e)}^2\right)^{\frac12}\\
&\le C\left(\sum_{e\in\mathcal{E}_h}\sum_{T\in\mathcal{T}_e}\|\triangle w\|_{L_2(T)}^2\right)^{\frac12}\left(\sum_{e\in\mathcal{E}_h}|e|^{-1}\left\|[[\frac{\partial v}{\partial n_e}]]\right\|_{L_2(e)}^2\right)^{\frac12}\\
&\le C\left(\sum_{T\in\mathcal{T}_h}\|\triangle w\|_{L_2(T)}^2\right)^{\frac12}\left(\sum_{e\in\mathcal{E}_h}|e|^{-1}\left\|[[\frac{\partial v}{\partial n_e}]]\right\|_{L_2(e)}^2\right)^{\frac12}.
\end{aligned}\tag{3.11}
$$

Here $\mathcal{T}_e$ is the set of the elements in $\mathcal{T}_h$ that share the common edge e.

Next we show the coercivity of $\mathcal{A}_h$. Similar to (2.7), we have

$$
\int_T m(\triangle+\tau)v(\triangle+\tau)v+\tau^2 vv\,\mathrm{d}x\ge\int_T C_1|\triangle v|^2+C_2|v|^2\mathrm{d}x,
$$

for some positive constants C_1 and C_2 depending on $m(x)$ and τ. Using the inequality of arithmetic and geometric means and the Cauchy-Schwarz inequality, we have that

$$
\begin{aligned}
\mathcal{A}_h(v,v) \;\geq\; & C_1 \sum_{T\in\mathcal{T}_h} \|\triangle u\|_{L^2(T)} - C\left(\sum_{T\in\mathcal{T}_h}\|\triangle v\|^2_{L_2(T)}\right)^{\frac{1}{2}} \left(\sum_{e\in\mathcal{E}_h}|e|^{-1}\left\|[[\frac{\partial v}{\partial n_e}]]\right\|^2_{L_2(e)}\right)^{\frac{1}{2}} \\
& + \sigma \sum_{e\in\mathcal{E}_h}|e|^{-1}\left\|[[\frac{\partial v}{\partial n_e}]]\right\|^2_{L_2(e)} \\
\geq\; & \frac{C_1}{2}\sum_{T\in\mathcal{T}_h}\|\triangle v\|^2_{L_2(T)} + \left(\sigma - \frac{C^2}{C_1}\right)\sum_{e\in\mathcal{E}_h}|e|^{-1}\left\|[[\frac{\partial v}{\partial n_e}]]\right\|^2_{L_2(e)}. \qquad (3.12)
\end{aligned}
$$

Provided σ is large enough, one has that

$$\mathcal{A}_h(v,v) \geq C\|v\|_h^2 \quad \text{for all } v\in V_h. \qquad (3.13)$$

Then the existence and uniqueness of the discrete problem follows immediately.

Let u be the exact solution and u_h be the discrete solution. We have the consistency relation

$$\mathcal{A}_h(u-u_h,v) = 0 \quad \text{for all } v\in V_h. \qquad (3.14)$$

Let $v\in V_h$ be arbitrary, we have that

$$
\begin{aligned}
\|u-u_h\|_h \;&\leq\; \|u-v\|_h + \|v-v_h\|_h \\
&\leq\; \|u-v\|_h + C\max_{w\in V_h\setminus\{0\}}\frac{\mathcal{A}_h(v-u_h,w)}{\|w\|_h} \\
&\leq\; \|u-v\|_h + C\max_{w\in V_h\setminus\{0\}}\frac{\mathcal{A}_h(v-u,w)}{\|w\|_h} \\
&\leq\; C\|u-v\|_h, \qquad (3.15)
\end{aligned}
$$

and hence

$$\|u-u_h\|_h \leq C\inf_{v\in V_h}\|u-v\|_h. \qquad (3.16)$$

Let $\Pi_h : C^0(\bar{\Omega}) \to V_h$ be the Lagrange nodal interpolation operator. Then we have that (Section 3.4 of [5])

$$\|u-\Pi_h u\|_h \leq Ch^\beta \|u\|_{H^{2+\beta}(\Omega)} \leq Ch^\beta |f|_{H^1(\Omega)}, \qquad (3.17)$$

where $\beta = \min\{\alpha, k-1\}$. Note that β is limited by the regularity of the solution and the degree of Lagrange elements.

Let $V = H_0^2(\Omega)$. Summarizing the approximation property and the error estimate, we obtain the following lemma.

Lemma 3.2. *(Quasi-optimality) We assume that*

$$\lim_{h\to 0}\inf_{v_h\in V_h}\|v-v_h\|_h = 0 \quad \textit{for all } v\in V. \qquad (3.18)$$

The discrete problem (3.6) has a unique solution and

$$\|u-u_h\|_h \leq Ch^{\beta}\|u\|_{H^{2+\beta}(\Omega)} \leq Ch^{\beta}|f|_{H^1(\Omega)}, \tag{3.19}$$

where C is a constant independent of the mesh size.

4 C^0 IPG for the eigenvalue problem

The C^0 IPG for the eigenvalue problem can be stated as follows. Find $u_h \in V_h$ and $\mu_h \in \mathbb{R}$ such that

$$\mathcal{A}_h(u_h,v) = \mu_h \mathcal{B}_h(u_h,v) \quad \text{for all } v \in V_h. \tag{4.1}$$

Following the abstract convergence theory developed in [11] and the spirit of DG method for the Laplace eigenvalue problem [1], we would like to show that the C^0IPG is "spectrally correct", namely,

- non-pollution of the spectrum: no discrete spurious eigenvalues;
- completeness of the spectrum: all eigenvalues smaller than a fixed value are approximated when the mesh is fine enough;
- non-pollution and completeness of the eigenspaces: there are no spurious eigenfunctions and the eigenspace approximations have the right dimension.

To carry out subsequent discussions, we recall some results of spectral theory (see [18]). We define two operators as follows:

$$T : H^1(\Omega) \to V, \quad \mathcal{A}(Tf,v) = \mathcal{B}(f,v) \quad \text{for all } v \in V, \tag{4.2}$$

$$T_h : H^1(\Omega) \to V_h, \quad \mathcal{A}_h(T_h f,v) = \mathcal{B}_h(f,v) \quad \text{for all } v \in V_h. \tag{4.3}$$

Since T is symmetric, positive definite and compact due to the compact embedding of V into $H_0^1(\Omega)$, classical spectral theorem implies that T has a sequence of positive eigenvalues $\{\lambda_j\}$ with zero being the only accumulation point. The inverse of $\{\lambda_j\}$, i.e., $\{\mu_j = 1/\lambda_j\}$ are the eigenvalues of (2.5) with ∞ being the only accumulation.

Let $\sigma(T)$ and $\rho(T)$ be the spectrum and resolvent sets of T. The resolvent operator is defined as

$$R_z(T) = (z-T)^{-1} \quad z \in \rho(T).$$

Similarly, we have $\sigma(T_h), \rho(T_h)$, and

$$R_z(T_h) = (z-T_h)^{-1} \quad z \in \rho(T_h).$$

In the rest of this section, we show that the C^0IPG is spectrally correct and prove the convergence rate.

4.1 Non-pollution of the spectrum

For non-pollution of the spectrum, we can show that for any open set containing $\sigma(T)$ also contains $\sigma(T_h)$ for h small enough. We first show that for z away from $\sigma(T)$, $z-T$ is bounded from below.

Lemma 4.1. *Let $z \in \rho(T), z \neq 0$. There exists a positive constant C only depending upon Ω and $|z|$ such that*

$$\|(z-T)f\|_h \geq C\|f\|_h \quad \textit{for all } f \in V(h). \tag{4.4}$$

Proof. Let $z \in \rho(T), z \neq 0$ be fixed and $f \in V(h)$. Set $g = (z-T)f$. Since $Tf \in V$, we have that $g \in V(h)$. Note that

$$T = ((\triangle+\tau)m(\triangle+\tau)+\tau^2)^{-1}\triangle = \tilde{T}^{-1}\triangle : H_0^1(\Omega) \to V \tag{4.5}$$

in the weak sense. Then $zf-g = Tf$ implies $\tilde{T}(zf-g) = \triangle f$. Hence $zf-g \in V$ is the solution of the problem

$$\begin{aligned} \tilde{T}(zf-g) - \frac{1}{z}\triangle(zf-g) &= \frac{\triangle g}{z} \quad \text{in } \Omega, \\ zf-g &= 0 \quad \text{on } \partial\Omega, \\ \frac{\partial}{\partial n}(zf-g) &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Since the above problem is a lower order perturbation of (2.3), we deduce that [15], for some C

$$\|zf-g\|_V \leq \frac{C}{|z|}\|\nabla g\|_{L^2(\Omega)} \leq \frac{C}{|z|}\|g\|_h. \tag{4.6}$$

Since $zf-g \in V$, we have that $\|zf-g\|_h \leq C\|zf-g\|_V$ and

$$\|zf-g\|_h \leq \frac{C}{|z|}\|g\|_h. \tag{4.7}$$

Using the triangle inequality, we obtain the desired result

$$\|f\|_h \leq \frac{1}{z}(\|zf-g\|_h + \|g\|_h) \leq C(|z|)\|g\|_h = C(|z|)\|(z-T)f\|_h. \tag{4.8}$$

□

Next we show that a similar property holds for T_h as well.

Lemma 4.2. *For $z \in \rho(T), z \neq 0$, there exists a positive constant C only depending on Ω and $|z|$ such that, for h small enough,*

$$\|(z-T_h)f\|_h \geq C\|f\|_h \quad \textit{for all } f \in V(h). \tag{4.9}$$

Proof. By triangle inequality, we have that

$$\|(z-T_h)f\|_h \geq \|(z-T)f\|_h - \|(T-T_h)f\|_h. \tag{4.10}$$

By Lemma 4.1, Lemma 3.1 and Lemma 3.2, we have

$$\|(z-T_h)f\|_h \geq C(|z|)\|f\|_h - Ch^{\beta}\|f\|_h, \tag{4.11}$$

where $C(|z|)$ is the constant in Lemma 4.1. Since $C(|z|)$ only depends on Ω and z, (4.9) is readily verified for h small enough. □

Lemma 4.3. *Let $F \subset \rho(T)$ be closed. There exists a positive constant C independent of h such that, for h small enough, we have*

$$\|R_z(T_h)\|_{\mathcal{L}(V(h),V(h))} \leq C \quad \textit{for all } z \in F. \tag{4.12}$$

Proof. Let $z \in F$ be fixed. Since $z \in \rho(T)$, we have that

$$\|R_z(T_h)\|_{\mathcal{L}(V(h),V(h))} = \sup_{g\in V(h),\|g\|_h=1} \|(z-T_h)^{-1}g\|_h. \tag{4.13}$$

Let $\|g\|_h = 1$ and $(z-T_h)^{-1}g = f$, we have

$$\|(z-T_h)f\|_h = \|g\|_h = 1. \tag{4.14}$$

From Lemma 4.2, for h small enough, we get

$$C\|f\|_h \leq \|(z-T_h)f\|_h = 1. \tag{4.15}$$

and the lemma follows immediately. □

Lemma 4.3 claims that, for any $z \in \rho(T)$ and h small enough, $(z-T_h)$ admits a bounded inverse operator from $V(h)$ to $V(h)$, i.e., $R_z(T_h)$ is well defined and continuous from $V(h)$ to $V(h)$. Thus we have shown the following theorem which implies non-pollution of the spectrum.

Theorem 4.1. *(Non-pollution of the spectrum) Let $A \subset \mathbb{C}$ be an open set containing $\sigma(T)$. Then, for h small enough, $\sigma(T_h) \subset A$.*

For fixed $z \in \rho(T)$ and $f \in V(h)$, we can write

$$\|zf - Tf\|_h \leq |z|\|f\|_h + \|Tf\|_h \leq |z|\|f\|_h + C\|f\|_h \leq C(|z|)\|f\|_h, \tag{4.16}$$

due to the stability estimate of the continuous problem and the Poincaré inequality of Lemma 3.1. Using Lemma 4.1, for all fixed $z \in \rho(T), z-T : V(h) \to V(h)$ is a continuous invertible operator with continuous inverse. A direct consequence of this fact is the analogue of Lemma 4.1: let $F \subset \rho(T)$ be closed; then, there exists a positive constant C independent of h such that

$$\|R_z(T)\|_{\mathcal{L}(V(h),V(h))} \leq C, \tag{4.17}$$

for all $z \in F$. From continuity of $T : H^1(\Omega) \to H^1(\Omega)$, if $F \subset \rho(T)$ is closed, there exists a positive constant C such that

$$\|R_z(T)\|_{\mathcal{L}(H^1(\Omega),H^1(\Omega))} \leq C, \tag{4.18}$$

for all $z \in F$.

4.2 Non-pollution and completeness of the eigenspaces

Let λ be an eigenvalue of T with algebraic multiplicity p. Denote by Γ a circle in the complex plane centered at λ such that no other eigenvalue lies inside Γ. Define the spectral projections E from $H^1(\Omega)$ into V and E_h from $H^1(\Omega)$ into V_h by (see [16])

$$E := \frac{1}{2\pi i}\int_\Gamma R_z(T)dz, \quad E_h := \frac{1}{2\pi i}\int_\Gamma R_z(T_h)dz. \tag{4.19}$$

Let X and Y be closed subspaces of $V(h)$. We define the "distance" between X and Y as

$$d(X,Y) = \max\{\delta_h(X,Y), \delta_h(Y,X)\}, \tag{4.20}$$

where

$$\delta_h(X,Y) := \sup_{x\in X, \|x\|=1} \inf_{y\in Y} \|x-y\|. \tag{4.21}$$

We first show that E_h converges to E in operator norm as $h\to 0$.

Theorem 4.2. *Let E and E_h be defined as in* (4.19)

$$\lim_{h\to 0} \|E-E_h\|_{\mathcal{L}(H^1(\Omega),V(h))} = 0. \tag{4.22}$$

Proof. It's easy to see that

$$(z-T)^{-1} - (z-T_h)^{-1} = (z-T_h)^{-1}(T-T_h)(z-T)^{-1}, \tag{4.23}$$

i.e.,

$$R_z(T) - R_z(T_h) = R_z(T_h)(T-T_h)R_z(T). \tag{4.24}$$

Let $f\in H_0^1(\Omega)$. We have

$$\begin{aligned} & \|R_z(T_h)(T-T_h)R_z(T)f\|_h \\ \leq \; & \|R_z(T_h)\|_{\mathcal{L}(V(h),V(h))}\|T-T_h\|_{\mathcal{L}(H^1(\Omega),V(h))}\|R_z(T)\|_{\mathcal{L}(H^1(\Omega),H^1(\Omega))}\|f\|_{H^1(\Omega)}. \end{aligned} \tag{4.25}$$

From Lemma 3.2 and Lemma 4.3 and (4.18), we obtain (4.22). $\square$

Theorem 4.3. *(Non-pollution of the eigenspace).*

$$\lim_{h\to 0} \delta_h(E_h(V_h), E(V)) = 0. \tag{4.26}$$

Proof. With $E(H^1(\Omega)) = E(V)$ and $E_h y_h = y_h$ for all $y_h \in E_h(V_h)$, we have

$$\begin{aligned} \sup_{y_h\in E_h(V_h), \|y_h\|_h=1} \inf_{x\in E(V)} \|y_h - x\|_h &= \sup_{y_h\in E_h(V_h), \|y\|_h=1} \inf_{x\in E(H^1(\Omega))} \|y_h - x\|_h \\ &= \sup_{y_h\in E_h(V_h), \|y\|_h=1} \inf_{x\in H^1(\Omega)} \|E_h y_h - Ex\|_h. \end{aligned} \tag{4.27}$$

Letting $x=y_h$ and using the discrete Poincaré inequality, we obtain

$$\begin{aligned}\sup_{y_h\in E_h(V_h),\|y\|_h=1}\inf_{x\in H^1(\Omega)}\|E_hy_h-Ex\|_h &\leq \sup_{y_h\in E_h(V_h),\|y\|_h=1}\|E_hy_h-Ey_h\|_h\\ &\leq \sup_{y_h\in E_h(V_h),\|y\|_h=1}\|E_h-E\|_{\mathcal{L}(H^1(\Omega),V(h))}\|y_h\|_h.\end{aligned}\tag{4.28}$$

Application of Theorem 4.2 completes the proof. □

Theorem 4.4. *(Completeness of the eigenspaces)*

$$\lim_{h\to 0}\delta_h(E(V),E_h(V_h))=0. \tag{4.29}$$

Proof.

$$\sup_{x\in E(V),\|x\|_h=1}\inf_{y_h\in E_h(V_h)}\|x-y_h\|_h=\sup_{x\in E(V),\|x\|_h=1}\inf_{y_h\in V_h}\|Ex-E_hy_h\|_h.$$

From quasi-optimality of V_h, there exists $x_h\in V_h$ such that

$$\lim_{h\to 0}\|x-x_h\|_h=0. \tag{4.30}$$

So we have

$$\begin{aligned}\inf_{y_h\in V_h}\|Ex-E_hy_h\|_h &\leq \|Ex-E_hx_h\|_h\\ &\leq \|E(x-x_h)\|_h+\|(E-E_h)x_h\|_h\\ &\leq C\|E\|_{\mathcal{L}(V(h),V(h))}\|x-x_h\|_h+\|E-E_h\|_{\mathcal{L}(V(h),V(h))}\|x_h\|_h.\end{aligned}\tag{4.31}$$

Since E is a projection, the first term goes to 0 as $h\to 0$. Using the fact that

$$\|E-E_h\|_{\mathcal{L}(V(h),V(h))}\leq\|E-E_h\|_{\mathcal{L}(H^1(\Omega),V(h))},$$

and Theorem 4.2, we have that

$$\|E-E_h\|_{\mathcal{L}(V(h),V(h))}\to 0 \quad \text{as } h\to 0.$$

Note that $E(V)$ is finite dimensional, point-wise convergences implies uniform convergence, which completes the proof. □

4.3 Completeness of the spectrum

Completeness of the spectrum is readily verified once we have completeness of the eigenspaces.

Theorem 4.5. *(Completeness of the spectrum). For all $z\in\sigma(T)$, there exists a family of $\{z_h\},z_h\in\sigma(T_h)$ such that*

$$\lim_{h\to 0}z_h=z. \tag{4.32}$$

Proof. Theorem 4.3 and Theorem 4.4 imply that $d(E(V),E_h(V_h))\to 0$ as $h\to 0$. Hence for h small enough, $E(V)$ and $E_h(V_h)$ have the same dimension. Let D_Γ be the domain bounded by Γ. If $D_\Gamma\cap\sigma(T)\neq\emptyset$, then, for h small enough, $D_\Gamma\cap\sigma(T_h)\neq\emptyset$. Since T only has a point spectrum, without loss of generality, one can choose D_Γ a disk with radius $\epsilon>0$ centered at z. Hence for h small enough, there must be an element in $\sigma(T_h)$ which is close enough to z (less than ϵ). The theorem follows consequently. □

4.4 Convergence Analysis

Let $s=\dim E$. It has been shown that, for h small enough, there are s eigenvalues of T_h such that

$$\lim_{h\to 0}\sup_{1\le i\le s}|\lambda-\lambda_{i,h}|=0. \tag{4.33}$$

Due to the approximation property of V_h (3.17), we have that

$$\delta(E(V),V_h)\le Ch^\beta. \tag{4.34}$$

Theorem 4.6. *For h small enough, we have that*

$$\sup_{1\le i\le n}|\lambda-\lambda_{i,h}|\le Ch^{2\beta}. \tag{4.35}$$

Proof. By (3.17), we have that

$$\begin{aligned}\|E-E_h\|_{\mathcal{L}(E(V),V(h))} &\le C\|T-T_h\|_{\mathcal{L}(E(V),V(h))}\\ &\le C\sup_{x\in E(V),\|x\|_h=1}\|Tx-T_hx\|_h\\ &\le Ch^\beta.\end{aligned}$$

Since E is a projection, for h small enough, $E_h|_{E(V)}:E(V)\to E_h(V_h)$ is an invertible mapping that we denote by $F_h=E_h|_{E(V)}$. Its inverse is uniformly bounded with respect to h.

Let $\tilde{T}=T|_{E(V)}$ and $\tilde{T}_h=F_h^{-1}T_hF_h:E(V)\to E(V)$. We have then [1]

$$\sup_{1\le i\le n}|\lambda-\lambda_{i,h}|\le C\|\tilde{T}-\tilde{T}_h\|_{\mathcal{L}(E(V),V(h))}. \tag{4.36}$$

Let $S_h=F_h^{-1}E_h:H^1(\Omega)\to V(h)$, which is a continuous operator. For all $x\in E(V)$, $S_hTx=\tilde{T}x$ and $S_hT_hx=\tilde{T}_hx$. So we have

$$(\tilde{T}-\tilde{T}_h)x=S_h(T-T_h)x\quad\text{for all } x\in E(V), \tag{4.37}$$

and

$$\begin{aligned}\|\tilde{T}-\tilde{T}_h\|_{\mathcal{L}(E(V),V(h))} &= \sup_{x\in E(V),\|x\|_h=1}\|\tilde{T}x-\tilde{T}_hx\|_h\\ &\le C\sup_{x\in E(V),\|x\|_h=1}\|Tx-T_hx\|_h\\ &\le Ch^\beta.\end{aligned}$$

It is clear that the problem considered is self-adjoint. Since the C^0 IPG is symmetric, following the reasoning used in [12], one actually has that

$$\sup_{1\leq i\leq n} |\lambda-\lambda_{i,h}| \leq Ch^{2\beta}. \tag{4.38}$$

□

5 Numerical examples

In this section, we present some preliminary examples using Lagrange elements. We choose two polygon domains: the unit square given by

$$(-1/2,1/2)\times(-1/2,1/2),$$

and an L-shaped domain given by

$$(-1/2,1/2)\times(-1/2,1/2)\setminus[0,1/2]\times[-1/2,0].$$

We generate initial quasi-uniform meshes with $h\approx 0.1$ for the two domains and uniformly refine them three times. Since there are no exact eigenvalues available, we define the relative error as

$$R_i = \frac{|\lambda_{h_i}-\lambda_{h_{i+1}}|}{\lambda_{h_{i+1}}},$$

where λ_{h_i} is the computed smallest eigenvalue on the mesh with size h_i. We set the penalty parameter $\sigma=20$ for all numerical examples according to the criteria in [17].

We set the function m to be $1/15$ and $\tau=4$. We first let $k=2$ and compute the smallest 6 eigenvalues for the two domains. In Table. 1, we show the smallest 6 eigenvalues for the unitsquare on a series of uniformly refined meshes. It is clear that all eigenvalues converge as the mesh size decreases. Similar behavior can be observed for the L-shaped domain (Table.2).

Table 1: The first 6 eigenvalues of the unit square ($m=1/15, k=2$)).

h	1/10	1/20	1/40	1/80
1st	3.81446397	3.67056460	3.62927378	3.61803042
2nd	6.46307953	6.05671782	5.93584577	5.90269901
3rd	6.47497291	6.05159762	5.93417395	5.90222194
4th	9.35899087	8.53033433	8.28667698	8.21942966
5th	11.38590197	10.30345017	10.00079742	9.91882214
6th	12.27331821	11.23008634	10.93236472	10.85063178

Table 2: The first 6 eigenvalues of the L-shaped domain ($m=1/15, k=2$).

h	1/10	1/20	1/40	1/80
1st	9.58933965	8.74791493	8.46921155	8.37478038
2nd	10.85920836	9.93745334	9.66267690	9.58390396
3rd	12.48756982	11.35129126	11.007555827	10.90958476
4th	15.17664789	13.39106860	12.86133934	12.71309166
5th	17.73632960	15.34549516	14.63833455	14.43565437
6th	22.60252968	19.43188655	18.49425640	18.21451526

In Fig. 1, we show the first and second eigenfunctions for the two domains. In Fig. 2, we plot relative errors for the first and the second eigenvalues against mesh sizes in log scale. For the unit square, we can see roughly the second order convergence is achieved for both eigenvalues. For the L-shaped domain, the convergence rate of the first eigenvalue is less than 2 due to the reentrant angle which leads to low regularity. The convergence rate of the second eigenvalue is higher indicating the second eigenfunction is smoother than the first one.

In Fig. 3, we repeat the plot for $k=3$. For the unit square, we see that the relative error is roughly of $O(h^4)$ for both eigenvalues. For the L-shaped domain, the convergence rate is less than $O(h^4)$ for both eigenvalues. However, the second eigenfunction has more regularity than the first eigenfunction which ends up with higher convergence rate. We note that, for compact self-adjoint operators, the order of convergence is related to the regularities of the eigenfunctions. If the eigenvalue of the multiplicity is more than one, the convergence is related to the approximation properties of the eigenspace [3].

Next we set $m=1/(7+x+y)$ and $\tau=4$. We let $k=2$ and show the first 6 eigenvalues for the unit square in Table. 3. For the values we have, the second and third values are the approximation of an exact eigenvalue with multiplicity 2. The plot of these two eigenvalues also supports our argument (see Fig. refeigfs2and3square).

Table 3: The first 6 eigenvalues of the unit square ($m=1/(7+x+y), k=2$).

h	1/10	1/20	1/40	1/80
1st	7.47404958	7.16350861	7.07416012	7.04980287
2nd	13.53510833	12.65674643	12.40540951	12.33699851
3rd	13.56136527	12.66649644	12.40864636	12.33801199
4th	19.87793508	18.09893827	17.57438280	17.42945049
5th	24.20522590	21.88523609	21.23824277	21.06329278
6th	25.98991545	23.76950307	23.13193451	22.95622682

Similar to the unit square, we show the results for the L-shaped domain in Table. 4.

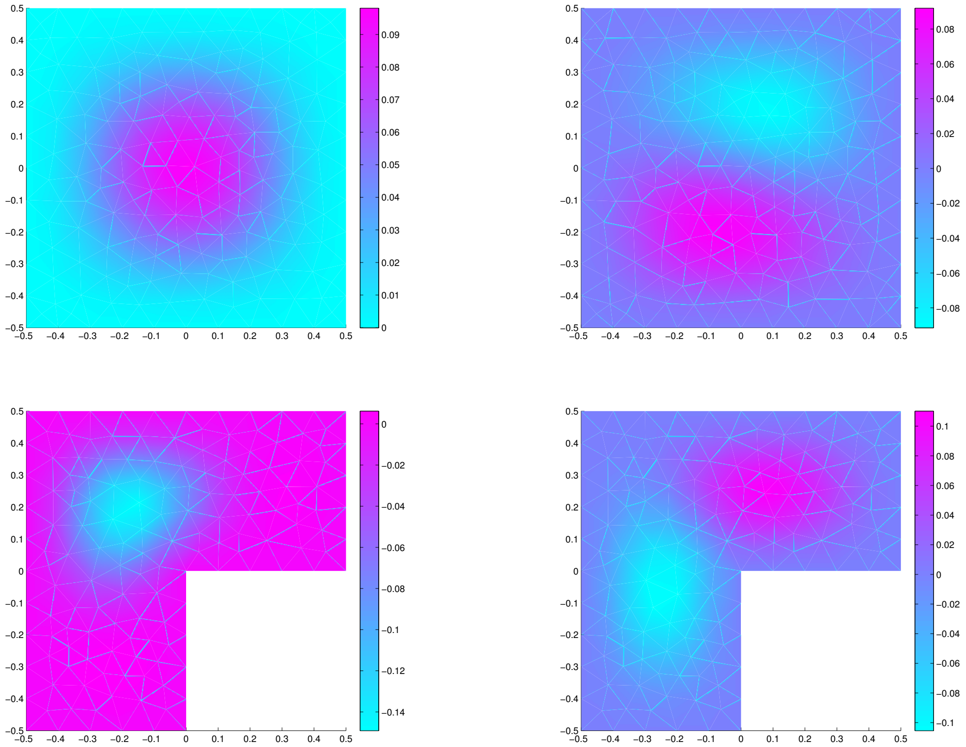


Figure 1: The first row: the first and the second eigenfunctions for the unit square. The second row: the first and the second eigenfunctions for the L-shaped domain.

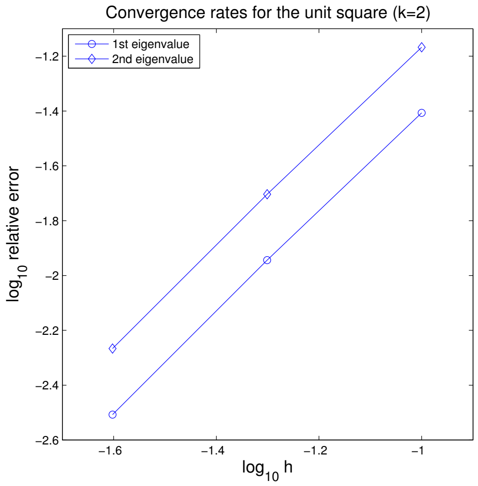
Acknowledgement

X. Ji is supported by the National Natural Science Foundation of China (No. 11271018, No. 91230203) and the Special Funds for National Basic Research Program of China (973 Program 2012CB025904, 863 Program 2012AA01A3094). Research of H. Geng is partly supported by the scholarship from China Scholarship Council (CSC) and the NSFC Grant (11371385) and NSFC Grant (11201506). The work of J. Sun is supported in part by NSF DMS-1016092/1321391. The work of L. Xu is partially supported by the NSFC Grant (11371385), the Start-up fund of Youth 1000 plan of China and that of Youth 100 plan of Chongqing University.

References

[1] P. F. Antonietti, A. Buffa and I. Perugia, Discontinuous Galerkin approximation of the Laplace eigenproblem, Comput. Methods Appl. Mech. Engrg., 195 (2006), 3483-3503.

[2] J. H. Argyris, I. Fried, and D. W. Scharpf. The TUBA family of plate elements for the matrix displacement method, Aero. J. Roy. Aero. Soc., 72 (1968), 701-709.



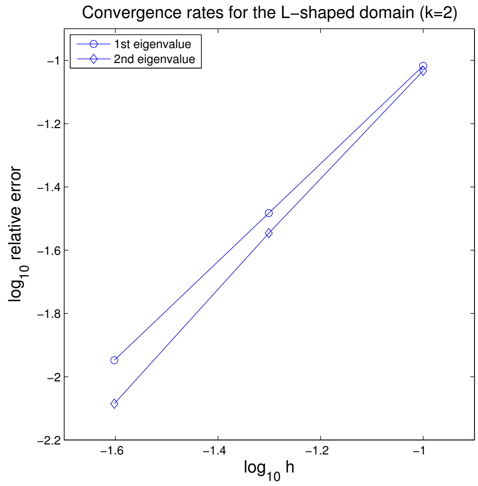


Figure 2: Relative errors of the first and second eigenvalues. Left: the unit square. Right: the L-shaped domain.

Table 4: The first 6 eigenvalues of the L-shaped domain ($m = 1/(7+x+y), k = 2$).

h	1/10	1/20	1/40	1/80
1st	20.16938036	18.40380909	17.81903688	17.61990139
2nd	22.76613875	20.80722663	20.22042030	20.05201473
3rd	26.73834158	24.26729777	23.53316984	23.32503037
4th	32.26383705	28.47643117	27.35769170	27.04459017
5th	38.42225900	33.12983642	31.54028543	31.08342150
6th	47.69016486	40.75278076	38.73679157	38.15608556

[3] I. Babuška and J. Osborn, Eigenvalue Problems, Handbook of Numerical Analysis, Vol. II, Finite Element Methods (Part 1), Edited by P.G. Ciarlet and J.L. Lions, Elsevier Science Publishers B.V. (North-Holland), 1991.

[4] D. Boffi, Finite element approximation of eigenvalue problems, Acta Numer., 19 (2010), 1-120.

[5] S. C. Brenner, C^0 interior penalty methods, in Frontiers in Numerical Analysis - Durham 2010, Lecture Notes in Computational Science and Engineering 85, Springer-Verlag, 2012, 79–147.

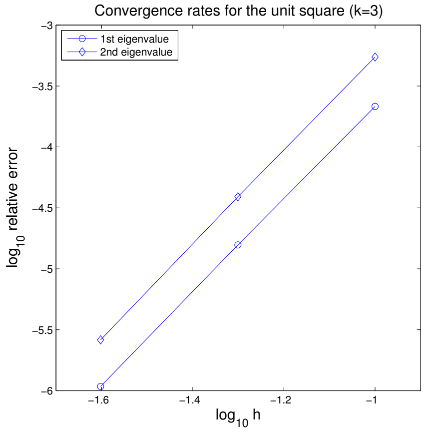
[6] S. C. Brenner and L. Sung, C^0 interior penalty methods for fourth order elliptic boundary value problems on polygonal domains, J. Sci. Comput., 22/23 (2005), 83-118.

[7] S. C. Brenner, P. B. Monk and J. Sun, C^0 IPG Method for Biharmonic Eigenvalue Problems, preprint.

[8] F. Cakoni and H. Haddar, On the existence of transmission eigenvalues in an inhomogenous medium, Appl. Anal., 88 (2009), 475-493.

[9] C. B. Davis, A partition of unity method with penalty for fourth order problems, J. Sci. Comput., 60 (2014), 228–248.

[10] P. G. Ciarlet, The Finite Element Method for Elliptic Problems, Classics in Applied Mathematics, 40, SIAM, Philadelphia, 2002.



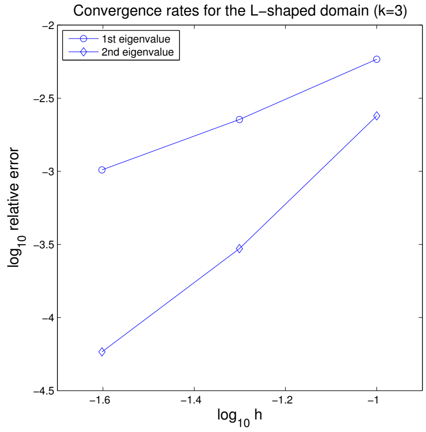


Figure 3: Relative errors of the first and second eigenvalues. Left: the unit square. Right: the L-shaped domain.

[11] J. Descloux, N. Nassif, and J. Rappaz, On spectral approximation Part 1. The problem of convergence, RAIRO Anal. Numér., 12 (1978), 97-112.

[12] J. Descloux, N. Nassif, and J. Rappaz, On spectral approximation Part 2. Error estimates for the Galerkin method convergence, RAIRO Anal. Numér., 12 (1978), 113-119.

[13] G. Engel, K. Garikipati, T. J. R. Hughes, M. G. Larson, L. Mazzei, and R. L. Taylor, Continuous/discontinuous finite element approximations of fourth order elliptic problems in structural and continuum mechanics with applications to thin beams and plates, and strain gradient elasticity, Comput. Methods Appl. Mech. Engrg., 191 (2002), 3669-3750.

[14] E. Georgoulis and P. Houston, Discontinuous Galerkin methods for the biharmonic problem, IMA J. Numer. Anal., 29 (2009), 573-594.

[15] P. Grisvard, Elliptic Problems in Nonsmooth Domains, Pitman Advanced Publishing Program, Boston, 1985.

[16] I. Ishihara, A mixed finite element method for the biharmonic eigenvalue problems of plate bending, Publ. Res. Inst. Math. Sci., Kyoto Univ., 14 (1978), 399-414.

[17] X. Ji, J. Sun and Y. Yang, Optimal penalty parameter for C0 IPDG, Applied Mathematics Letters, 37(2014), 112-117.

[18] T. Kato, Perturbation Theory of Linear Operators, Springer-Verlag, 1966.

[19] L. Morley. The triangular equilibrium problem in the solution of plate bending problems, Aero. Quart., 19 (1968), 149-169.

[20] J. Sun, Iterative methods for transmission eigenvalues, SIAM J. Numer. Anal., 49 (2011), 1860-1874.

[21] J. Sun, A new family of high regularity elements, Numer. Meth. Part. D. E., 28 (2012), 1-16.

[22] G. N. Wells and N. T. Dung, A C^0 discontinuous Galerkin formulation for Kirchhoff plates, Comput. Meth. Appl. Mech. Engrg., 196 (2007), 3370-3380.

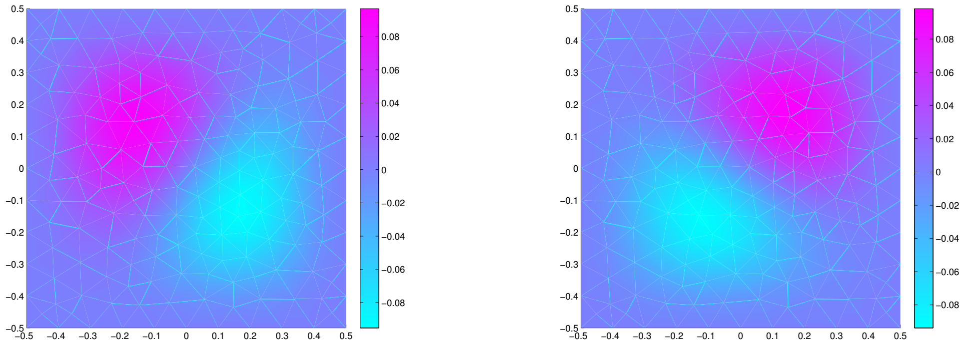


Figure 4: The second and third eigenfunctions of the unit square.