C^0IPG methods for the transmission eigenvalue problem

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Abstract

We consider a non-self-adjoint fourth order eigenvalue problem using a discontinuous Galerkin (DG) method. For high order problems, DG methods are competitive since they use simple basis functions and have less degrees of freedom. The numerical implementation is much easier compared with classical finite element methods. In this paper, we propose an interior penalty discontinuous Galerkin method using C^0 Lagrange elements (C^0IPG) for the transmission eigenvalue problem and prove the optimal convergence. The method is applied to various examples and its effectiveness is validated.

Keywords: transmission eigenvalues, high order eigenvalue problem, discontinuous Galerkin method

1. Introduction

The transmission eigenvalue problem has come to play an important role in the inverse scattering theory [7, 4]. It is not only used for estimating the material properties of the scattering objects, but also for the uniqueness and reconstruction in the inverse scattering theory. We refer the reader to the special issue of Inverse Problems on the transmission eigenvalues (Number 10, 2013) and references therein for the rapid development of this research area.

While the theory is still developing, many researchers have already started working on numerical schemes for the transmission eigenvalues. In [6], Colton et al. proposed three finite element methods. A first numerical method with a rigorous convergence analysis was introduced in [18] by Sun, in which transmission eigenvalues are computed as roots of a nonlinear function which are eigenvalues of the associated positive definite fourth order problem. However, the method can only compute real transmission eigenvalues. In [13], Ji et al. proposed a simple mixed finite element method. An and Shen [1] used spectral element methods for radially stratified media. In [16], Kleefeld employed a method based on the complex-valued contour integrals. A mixed finite element method using Argyris elements is proposed in [5], where error estimates for complex eigenpairs are proved. We refer the readers to [8, 15] for other methods for transmission eigenvalues.

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Due to the higher order and non-self-adjoint nature of the problem, the above methods either are computationally expensive or need stringent conditions on the domain geometry and physical properties. Recently, an interior penalty discontinuous Galerkin method using C^0 Lagrange elements (C^0IPG) became popular for the high order elliptic problems [9, 2]. The method has a good hierarchy and does not need to enforce the jump of the solution since the finite element space is H^1-conforming. Brenner et al. employed the C^0IPG to compute biharmonic eigenvalue problems [3]. They proved the converge of C^0IPG for biharmonic eigenvalue problems and compared it with the Argyris element, the mixed method, and the Morley element.

In this paper, we employ C^0IPG methods for the transmission eigenvalue problems using the mixed formulation proposed in [5]. We prove the optimal error estimate and validate the method by numerical examples. The rest of the paper is organized as follows. In section 2, we introduce the mixed fourth order transmission eigenvalue problem. Section 3 describes the C^0IPG and shows the convergence of the discrete problem using Osborn's perturbation theory [17]. Numerical examples are given in Section 4.

2. The transmission eigenvalue problem

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$ with unit outward normal ν. Let $n(x)$ be a real valued C^2 function satisfying $n(x) - 1 > \gamma > 0$ (or $n(x) - 1 < 0$), where γ is a constant. We consider the transmission eigenvalue problem of finding $u \in H_0^2(\Omega)$ and $\tau \in \mathbb{C}$ such that

$$(\Delta + \tau n(x)) \frac{1}{n(x)-1} (\Delta + \tau) u = 0 \quad \text{in } \Omega, \tag{2.1}$$

with boundary conditions

$$u = 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{2.2}$$

Denote the $L^2(\Omega)$ inner product by (u, v). The variational formulation of the eigenvalue problem can be stated as follows. Find a nontrivial transmission eigenfunction $u \in H_0^2(\Omega)$ and the corresponding eigenvalue $\tau \in \mathbb{C}$ such that

$$\left(\frac{1}{n(x)-1} (\Delta u + \tau u), (\Delta v + \tau n(x) v) \right) = 0 \quad \forall v \in H_0^2(\Omega). \tag{2.3}$$

In the rest of this paper, we assume that $n(x) > n_0 > 1$ a.e., where n_0 is a constant. Similar result holds for $n(x) < 1$.

Expanding (2.3), we have

$$(\Delta u, \Delta v)_{n-1} = -\tau \left((u, \Delta v)_{n-1} + (\Delta u, nv)_{n-1} + \tau (nu, v)_{n-1} \right) \quad \forall v \in H_0^2(\Omega). \tag{2.4}$$

where

$$(u, v)_{n-1} = \int_T \frac{u \bar{v}}{n(x)-1} dA, \tag{2.5}$$

and $\bar{v}$ denotes the complex conjugate of v. Using Poincaré's inequality, it is easy to show that $\tau = 0$ is not an eigenvalue [18].

Let $w \in H_0^1(\Omega)$ such that $\Delta w = \tau u/(n(x)-1)$ in Ω in the weak sense. Following [5], we can obtain the following equivalent problem. Find a constant $\tau \in \mathbb{C}$ and a nontrivial pair of functions $(u, w) \in H_0^2(\Omega) \times H_0^1(\Omega)$ such that

$$(\Delta u, \Delta v)_{n-1} = -\tau\left((u, \Delta v)_{n-1} + (\Delta u, nv)_{n-1} - (\nabla w, \nabla v)\right) \qquad \forall v \in H_0^2(\Omega). \tag{2.6a}$$

$$(\nabla w, \nabla z) = -\tau(nu, z)_{n-1} \qquad \forall z \in H_0^1(\Omega). \tag{2.6b}$$

To solve the new non-self-adjoint eigenvalue problem (2.6a) - (2.6b), we define the sesquilinear form A on $\left(H_0^2(\Omega) \times H_0^1(\Omega)\right) \times \left(H_0^2(\Omega) \times H_0^1(\Omega)\right)$ as follows:

$$A\left((u, w), (v, z)\right) = a(u, v) + c(w, z), \tag{2.7}$$

where

$$a(u, v) = (\Delta u, \Delta v)_{n-1}, \tag{2.8a}$$

$$b_1(u, v) = (u, \Delta v)_{n-1} + (\Delta u, nv)_{n-1}, \tag{2.8b}$$

$$b_2(w, v) = -(\nabla w, \nabla v), \tag{2.8c}$$

$$b_3(u, z) = (nu, z)_{n-1}, \tag{2.8d}$$

$$c(w, z) = (\nabla w, \nabla z). \tag{2.8e}$$

Here $u, v \in H_0^2(\Omega), z, w \in H_0^1(\Omega)$. Note that A is a inner product on $H_0^2(\Omega) \times H_0^1(\Omega)$.

Then the eigenvalue problem is to find $\lambda \in \mathbb{C}$ and non-trivial $(u, w) \in H_0^2(\Omega) \times H_0^1(\Omega)$ such that

$$\lambda A\left((u, w), (v, z)\right) = b_1(u, v) + b_2(w, v) + b_3(u, z), \quad \forall (v, z) \in H_0^2(\Omega) \times H_0^1(\Omega), \tag{2.9}$$

where $\lambda = -1/\tau$. Note that $\tau = 0$ is not a transmission eigenvalue.

We define the operator T: $H_0^2(\Omega) \times H_0^1(\Omega) \to H_0^2(\Omega) \times H_0^1(\Omega)$ by

$$A\left(T(u, w), (v, z)\right) = b_1(u, v) + b_2(w, v) + b_3(u, z), \quad \forall (v, z) \in H_0^2(\Omega) \times H_0^1(\Omega). \tag{2.10}$$

Then we seek $\lambda \in \mathbb{C}$ and non-trivial $(u, w) \in H_0^2(\Omega) \times H_0^1(\Omega)$ such that

$$\lambda(u, w) = T(u, w). \tag{2.11}$$

No spurious eigenvalues are introduced into the system since if $\lambda \neq 0$, $(0, w)$ is not an eigenfunction of this system.

From the definition of T, we have the corresponding source problem. Given (u, w), find $(f, g) \in H_0^2(\Omega) \times H_0^1(\Omega)$ such that

$$A\left((f, g), (v, z)\right) = b_1(u, v) + b_2(w, v) + b_3(u, z) \quad \forall (v, z) \in H_0^2(\Omega) \times H_0^1(\Omega). \tag{2.12}$$

Since A is an inner product on $H_0^2(\Omega) \times H_0^1(\Omega)$, the above problem is well-posed.

We also define the adjoint operator T^*: $H_0^2(\Omega) \times H_0^1(\Omega) \to H_0^2(\Omega) \times H_0^1(\Omega)$ by

$$A\left((u,w), T^*(v,z)\right) = b_1(u,v) + b_2(w,v) + b_3(u,z) \quad \forall (u,w) \in H_0^2(\Omega) \times H_0^1(\Omega). \tag{2.13}$$

The corresponding source problem is as follows. Given (v,z), find $(f,g) \in H_0^2(\Omega) \times H_0^1(\Omega)$ such that

$$A\left((u,w), (f,g)\right) = b_1(u,v) + b_2(w,v) + b_3(u,z) \quad \forall (u,w) \in H_0^2(\Omega) \times H_0^1(\Omega). \tag{2.14}$$

This source problem is also well-posed due to the same reason that A is an inner product.

3. C^{0}IPG for the transmission eigenvalue problem

Now we describe the C^{0}IPG method for (2.6a)-(2.6b). Let $\mathcal{T}_h$ be a regular triangulation for Ω. Let $V_h^1 \subset H_0^1(\Omega)$ and $V_h^2 \subset H_0^1(\Omega)$ be the P_k and $P_{k-1}(k \geq 2)$ Lagrange finite element spaces associated with $\mathcal{T}_h$, respectively. For an interior edge e shared by two triangles $T_\pm$, we define the jump and average as

$$\left[\!\left[\frac{\partial u}{\partial \nu_e} \right]\!\right] = \nu_e \cdot (\nabla u_+ - \nabla u_-), \quad \{\!\{ m \triangle u \}\!\} = \frac{1}{2}(m_- \triangle u_- + m_+ \triangle u_+), \tag{3.15}$$

where $u_\pm = u|_{T_\pm}$ and ν_e points from T_- to T_+. For a boundary edge e, we take ν_e to be the unit normal pointing towards the outside of Ω and define

$$\left[\!\left[\frac{\partial u}{\partial \nu_e} \right]\!\right] = -\nu_e \cdot \nabla u_T, \quad \{\!\{ m \triangle u \}\!\} = m \triangle u. \tag{3.16}$$

Let $b_1'(u_h, v_h)$ be defined as

$$b_1'(u_h, v_h) = -\left(\nabla \frac{u_h}{n-1}, \nabla v_h \right) - \left(\nabla u_h, \nabla \frac{n v_h}{n-1} \right) \quad \forall u_h, v_h \in V_h^1. \tag{3.17}$$

The discrete eigenvalue problem is to find $(u_h, w_h) \in V_h^1 \times V_h^2$ and $\lambda_h \in \mathbb{C}$ such that

$$\lambda_h \mathcal{A}_h\left((u_h, w_h), (v_h, z_h)\right) = b_1'(u_h, v_h) + b_2(w_h, v_h) + b_3(u_h, z_h) \quad \forall (v_h, z_h) \in V_h^1 \times V_h^2, \tag{3.18}$$

where

$$\begin{aligned} \mathcal{A}_h\left((u,w),(v,z)\right) = & \sum_{T \in \mathcal{T}_h} \int_T \frac{1}{n-1} \Delta u \Delta v dx + \sum_{e \in \mathcal{E}_h} \int_e \left\{\!\!\left\{ \frac{1}{n-1} \triangle u \right\}\!\!\right\} \left[\!\left[\frac{\partial v}{\partial n_e} \right]\!\right] + \left\{\!\!\left\{ \frac{1}{n-1} \triangle v \right\}\!\!\right\} \left[\!\left[\frac{\partial u}{\partial n_e} \right]\!\right] ds \\ & + \sigma \sum_{e \in \mathcal{E}_h} \frac{1}{|e|} \int_e \left[\!\left[\frac{\partial u}{\partial n_e} \right]\!\right] \left[\!\left[\frac{\partial v}{\partial n_e} \right]\!\right] ds + \sum_{T \in \mathcal{T}_h} \int_T \nabla w \nabla z dx. \end{aligned} \tag{3.19}$$

Here $\sigma > 0$ is the penalty parameter and $\mathcal{E}_h$ is the set of all the edges of $\mathcal{T}_h$.

Next we define an approximation to the operator T, denoted by $T_h : H_0^2(\Omega) \times H_0^1(\Omega) \to V_h^1 \times V_h^2$ such that for $(p,q) \in H_0^2(\Omega) \times H_0^1(\Omega)$, $T_h(p,q) \in V_h^1 \times V_h^2$, we have

$$\mathcal{A}_h\left(T_h(p,q), (v_h, z_h)\right) = b_1'(p, v_h) + b_2(q, v_h) + b_3(q, z_h) \quad \forall (v_h, z_h) \in V_h^1 \times V_h^2. \tag{3.20}$$

T_h is the solution operator of the discrete source problem. Given (p,q), find $(u_h,w_h) \in V_h^1 \times V_h^2$ such that

$$\mathcal{A}_h\left((u_h,w_h),(v_h,z_h)\right) = b_1'(p,v_h) + b_2(q,v_h) + b_3(q,z_h) \quad \forall (v_h,z_h) \in V_h^1 \times V_h^2. \tag{3.21}$$

We need to check the well-posedness of the discrete source problem, i.e., the existence of T_h. To this end, we define the mesh dependent norm $\|\cdot\|_h$ on $V_h^1 \times V_h^2$ as

$$\|(u,w)\|_h^2 = \sum_{T\in\mathcal{T}_h} \|\triangle u\|_{L_2(T)}^2 + \sigma \sum_{e\in\mathcal{E}_h} \frac{1}{|e|} \left\| \left[\!\!\left[\frac{\partial u}{\partial n_e}\right]\!\!\right] \right\|_{L^2(e)}^2 + \sum_{T\in\mathcal{T}_h} \|\nabla w\|_{L_2(T)}^2. \tag{3.22}$$

The bilinear form $\mathcal{A}_h(\cdot,\cdot)$ is bounded

$$|\mathcal{A}_h((u,w),(v,z))| \le C\|(u,w)\|_h \|(v,z)\|_h \quad \forall u,v \in V_h^1, w,z \in V_h^2. \tag{3.23}$$

This is due to the Poincaré inequality, standard inverse estimates and the Cauchy-Schwarz inequality since

$$\begin{aligned}
\sum_{e\in\mathcal{E}_h} \left| \int_e \left\{\!\!\left\{\frac{1}{n-1}\triangle u\right\}\!\!\right\} \left[\!\!\left[\frac{\partial v}{\partial n_e}\right]\!\!\right] ds \right| &\le \left(\sum_{\frac{1}{n-1}\in\mathcal{E}_h} |e| \left\| \left\{\!\!\left\{\frac{1}{n-1}\triangle u\right\}\!\!\right\} \right\|_{L_2(e)}^2 \right)^{\frac{1}{2}} \left(\sum_{e\in\mathcal{E}_h} |e|^{-1} \left\| \left[\!\!\left[\frac{\partial v}{\partial n_e}\right]\!\!\right] \right\|_{L_2(e)}^2 \right)^{\frac{1}{2}} \\
&\le C \left(\sum_{e\in\mathcal{E}_h} \sum_{T\in\mathcal{T}_e} \|\triangle u\|_{L_2(T)}^2 \right)^{\frac{1}{2}} \left(\sum_{e\in\mathcal{E}_h} |e|^{-1} \left\| \left[\!\!\left[\frac{\partial v}{\partial n_e}\right]\!\!\right] \right\|_{L_2(e)}^2 \right)^{\frac{1}{2}} \\
&\le C \left(\sum_{T\in\mathcal{T}_h} \|\triangle u\|_{L_2(T)}^2 \right)^{\frac{1}{2}} \left(\sum_{e\in\mathcal{E}_h} |e|^{-1} \left\| \left[\!\!\left[\frac{\partial v}{\partial n_e}\right]\!\!\right] \right\|_{L_2(e)}^2 \right)^{\frac{1}{2}}.
\end{aligned} \tag{3.24}$$

Here $\mathcal{T}_e$ is the set of the elements in $\mathcal{T}_h$ sharing the common edge e.

Next, we show the coercivity of $\mathcal{A}_h$. Using the inequality of arithmetic-geometric means and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\mathcal{A}_h((u,w),(u,w)) &\ge C_1 \sum_{T\in\mathcal{T}_h} \|\triangle u\|_{L^2(T)} - C \left(\sum_{T\in\mathcal{T}_h} \|\triangle u\|_{L_2(T)}^2 \right)^{\frac{1}{2}} \left(\sum_{e\in\mathcal{E}_h} |e|^{-1} \left\| \left[\!\!\left[\frac{\partial u}{\partial n_e}\right]\!\!\right] \right\|_{L_2(e)}^2 \right)^{\frac{1}{2}} \\
&\quad + \sigma \sum_{e\in\mathcal{E}_h} |e|^{-1} \left\| \left[\!\!\left[\frac{\partial u}{\partial n_e}\right]\!\!\right] \right\|_{L_2(e)}^2 + \sum_{T\in\mathcal{T}_h} \|\nabla w\|_{L_2(T)}^2 \\
&\ge \frac{C_1}{2} \sum_{T\in\mathcal{T}_h} \|\triangle u\|_{L_2(T)}^2 + \left(\sigma - \frac{C^2}{C_1}\right) \sum_{e\in\mathcal{E}_h} |e|^{-1} \left\| \left[\!\!\left[\frac{\partial u}{\partial n_e}\right]\!\!\right] \right\|_{L_2(e)}^2 + \sum_{T\in\mathcal{T}_h} \|\nabla w\|_{L_2(T)}^2.
\end{aligned}$$

With σ large enough, one has that

$$\mathcal{A}_h((u,w),(u,w)) \ge C\|(u,w)\|_h^2 \quad \forall (u,w) \in V_h^1 \times V_h^2. \tag{3.25}$$

Then the existence and uniqueness of the discrete source problem follows immediately.

Similarly, we can define the approximation T_h^* to the operator T^*, T_h^*: $H_0^2(\Omega) \times H_0^1(\Omega) \to V_h^1 \times V_h^2$ such that, for $(p,q) \in H_0^2(\Omega) \times H_0^1(\Omega)$,

$$A_h\left((u_h,w_h), T_h^*(p,q)\right) = b_1'(u_h,p) + b_2(w_h,p) + b_3(w_h,q) \quad \forall (u_h,w_h) \in V_h^1 \times V_h^2. \tag{3.26}$$

From the definition of T_h^*, we have the discrete adjoint problem. Given (p,q), find $(v_h,z_h) \in V_h^1 \times V_h^2$ such that

$$A_h\left((u_h,w_h),(v_h,z_h)\right) = b_1'(u_h,p) + b_2(w_h,p) + b_3(w_h,q) \quad \forall (u_h,w_h) \in V_h^1 \times V_h^2. \tag{3.27}$$

The proof of the existence and uniqueness is similar to the original problem.

For the convergence analysis of the C^{0}IPG method for the transmission eigenvalue problem, we need the following result for T.

Theorem 3.1. *Let $n(x)$ be a real valued C^2 function with $n(x)-1>\gamma>0$ in Ω. Under the conditions on the domain and the Lagrange finite element spaces, $T_h \to T$ as $h \to 0$ in norm. In particular, we have*

$$\|T - T_h\|_{\mathcal{L}(L^2(\Omega)\times L^2(\Omega), L^2(\Omega)\times L^2(\Omega))} \le Ch^{min(\alpha,2\beta)}, \tag{3.28}$$

where $min(\alpha,2\beta) > 0$ and α,β depend on the interior angles of the Lipschitz polygon as described in the proof. Similarly, $\|T_h^* - T^*\| \to 0$ as $h \to 0$ and

$$\|T^* - T_h^*\|_{\mathcal{L}(L^2(\Omega)\times L^2(\Omega), L^2(\Omega)\times L^2(\Omega))} \le Ch^{min(\alpha,2\beta)}. \tag{3.29}$$

Proof. Letting $T(u,v) = (f,g)$ and $T_h(u,v) = (f_h,g_h)$, we obtain that

$$\|(T - T_h)(u,w)\| \le \|f - f_h\| + \|g - g_h\|, \tag{3.30}$$

where $||\cdot||$ denotes the L^2 norm. We have that

$$(\nabla g, \nabla z) = (nu, z)_{n-1}. \tag{3.31}$$

Since $n/(n-1) \in L^\infty(\Omega)$ and Ω is a Lipschitz polygon, classical regularity theory for elliptic equations implies that there exists an $\alpha_0 > 0$ such that

$$\|g\|_{H^{1+\alpha}(\Omega)} \le C\|nu/(n-1)\|_{H^{-1+\alpha}(\Omega)}, \tag{3.32}$$

where $\alpha > \alpha_0 \ge 1/2$ and the regularity parameter α_0 depends on the interior angles of the polygon. When the domain is convex, $\alpha = 1$ [10]. Then the same argument in the proof of Lemma 1 in [4] implies that

$$\|g - g_h\| \le Ch^\alpha \|u\|. \tag{3.33}$$

Next we derive an estimate for f, which satisfies the following equation

$$(\Delta f, \Delta v)_{n-1} = (u, \Delta v)_{n-1} + (\Delta u, nv)_{n-1} - (\nabla w, \nabla v) \quad \forall v \in H_0^2(\Omega). \tag{3.34}$$

Since n is smooth and $n-1$ is bounded away from 0, the same regularity as the biharmonic equation holds. In addition, we can use the result of Lemma 3.1 in [3], i.e.,

$$\|f - f_h\| \le Ch^{2\beta}\left(\|u\| + \|w\|\right) \tag{3.35}$$

where $\beta \in (1/2, 1]$ is the elliptic regularity for the biharmonic equation. If Ω is convex, $\beta = 1$ [**?**].

Putting (3.33) and (3.35) together, we have that

$$\|T - T_h\| \leq Ch^{\min\{\alpha, 2\beta\}}. \tag{3.36}$$

The same argument holds for T^* and T_h^*.

□

Now we are ready to prove the convergence of the eigenvalue problem, i.e., find λ_h and non-trivial eigenfunctions $(u_h, w_h) \in V_h^1 \times V_h^2$ satisfying

$$\lambda_h(u_h, w_h) = T_h(u_h, w_h), \tag{3.37}$$

when h is small enough.

Let X be a Hilbert space and S: $X \to X$ be a compact operator. Let λ be a non-zero eigenvalue of S with algebraic multiplicity p. Denote by Γ a circle in the complex plane centered at λ such that no other eigenvalue lies inside Γ. Define the spectral projection E as

$$E := \frac{1}{2\pi i} \int_\Gamma (z - S)^{-1} dz, \tag{3.38}$$

and $R(E)$ denotes the range of E and its dimension is p (see, e.g, [14]). In a similar way, define $R(E^*)$ the range of the spectral projection E^* for the Hilbert adjoint S^* of S whose eigenvalue is $\bar{\lambda}$.

Let S_h: $X \to X$ denote a sequence of compact operators for $h > 0$. In [17], Osborn gives the convergence conditions under which the eigenvalues of S_h converge to those of S. Suppose $\lambda_{h,1}, \cdots, \lambda_{h,p}$ converge to an eigenvalue λ of S with multiplicity p, we define

$$\hat{\lambda}_h = \frac{1}{p} \sum_{j=1}^{m} \lambda_{h,j}. \tag{3.39}$$

Lemma 3.2. *([17, 5]). Suppose $S_h \to S$ in norm and $S_h^* \to S^*$ in norm. Let $\phi_1, \cdots, \phi_p$ be any basis for $R(E)$ and let $\phi_1^*, \cdots, \phi_p^*$ be the dual basis. Then there exists a constant C such that*

$$|\lambda - \hat{\lambda}_h| \leq \frac{1}{p} \sum_{j=1}^{m} |\langle (S - S_h)\phi_j, \phi_j^* \rangle| + C\|(S - S_h)|_{R(E)}\| \|(S^* - S_h^*)_{R(E^*)}\|, \tag{3.40}$$

where $\langle (S - S_h)\phi_j, \phi_j^* \rangle$ denotes the Hilbert space duality pairing.

Theorem 3.3. *Under the assumptions of Theorem 3.1, there is a constant C such that*

$$|\lambda - \hat{\lambda}_h| = O(h^{2min(\alpha, 2\beta)}), \tag{3.41}$$

where $min(\alpha, 2\beta) > 0$ and α, β are the same as the Theorem 3.1.

Proof. It is straightforward to verify the Galerkin orthogonality of A_h (see (4.16) of [**?**]):

$$A_h((T - T_h)(u, w), (v_h, z_h)) = 0, \quad \forall (v_h, z_h) \in V_h^1 \times V_h^2. \tag{3.42}$$

The rest of the proof follows exactly the proof of Theorem 3.3 in [5] with A_h replacing A. □

4. Numerical examples

In this section, we present some numerical examples using Lagrange elements. Let $\{\phi_i^t\}_{i=1}^{N_{loc}}$ and $\{\varphi_i^t\}_{i=1}^{M_{loc}}$ be the local basis for V_h^1 and V_h^2, respectively, where $N_{loc} = \frac{(k+1)(k+2)}{2}$, $M_{loc} = \frac{k(k+1)}{2}$. We define the following matrices

$$A_{ij} = \sum_{t=1}^{N_t} A_{ij}^t = \sum_{t=1}^{N_t} \frac{1}{n-1}(\Delta\phi_j^t, \Delta\phi_i^t), \tag{4.43}$$

$$S_{ij}^1 = -\sum_{t=1}^{N_t} \left(\nabla\phi_j^t, \nabla\frac{\phi_i^t}{n-1}\right), \tag{4.44}$$

$$S_{ij}^2 = -\sum_{t=1}^{N_t} (\nabla\frac{n\phi_j^t}{n-1}, \nabla\phi_i^t), \tag{4.45}$$

$$S_{ij} = \sum_{t=1}^{N_t} (\nabla\varphi_j^t, \nabla\phi_i^t), \tag{4.46}$$

$$S'_{ij} = \sum_{t=1}^{N_t} (\nabla\varphi_j^t, \nabla\varphi_i^t), \tag{4.47}$$

$$M_{ij} = \sum_{t=1}^{N_t} (n\phi_j^t, \varphi_i^t)_{n-1}, \tag{4.48}$$

where $(\phi_j^t, \phi_i^t) = \int_{T_t} \phi_j^t \phi_i^t dx$ and N_t is the number of triangles in $\mathcal{T}_h$.

For an interior edge e shared by two triangles T_1 and T_2, where ν_e points from T_2 to T_1, we define

$$P_j^e = \frac{1}{2(n-1)}\left(\triangle\phi_j^{t,1} + \triangle\phi_j^{t,2}\right), \quad Q_j^e = \nu_e \cdot \left(\nabla\phi_j^{t,1} - \nabla\phi_j^{t,2}\right). \tag{4.49}$$

For a boundary edge e which is an edge of the triangle T, we take ν_e to be the unit normal pointing towards the outside of Ω

$$P_j^e = \frac{1}{n-1}(\triangle\phi_j^t), \quad Q_j^e = -\nu_e \cdot \left(\nabla\phi_j^t\right). \tag{4.50}$$

Matrix J is defined as

$$J_{ij} = \sum_{e\in\mathcal{E}_h} \int_e (P_j^e Q_i^e + P_i^e Q_j^e) ds + \sigma \sum_{e\in\mathcal{E}_h} \frac{1}{|e|} \int_e Q_i^e Q_j^e ds. \tag{4.51}$$

We write $u_h = \sum_{t=1}^{N_t} \sum_{i=1}^{N_{loc}} u_i^t \phi_i^t$ and define $\mathbf{u} = (u_1^1, u_2^1, \ldots, u_{N_{loc}}^1, \ldots, u_1^{N_t}, u_2^{N_t}, \ldots, u_{N_{loc}}^{N_t})^T$. Similarly, $w_h = \sum_{t=1}^{N_t} \sum_{i=1}^{M_{loc}} w_i^t \varphi_i^t$ and $\mathbf{w} = (w_1^1, w_2^1, \ldots, w_{N_{loc}}^1, \ldots, w_1^{N_t}, w_2^{N_t}, \ldots, w_{N_{loc}}^{N_t})^T$.

The matrix eigenvalue problem is then given by $\mathcal{A}\mathbf{x} = \tau\mathcal{B}\mathbf{x}$ where

$$\mathcal{A} = \begin{pmatrix} A+J & 0 \\ 0 & S' \end{pmatrix}, \ \mathcal{B} = -\begin{pmatrix} S^1+S^2 & -S \\ M & 0 \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} \mathbf{u} \\ \mathbf{w} \end{pmatrix}. \tag{4.52}$$

Shape	1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}	6^{th}
Unit square	3.75761	6.56741	6.58446	9.38889	11.44229	14.39394
L-shape	10.16265	11.24956	13.57216	15.20801	17.65660	22.56712
Circle	4.09824	7.39857	7.46921	11.96711	12.06806	16.89126
Triangle	3.48391	5.59152	5.61961	8.90951	8.95417	9.56285

Table 1: The first six transmission eigenvalues when $k = 2$, the index of refraction n is 16, the mesh size is $h \approx 0.1$.

To compute the generalized eigenvalues of this systems, we use $'eigs'$ in Matlab. In the experiment, we choose $\sigma = 20$. We refer the readers to [12] for some discussion on how to choose σ.

We choose four test domains: the disk with radius $1/2$ centered at the origin, the triangle whose vertices are given by $(-\sqrt{3}/2, -1/2)$, $(0, 1)$ and $(\sqrt{3}/2, -1/2)$, the unit square given by

$$(-1/2, 1/2) \times (-1/2, 1/2), \tag{4.53}$$

and an L-shaped domain given by

$$(-1/2, 1/2) \times (-1/2, 1/2) \setminus [0, 1/2] \times [-1/2, 0]. \tag{4.54}$$

For each domain, we generate an initial coarse triangular mesh and uniformly refine the mesh to perform a convergence study. For the disk, the exact error can be obtained with precise estimates of transmission eigenvalues via special functions [6]. For the other domains, we define the relative error as

$$\text{Relative Error} = \frac{|\lambda_{h_i} - \lambda_{h_{i+1}}|}{\lambda_{h_{i+1}}}, \tag{4.55}$$

where λ_{h_i} is the eigenvalue on the mesh with size h_i.

For simplicity, we use constant index of refraction in the following tests. We first choose $n = 16$ and $k = 2$. Table 1 shows the first six eigenvalues. They are consistent with the values in [6]. Note that we are computing the square of the value in [6]. We choose the first and second eigenvalues for all domains and show the errors against mesh size in Figure 1 and Figure 2. For the unit square and triangle domain, the second order convergence is roughly obtained. For the L-shaped domain, the convergence rate is less than 2 due to the fact that the reentrant corner leads to low regularity. The convergence rate of the first eigenvalue is lower indicating that the second eigenfunction is smoother than the first one. For the circle domain, the convergence rate is nearly 2.

Next we choose $k = 3$ and show the first six eigenvalues in Table 2. They are also consistent with the values in [6]. The convergence order of the first eigenvalue is shown in Figure 3. As expected, for the unit square and triangle domain, the relative error is roughly $O(h^4)$. For the L-shape domain, the convergence rate is still less than 2, which is the same with $k = 2$. However, the actual error is much smaller. Even though the eigenfunctions are smooth on the circle, the convergence order for eigenfunction does not increase to 4. This is due to the fact that we use triangles for the disc.

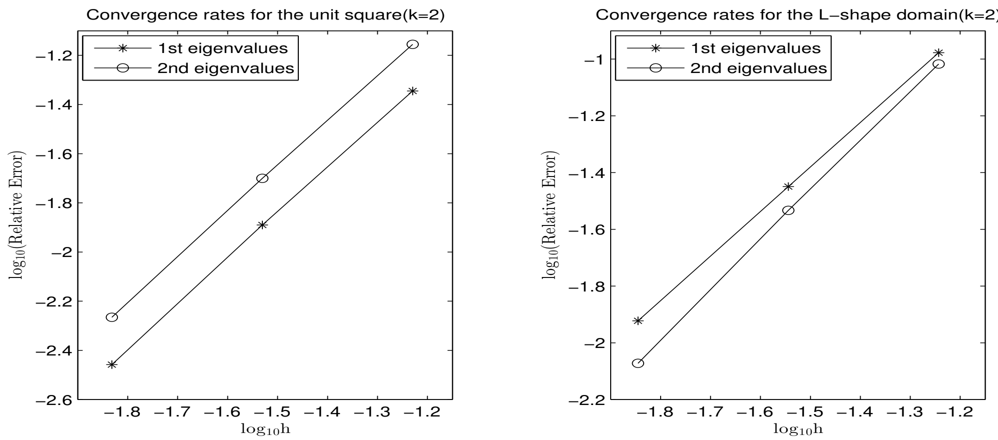

Figure 1: Relative errors of the the fist and second transmission eigenvalues for the unit square and L-shape domain.

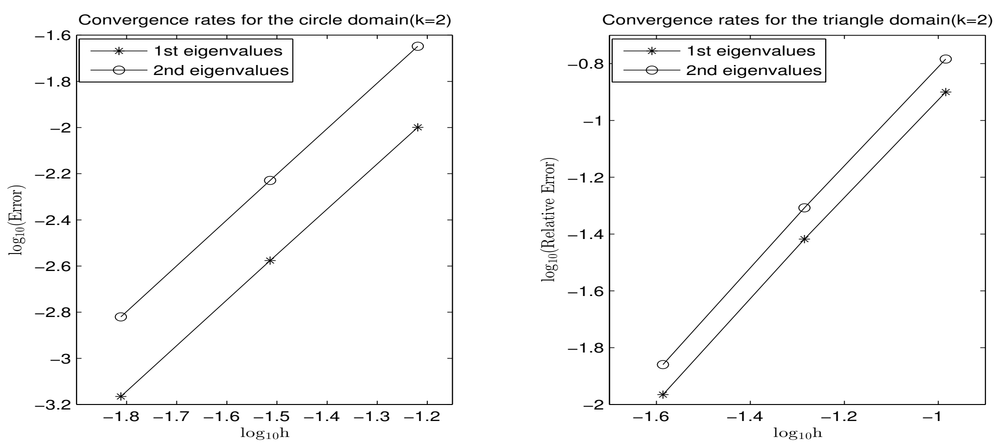

Figure 2: Errors of the the first and second transmission eigenvalues for the circle and triangle domain.

Table 3 shows the convergence rate for different domains with $k = 2, n = 20$, which is similar to the case when $k = 2, n = 16$ (Figure 1 and Figure 2).

The results for $k = 2, n = 4$ and $k = 3, n = 4$ are shown in Table 4 and Table 5, respectively. Note that the transmission eigenvalue with the smallest norm is complex.

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Shape	1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}	6^{th}
Unit square	3.53389	5.97799	5.97804	8.227707	9.87477	12.07843
L-shape	8.74377	9.87596	11.65633	12.74990	14.36241	18.08487
Circle	3.98005	6.87777	6.87753	10.49549	10.49481	14.12837
Triangle	3.30729	5.23230	5.23250	8.05752	8.05808	8.54437

Table 2: The first six transmission eigenvalues when $k = 3$, the index of refraction n is 16, the mesh size is $h \approx 0.1$.

Unit square	Relative error	order	L-shape	Relative error	order	Circle	Error	order
2.94847			7.95648			3.22042	3.6012E-2	
2.82444	4.3914E-2		7.21806	1.0230E-1		3.13876	9.7431E-3	1.97
2.78948	1.2533E-2	1.81	6.97655	3.4617E-2	1.56	3.11648	2.5732E-3	1.96
2.78005	3.3904E-3	1.90	6.89572	1.1722E-2	1.56	3.11054	6.6289E-4	1.98

Table 3: Relative errors of the first eigenvalues when $k = 2, n = 20$.

Shape	Base mesh	1 refinement	2 refinement	3 refinement
unit square	17.01313-10.54955i	16.93894-10.00686i	16.93100-9.85631i	16.93063-9.81662i
Number of DoFs	842	3242	12722	50402
L-shape	33.76839-32.99526i	33.15021-30.65816i	32.96912-29.89779i	32.89667-29.62940i
Number of DoFs	707	2702	10562	41762
Circle	19.28901-10.90883i	19.26496-10.62636i	19.28593-10.55122i	19.29511- 10.53133i
Number of DoFs	685	2638	10354	41026
Triangle	15.39951-11.45527i	15.21996-10.21911i	15.27216-9.86583i	15.30332-9.77128i
Number of DoFs	381	1425	5508	21654

Table 4: The first transmission eigenvalues for different domains when $k = 2, n = 4$.

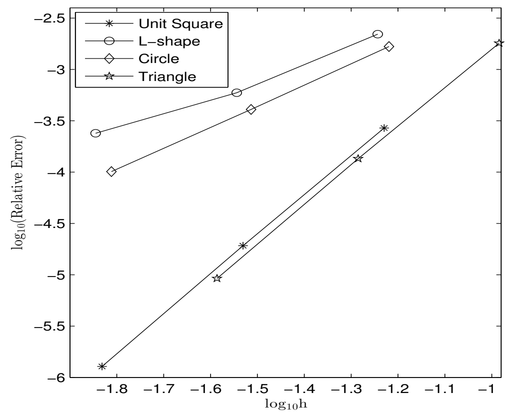

Figure 3: Relative errors of the first eigenvalues when $k = 3, n = 16$.

Shape	Base mesh	1 refinement	2 refinement	3 refinement
unit square	16.92849-9.80552i	16.93061-9.80316i	16.93078-9.80299i	16.93079-9.80298i
Number of DoFs	2130	8314	32850	130594
L-shape	32.84885-29.52350i	32.85364-29.48811i	32.84831-29.47366i	32.84568-29.46678i
Number of DoFs	1779	6910	27234	108130
Circle	19.42700- 10.60163i	19.33056-10.54220i	19.30674-10.52871i	19.30084-10.52547i
Number of DoFs	1733	6766	26738	106306
Triangle	15.29910-9.75092i	15.31576-9.73941i	15.31707-9.73849i	15.31716-9.73842i
Number of DoFs	944	3615	14144	55950

Table 5: The first transmission eigenvalues for different domains when $k = 3, n = 4$.

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