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Article

A Multistep Reciprocity Gap Functional Method for the Inverse Problem in a Multi-layered Medium

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Dedicated to Professor Robert P. Gilbert on the occasion of his 80^{th} birthday.

We introduce a multistep reciprocity gap functional method to reconstruct the support of a target embedded in a multi-layered medium from near field measurements at a fixed frequency. The approach is based on the use of the standard reciprocity gap functional method for homogeneous background to strip the layers and recover the data at each interface by mean of the so-called interior transmission problem and then, at the last step, to reconstruct the support of the scatterer. The advantage of this approach is that it avoids computing the Green's function for multi-layered medium. Numerical examples are given, showing the performance of our reconstruction algorithm.

Keywords: Inverse scattering, buried objects, multilayered media, reciprocity gap functional method, inhomogeneous media

AMS Subject Classification: 35R30, 35Q60, 35J40, 78A25

1. Introduction

The problem of identifying objects embedded in a known background medium has drawn increasing attention in recent years because its important applications in underground imaging [5], [7], [20], [21], submarine imaging [17], [18], [19] and non-destructive testing [3]. The model we have in mind is the detection of objects imbedded in a known inhomogeneous background that consists of several homogeneous (possibly absorbing) layers using time harmonic acoustic or electromagnetic interrogation at a fixed frequency. More precisely, from a knowledge of the total field due to point sources measured on an accessible surface, the aim is to find the support of the buried target without any a priori knowledge of its physical properties. For sake of presentation here we consider only penetrable obstacles, but our discussion can apply verbatim to impenetrable obstacles also. Many methods have been developed in the literature to solve the inverse problem for buried object (see [3], [5], [7], [17], [18], [19], [20], [21], [22] and the references therein). One of the

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challenges in solving this problem is the evaluation of the Green's function for the background medium, task that is analytically complicated and computationally expensive [7], [5], [15], [16], [24]. This paper suggests an approach that avoids the need to compute the Green's function of the background media, and instead uses fundamental solution for homogenous media.

Our method for solving the above mentioned inverse scattering problem is a version of the linear sampling method [11] based on the reciprocity gap functional [1] introduced in [10] (see also [5] for a more complete analysis in the vector case). This method is based on the study of an ill-posed linear integral equation involving the measured Cauchy data of the total field on an accessible surface surrounding the target, whose solution serves as an indicator function for the support of the unknown scattering object. It is important to mention that this method doesn't require any a priori information on the physical properties of the scatterer. Owning to the use of both Cauchy data, the reciprocity gap functional method requires only the knowledge of a Green's function associated with the bounded region surrounded by the measurement surface. In [13], [14] this method has been generalized to more complicated penetrable and impenetrable partially coated objects.

Here we propose a multistep reciprocity gap functional method which only needs the fundamental solution for homogeneous media at each step. More specifically, we proceed from layer to layer till we reach the last layer containing the scatterer by using the reciprocity gap functional method for one homogeneous layer. This way by means of the so-called interior transmission problem we are able to recover a dense set of data on the nearest interface which is then used to set up the next step of the reciprocity gap equation. We introduce and analyze this idea for the simplest case of a bounded two-layered medium and for an inhomogeneous isotropic object. Generalization to the case of bounded multi-layered medium and other type of scatterers can be done in a straightforward manner. Although, the reciprocity gap functional method does not apply to unbounded layered medium, we show how to adapt our ideas to this case which is important for underground imaging.

The plan of the paper is the following. In the next section we formulate the problem and set the notations. We then proceed in Section 3 with recalling the reciprocity gap functional method and introducing our multistep approach for solving the inverse problem. Finally, in Section 4 we provide several numerical examples based on the algorithm developed in the paper.

2. Formulation of the Inverse Problem

We consider the inverse scattering problem for an inhomogeneous medium embedded in a piece-wise homogenous background at a fixed frequency. In our configuration we assume that (possibly part of) a smooth boundary $\partial\Omega$ of an open bounded region $\Omega \in \mathbb{R}^d$, $d = 2, 3$ is accessible for the measurements. Let $\Omega^c := \mathbb{R}^d \setminus \overline{\Omega}$ denote the exterior of Ω. To avoid technicalities we introduce our ideas in a simpler configuration of a two-layered medium. The generalization of our approach to multi-layered medium can be done in a straightforward manner. More specifically, we assume that the index of refraction of the background medium is given by

$$N_b(x) = \begin{cases} 1 & \text{in } \Omega^c, \\ n_0 & \text{in } \Omega \setminus S, \\ n_1 & \text{in } S, \end{cases} \tag{1}$$

where $S \subsetneq \Omega$ is another open bounded region with piecewise smooth boundary ∂S included entirely in Ω, and the complex constants n_0 and n_1 are such that

$\Re(n_0) > 0$, $\Re(n_1) > 0$ and $\Im(n_0) \geq 0$, $\Im(n_0) \geq 0$. As it will become clear later, the background medium outside Ω does not play any role in our inversion scheme, hence without loss of generality we take it to be homogeneous for sake of simplicity. Consider now the scatterer to be an inhomogeneous medium with support $D \subset S$ entirely immersed in the homogeneous region S. Let $n \in L^\infty(D)$ denote the index of refraction of the scatterer and let us assume that $\Re(n) \geq \alpha > 0$ and $\Im(n) \geq 0$ almost everywhere in D. We denote by

$$N(x) = \begin{cases} 1 & \text{in } \Omega^c, \\ n_0 & \text{in } \Omega \setminus S, \\ n_1 & \text{in } S \setminus D, \\ n(x) & \text{in } D, \end{cases} \tag{2}$$

the index of refraction of the background medium together with the obstacle D. Suppose now that a point source incident field $u^i(\cdot, x_0)$ located at $x_0 \in \Omega^c$ given by

$$u^i(x, x_0) = \begin{cases} \frac{i}{4} H_0^{(1)}(k|x - x_0|) & \text{in } \mathbb{R}^2 \\ \dfrac{e^{ik|x-x_0|}}{4\pi|x - x_0|} & \text{in } \mathbb{R}^3 \end{cases}$$

is applied, where $H_0^{(1)}$ is the Hankel function of order zero and $k > 0$ is the wave number. Then the total field $u \in H^1_{loc}(\mathbb{R}^d \setminus \{x_0\})$, $d = 2, 3$, is a solution to the following scattering problem

$$\begin{cases} \Delta u + k^2 N(x) u = 0 \text{ in } \mathbb{R}^d \setminus \{x_0\}, \\ u(x, x_0) = \mathbb{G}(x, x_0) + u^s(x, x_0), \\ u^s(x, x_0) \ \text{ satisfies (SRC)}, \end{cases} \tag{3}$$

where u^s is the scattered field due to the inhomogeneity and the Sommerfeld Radiation Condition (SRC) is given by

$$\begin{cases} \lim\limits_{r \to +\infty} \sqrt{r} \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0 & \text{in } \mathbb{R}^2 \\ \lim\limits_{r \to +\infty} r \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0 & \text{in } \mathbb{R}^3, \end{cases} \tag{4}$$

where $r = |x|$. Here $\mathbb{G}(x, x_0) = u^i(x, x_0) + u^s_b(x, x_0)$ denotes the Green's function of the background medium which satisfies

$$\Delta \mathbb{G}(x, x_0) + k^2 N_b(x) \mathbb{G}(x, x_0) = -\delta(x - x_0) \tag{5}$$

where the regular function $u^s_b \in H^1_{loc}(\mathbb{R}^d)$ is the scattered field due to the background. Hereafter, we denote by ν the unit normal to the boundary of a bounded region oriented always toward the exterior to this region. Note that in (3) continuity of u and $\partial u / \partial \nu$ is assumed across interfaces of discontinuities of N.

To formulate our inverse problem, we consider an analytic closed manifold C surrounding Ω and denote by $C_0 \subset C$ an open subset of this manifold. The *inverse problem* we are dealing with is to determine the support D of the inhomogeneous medium from a knowledge of the Cauchy data $\left\{ u(\cdot, x_0), \frac{\partial u(\cdot, x_0)}{\partial \nu} \text{ on } \partial\Omega \right\}$ of the total field $u(\cdot, x_0)$ satisfying (3) due to point sources $u^i(\cdot, x_0)$ for all $x_0 \in C_0$ (see Figure 1). By analyticity, without loss of generality we assume that the Cauchy data of

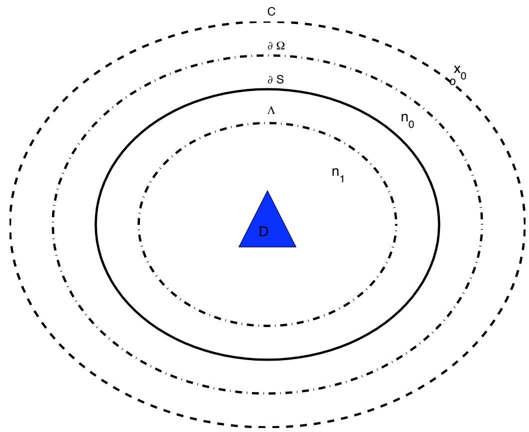

Figure 1. The configuration of the model: point sources x_0 are located on part of the analytic curve C, whereas the corresponding Cauchy data is recorded on the accessible boundary $\partial\Omega$. Inside $\partial\Omega$, the domain S has a different refractive index n_1 compare to the refractive index n_0 in the annulus bounded by $\partial\Omega$ and ∂S. The unknown target D has refractive index $n(x)$. The phantom sources z_0 are located on the curve Λ.

the total field are known for all $x_0 \in C$. Our goal is to use the reciprocity gap functional method to reconstruct D using only fundamental solutions of homogeneous media instead of the Green's function of the inhomogeneous background. We use an auxiliary interior transmission problem to recover the Cauchy data on the unaccessible interface between the two layers, avoiding this way the computation of multilayered Green's function. As we remark later, this idea can apply to background media with more than two homogeneous layers. We refer the reader to Section 4 for potential applicability of our method to more interesting inverse problems in underground imaging. We finish this section by noting that, although excluded from this discussion, our approach can also be used to find the support of impenetrable targets with different type of boundary conditions [10], [13], [14] .

3. The Reciprocity Gap Functional

Our inversion method for reconstructing the support D of the inhomogeneity is based on the Reciprocity Gap Functional Method. For sake of reader's convenience we provide here a brief discussion of this method in its original formulation (for more details see [5, 10, 13, 14, 22]). To this end, we define the reciprocity gap functional $\mathcal{R}_{\partial\Omega}$ by

$$\mathcal{R}_{\partial\Omega}(u, v) = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) ds,$$

with

$$u \in \mathcal{U} = \{u(\cdot, x_0) \text{ solutions to (3) for all } x_0 \in C\}$$
$$v \in \mathbb{H}(\Omega) = \left\{v \in L^2(\Omega) \,:\, \Delta v + k^2 N_b(x) v = 0 \text{ in } \Omega\right\}$$

where the equation is satisfied in the distributional sense. Since u depends on the point source $x_0 \in C$, the reciprocity gap functional can be seen as an operator

$$R\,:\,\mathbb{H}(\Omega) \longrightarrow L^2(C)$$

defined by

$$R(v)(x_0) = \mathcal{R}_{\partial\Omega}(u, v).$$

It can be shown exactly in the same way as in [5, 13, 14] that R is injective and has dense range provided that $k > 0$ is not such that the following homogeneous *interior transmission problem*

$$\begin{cases} \Delta v + k^2 n_1 v = 0 & \text{in } D \\ \Delta w + k^2 n(x) w = 0 & \text{in } D, \\ w - v = 0 & \text{on } \partial D, \\ \dfrac{\partial w}{\partial \nu} - \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial D. \end{cases} \tag{6}$$

has a nontrivial solution. Such values of $k > 0$ are called *transmission eigenvalues* corresponding to (6). The Fredholm property of (6) and the discreteness of transmission eigenvalues are studied in [23] and [12], whereas the existence of transmission eigenvalues for n_1 and $n(x)$ both real is shown in [6]. Note that if either $\Im(n_1) > 0$ or $\Im(n) > 0$ there are no transmission eigenvalues [4].

By means of the reciprocity gap functional operator R we can construct an integral equation of the first kind whose solution is an indicator function for D. Indeed, let $z \in \Omega$ consider $\mathbb{G}(\cdot, z)$ the Green's function of the background medium (5) with source located at z. We note that for the gap reciprocity functional method the actual index of refraction outside Ω does not play any role, therefore simplifying assumptions can be used in the construction of $\mathbb{G}(\cdot, z)$ as long as (5) is satisfied (see [22]). Then, if $\{\mathcal{S}\varphi\}$ for φ in some appropriate Hilbert space, form a dense subset of $\mathbb{H}(\Omega)$, we look for a solution $\varphi := \varphi_z$, $z \in \Omega$ to the first kind integral equation

$$R(\mathcal{S}\varphi)(x_0) = \mathcal{R}_{\partial\Omega}(u(\cdot, x_0), \mathbb{G}(\cdot, z)) \qquad \forall x_0 \in C, \tag{7}$$

where

$$R(\mathcal{S}\varphi)(x_0) := \mathcal{R}_{\partial\Omega}(u(\cdot, x_0), \mathcal{S}\varphi).$$

To fix our ideas we take the family of single layer potentials given by

$$(\mathcal{S}\varphi)(y) := \int_\Sigma \varphi(x)\, \mathbb{G}(x, y)\, ds_x \qquad \varphi \in L^2(\Sigma)$$

where Σ is a close or open manifold lying in the exterior of $\partial\Omega$ (possibly equal to $\partial\Omega$), which forms a dense set in $\mathbb{H}(\Omega)$. Note that many other choices of such one parametric family of solutions are possible. The following theorem holds true [5, 10, 13, 14, 22].

Theorem 3.1: *Assume that k is not transmission eigenvalue corresponding to (6). Then* *(i) for $z \in D$ and a given $\varepsilon > 0$, there exists a $\varphi_z^\varepsilon \in L^2(\Sigma)$ such that*

$$\|R(\mathcal{S}\varphi_z^\varepsilon) - \mathcal{R}_{\partial\Omega}(u, \mathbb{G}(\cdot, z))\|_{L^2(C)} < \varepsilon \tag{8}$$

and $\mathcal{S}\varphi_z^\varepsilon$ converges in the $L^2(D)$–norm to $v := v_z$ as $\varepsilon \to 0$ where (v, w) is the unique solution of (6) with $f = \mathbb{G}(\cdot, z)|_{\partial D}$ and $h = \frac{\partial \mathbb{G}(\cdot, z_0)}{\partial \nu}|_{\partial D}$;

(ii) for a fixed $\varepsilon > 0$, we have that

$$\lim_{z \to \partial D} \|\mathcal{S}\varphi_z^\varepsilon\|_{L^2(D)} = \infty \qquad \textit{and} \qquad \lim_{z \to \partial D} \|\varphi_z^\varepsilon\|_{L^2(\partial\Omega)} = \infty;$$

(iii) for $z \in \Omega \setminus \overline{D}$ and a given $\varepsilon > 0$, every $\varphi_z^\varepsilon \in L^2(\Sigma)$ that satisfies

$$\|R(\mathcal{S}\varphi_z^\varepsilon) - \mathcal{R}_{\partial\Omega}(u, \mathbb{G}(\cdot, z))\|_{L^2(C)} < \varepsilon$$

is such that

$$\lim_{\varepsilon \to 0} \|\mathcal{S}\varphi_z^\varepsilon\|_{L^2(D)} = \infty \qquad \textit{and} \qquad \lim_{\varepsilon \to 0} \|\varphi_z^\varepsilon\|_{L^2(\Sigma)} = \infty.$$

The above theorem validates that φ_z^ε can be used as an indicator function for D.

The main drawback of the original reciprocity gap functional method is that one needs to compute a Green's function $\mathbb{G}(\cdot, \cdot)$ for the multi-layered medium inside Ω, which is not an easy a task, despite any simplifying assumptions we can make outside Ω. In this paper we propose a *multistep reciprocity gap functional method* which uses only the fundamental solution for a homogeneous medium. Roughly speaking, the idea is to use the first part of Theorem 3.1 multiple times going layer by layer to recover the Cauchy data at up to the last layer interface and then once again Theorem 3.1 for the last homogeneous layer containing the scatterer to recover D. In the following we present our idea for the simplified two-layered model Ω introduced in Section 2.

To begin, we recall that $\overline{D} \subset S$ and consider a closed curve Λ inside S containing the scatterer D. We play the above scenario considering S as being now the scattering object. Thus, we start by defining the corresponding reciprocity gap functional

$$\mathcal{R}^{(0)}_{\partial\Omega}(u, v) = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) ds,$$

with

$$\begin{aligned} &u \in \mathcal{U} = \{u(\cdot, x_0) \text{ solutions to (3) for all } x_0 \in C\} \\ &v \in \mathbb{H}_0(\Omega) = \left\{ v \in L^2(\Omega) \,:\, \Delta v + k^2 n_0 v = 0 \text{ in } \Omega \right\} \end{aligned}$$

where n_0 is the constant index of refraction in the annulus bounded by $\partial\Omega$ and ∂S and the equation is satisfied in the distributional sense. Setting $k_0 := \sqrt{n_0}k$, we now consider the fundamental solution $\Phi_{k_0}(\cdot, z_0)$ of the Helmholtz operator $(\Delta + k^2 n_0)$ with singularity at z_0, which is given by

$$\Phi_{k_0}(x, z_0) = \begin{cases} \frac{i}{4} H_0^{(1)}(k\sqrt{n_0}|x - z_0|) & \text{in } \mathbb{R}^2 \\[2ex] \dfrac{e^{ik\sqrt{n_0}|x - z_0|}}{4\pi|x - x_0|} & \text{in } \mathbb{R}^3 \end{cases}$$

Hence for $z_0 \in \Lambda$, $v_{z_0} \in \mathbb{H}_0(\Omega)$ satisfies

$$\mathcal{R}^{(0)}_{\partial\Omega}(u, v_{z_0}) = \mathcal{R}^{(0)}_{\partial\Omega}(u, \Phi_{k_0}(\cdot, z_0)) \tag{9}$$

if there is a solution (v_{z_0}, w_{z_0}) of the interior transmission problem

$$\begin{cases} \Delta v_{z_0} + k^2 n_0 v_{z_0} = 0 & \text{in } S \\ \Delta w_{z_0} + k^2 N(x) w_{z_0} = 0 & \text{in } S, \\ w_{z_0} - v_{z_0} = \Phi_{k_0}(\cdot, z_0) & \text{on } \partial S, \\ \dfrac{\partial w_{z_0}}{\partial \nu} - \dfrac{\partial v_{z_0}}{\partial \nu} = \dfrac{\Phi_{k_0}(\cdot, z_0)}{\partial \nu} & \text{on } \partial S. \end{cases} \tag{10}$$

such that v_{z_0} can be extended in $\mathbb{H}_0(\Omega)$. Note that (10) has always a unique solution $v_{z_0} \in L^2(S)$, $w_{z_0} \in L^2(S)$ such that $v_{z_0} - w_{z_0} \in H^2(S)$, provided that $k > 0$ is not a corresponding transmission eigenvalue [12], [23]. In particular, if a solution v_{z_0} to (9) exists for $z_0 \in \Lambda$, we have in principle the Cauchy data for w_{z_0} on ∂S

$$w_{z_0} = v_{z_0} - \Phi_{k_0}(\cdot, z_0), \qquad \frac{\partial w_{z_0}}{\partial \nu} = \frac{\partial v_{z_0}}{\partial \nu} + \frac{\Phi_{k_0}(\cdot, z_0)}{\partial \nu}, \qquad \text{for all } \ z_0 \in \Lambda. \tag{11}$$

Of course, it is not possible in general to extend v_{z_0} in $\mathbb{H}_0(\Omega)$, if (v_{z_0}, w_{z_0}) solves (10). Instead, by a denseness argument, for any $\epsilon > 0$, v_{z_0} can be approximated in the $L^2(S)$-norm by either a single-layer potential [4]

$$(\mathcal{S}^{(0)} \varphi^{\epsilon}_{z_0})(x) := \int_{\Sigma} \varphi^{\epsilon}_{z_0}(y)\, \Phi_{k_0}(x, y)\, ds_y \qquad \varphi^{\epsilon}_{z_0} \in L^2(\Sigma)$$

where Σ is a close or open manifold lying in the exterior of $\partial\Omega$ (possibly equal to $\partial\Omega$), or by a Herglotz wave function [9]

$$v^{(0)}_{g^{\epsilon}_{z_0}}(x) = \int_{\mathbb{S}} \mathrm{e}^{ik_0 x\cdot\hat{y}} g^{\epsilon}_{z_0}(\hat{y}) ds(\hat{y}) \qquad g^{\epsilon}_{z_0} \in L^2(\mathbb{S})$$

where $\hat{y} = y/|y| \in \mathbb{S}$ with $\mathbb{S} := \{x \in \mathbb{R}^d$ such that $|x| = 1,\ d = 2, 3\}$ being the unit sphere. According to Theorem 3.1, either $\varphi^{\epsilon}_{z_0} \in L^2(\Sigma)$ or $g^{\epsilon}_{z_0} \in L^2(\mathbb{S})$ is an *approximate solution* to

$$\mathcal{R}^{(0)}_{\partial\Omega}(u(\cdot, x_0), \mathcal{S}^{(0)}\varphi) = \mathcal{R}^{(0)}_{\partial\Omega}(u(\cdot, x_0), \Phi_{k_0}(\cdot, z_0)) \tag{12}$$

$$\mathcal{R}^{(0)}_{\partial\Omega}(u(\cdot, x_0), v^{(0)}_g) = \mathcal{R}^{(0)}_{\partial\Omega}(u(\cdot, x_0), \Phi_{k_0}(\cdot, z_0)) \tag{13}$$

with discrepancy $\epsilon > 0$ in the $L^2(C)$-norm. In other words this means that the approximate solution of (12) and (13) provide an approximation for the Cauchy data of w_{z_0} on ∂S.

Remark 1 : Although here we assume that the interface ∂S is known, it is possible to reconstruct ∂S by mean of the approximate solution of (12) or (13), $\varphi^{\epsilon}_{z_0} \in L^2(\partial\Omega)$ or $g^{\epsilon}_{z_0} \in L^2(\mathbb{S})$ respectively, using their behavior stated in Theorem 3.1 formulated here for $\mathcal{R}^{(0)}_{\partial\Omega}$.

Now, having available the Cauchy data (11) for w_{z_0} on the interface ∂S with the phantom sources $z_0$located on Λ (which we have a priori assumed contains the scatterer D and is contained in S), we can proceed with a new step of the reciprocity gap functional method with measurements on ∂S to recover the shape of the unknown target D. Note that here the measurements manifold ∂S contains inside the source manifold Λ, which is the reversed situation to what we typically have in the standard setting of the reciprocity gap functional method. Hence, we can now define a new reciprocity gap functional with sources located inside our domain S. In particular

$$\mathcal{R}^{(1)}_{\partial S}(w,p) = \int_{\partial S} \left(w\frac{\partial p}{\partial \nu} - p\frac{\partial w}{\partial \nu} \right) ds, \tag{14}$$

with

$$w \in \mathcal{W} = \{w(\cdot, z_0) := w_{z_0}(\cdot) \text{ where } (v_{z_0}, w_{z_0}) \text{ solves (10) for } z_0 \in \Lambda\}$$

and

$$p \in \mathbb{H}_1(S) = \left\{p \in L^2(S) \,:\, \Delta p + k_1^2 p = 0 \text{ in } S\right\},$$

where $k_1 = k\sqrt{n_1}$, n_1 is the index of refraction of the medium S and the equation is satisfied in the distributional sense. Note that since both w and p are in $L^2(S)$ with Laplacian in $L^2(S)$, the integral in (14) is well defined by the Green's formula [4]. The relation (14) defines the reciprocity gap operator $R^{(1)} : \mathbb{H}_1(S) \to L^2(\Lambda)$ by

$$R^{(1)}(p)(z_0) := \mathcal{R}^{(1)}_{\partial S}(w(\cdot, z_0), p). \tag{15}$$

The next goal is to study properties of the reciprocity gap operator $R^{(1)} : \mathbb{H}_1(S) \to L^2(\Lambda)$. To this end we can prove the following theorem.

Theorem 3.2: *Assume that k is not a transmission eigenvalue corresponding to (10) and $k\sqrt{n_0}$ is not a Dirichlet eigenvalue of $-\Delta$ inside Λ. Then the reciprocity gap operator $R^{(1)} : \mathbb{H}_1(S) \to L^2(\Lambda)$ is injective with dense range.*

Proof: Let $R^{(1)}p = 0$ which means that $\mathcal{R}^{(1)}_{\partial S}(w(\cdot, z_0, p) = 0$ for all $z_0 \in \Lambda$. Using Green's formula

$$\begin{aligned} 0 &= \int_{\partial S} \left(w(\cdot, z_0)\frac{\partial p}{\partial \nu} - p\frac{\partial w(\cdot, z_0)}{\partial \nu} \right) ds \\ &= \int_{\partial D} \left(w(\cdot, z_0)\frac{\partial p}{\partial \nu} - p\frac{\partial w(\cdot, z_0)}{\partial \nu} \right) ds = k^2 \int_D (n - n_1) w(\cdot, z_0) p dx \end{aligned} \tag{16}$$

Now let $\tilde{w}, \tilde{v}$, $d = 2, 3$, be the solution of the interior transmission problem (which exists due to the assumption on k [23])

$$\Delta \tilde{w} + k^2 N(x)\, \tilde{w} = k^2 (N(x) - N_b(x)) p \qquad \text{in} \quad S \tag{17}$$

$$\Delta \tilde{v} + k^2 n_0 \tilde{v} = 0 \qquad \text{in} \quad S \tag{18}$$

$$\tilde{w} = \tilde{v} \qquad \text{on} \quad \partial S \tag{19}$$

$$\frac{\partial \tilde{w}}{\partial \nu} = \frac{\partial \tilde{v}}{\partial \nu} \qquad \text{on} \quad \partial S \tag{20}$$

where $N(x)$ is given by (2) and $N_b(x)$ is given by (1) (note that $N - N_b = n - n_1$ in D and $N - N_b = 0$ in $S \setminus \overline{D}$). Then (16) can be rewritten as

$$\int_D \left(\Delta\tilde{w} + k^2 n\,\tilde{w}\right) w(\cdot, z_0)\,dx = 0. \tag{21}$$

Using Green's formula and the equation satisfied by $w(\cdot, z_0) := w_{z_0}$ in (10), we obtain

$$\begin{aligned} 0 = \int_D \left(\Delta\tilde{w} + k^2 n\,\tilde{w}\right) w(\cdot, z_0)\,dx &= \int_D \left(\Delta w(\cdot, z_0) + k^2 n w(\cdot, z_0)\right) \tilde{w}\,dx \\ + \int_{\partial D} \left(w(\cdot, z_0)\frac{\partial \tilde{w}}{\partial \nu} - \tilde{w}\frac{\partial w(\cdot, z_0)}{\partial \nu}\right) ds &= \int_{\partial S} \left(w(\cdot, z_0)\frac{\partial \tilde{w}}{\partial \nu} - \tilde{w}\frac{\partial w(\cdot, z_0)}{\partial \nu}\right) ds. \end{aligned} \tag{22}$$

Using now the boundary conditions for $w(\cdot, z_0) := w_{z_0}$ in (10) and the boundary conditions for $\tilde{w}$, we have

$$\begin{aligned} 0 = \int_{\partial S} \left(w(\cdot, z_0)\frac{\partial \tilde{w}}{\partial \nu} - \tilde{w}\frac{\partial w(\cdot, z_0)}{\partial \nu}\right) ds &= \int_{\partial S} \left(\Phi_{k_0}(\cdot, z_0)\frac{\partial \tilde{v}}{\partial \nu} - \tilde{v}\frac{\partial \Phi_{k_0}(\cdot, z_0)}{\partial \nu}\right) ds \\ + \int_{\partial S} \left(v_{z_0}\frac{\partial \tilde{v}}{\partial \nu} - \tilde{v}\frac{\partial v_{z_0}}{\partial \nu}\right) ds &= \tilde{v}(z_0) \end{aligned}$$

Hence, $\tilde{v}(z_0) = 0$ for all $z_0 \in \Lambda$ and since $\tilde{v}$ is a solution of $\Delta\tilde{v} + k^2 n_0 \tilde{v} = 0$ inside Λ, we can conclude that $\tilde{v} = 0$ inside the domain bounded by Λ (since we have assumed that Λ is such that that $k\sqrt{n_0}$ is not a Dirichlet eigenvalue). Finally, by the unique continuation principle, we have that $\tilde{v} = 0$ in S and from Holmgren's Theorem, $\tilde{w} = 0$ in S also. Hence from (17), $p = 0$ which proves injectivity.

Next, to prove that $R^{(1)} : \mathbb{H}_1(S) \to L^2(\Lambda)$ has dense range, we consider $\beta \in L^2(\Lambda)$ such that

$$(R^{(1)}p, \beta)_{L^2(\Lambda)} = 0 \qquad \text{for all } v \in \mathbb{H}_1(S).$$

Then from (14) and (15) and the bi-linearity of $\mathcal{R}^{(1)}$, we have that

$$0 = (R^{(1)}p, \beta)_{L^2(\Lambda)} = \mathcal{R}^{(1)}(\mathcal{Q}, p)$$

where

$$\mathcal{Q}(x) = \int_\Lambda \beta(z_0) w(x, z_0)\,ds_{z_0}$$

and $w(\cdot, z_0) := w_{z_0}$ with v_{z_0}, w_{z_0} being the solution of (10). Let us denote by

$$\mathcal{P}(x) = \int_\Lambda \beta(z_0)(\Phi_{k_0}(x, z_0) + v_{z_0})ds_{z_0}.$$

Hence from Green's formula we have that

$$0 = \mathcal{R}^{(1)}(\mathcal{Q}, p) = k^2 \int\limits_D (n - n_1)\mathcal{Q}\, p\, dx \qquad \text{for all } \; p \in \mathbb{H}_1(S)$$

which implies that $\mathcal{Q} = 0$ in D (see [3], [5]). Hence by the unique continuation principle $\mathcal{Q} = 0$ in S since $\mathcal{Q}$ satisfies $\Delta\mathcal{Q} + k^2 N(x)\mathcal{Q} = 0$ in S. But due to (11) and superposition principle we have that $\mathcal{Q} = \mathcal{P} = 0$ in the annulus bounded by ∂S and Λ. Owning to the continuity of the single layer potential we have that $\mathcal{P}$ is continuous across Λ and thus, due to the assumption on k, $\mathcal{P} = 0$ inside Λ by uniqueness of the Dirichlet problem inside Λ. Finally, by the jump property of the normal derivative of single layer potential across Λ we obtain that $\beta = 0$ which proves the theorem. □

Remark 2 : The values of $k > 0$ excluded in Theorem 3.2 form at most a discrete set with $+\infty$ as the only accumulation point [4], [6], [23]. Furthermore if either $\Im(N) > 0$ inside D or $\Im(n_0) > 0$ there are no transmission eigenvalues corresponding to (10) [4]. The assumption that $k\sqrt{n_0}$ is not a Dirichlet eigenvalue of $-\Delta$ inside Λ is not a restriction since we can choose Λ to satisfy this assumption the known n_0.

In order to construct an indicator function for the support of the unknown inhomogeneity we parametrize the space $\mathbb{H}_1(S)$ by means of one parametric dense subspace. In particular, one can choose either the set of single-layer potentials

$$(\mathcal{S}^{(1)}\varphi)(x) := \int_\Sigma \varphi(y)\, \Phi_{k_1}(x, y)\, ds_y \qquad \varphi \in L^2(\Sigma) \tag{23}$$

where Σ is a close or open manifold lying in the exterior of S, or the set of Herglotz wave functions [9]

$$v_g^{(1)}(x) = \int_{\mathbb{S}} \mathrm{e}^{ik_1 x\cdot\hat{y}} g(\hat{y}) ds(\hat{y}) \qquad g \in L^2(\mathbb{S}) \tag{24}$$

where $\hat{y} = y/|y| \in \mathbb{S}$ with $\mathbb{S} := \{x \in \mathbb{R}^d$ such that $|x| = 1,\ d = 2,3\}$ being the unit sphere. Both the above form dense subspace of $\mathbb{H}_1(S)$ with respect to the $L^2(S)$-norm [2], [9]. With the help of the reciprocity gap functional operator we can construct the following integral operators

$$\mathbb{R} : L^2(\Sigma) \to L^2(\Lambda), \qquad \mathbb{R}(\varphi)(z_0) := R^{(1)}(\mathcal{S}^{(1)}\varphi)(z_0)$$

if (23) is used, and

$$\mathbb{R} : L^2(\mathbb{S}) \to L^2(\Lambda), \qquad \mathbb{R}(g)(z_0) := R^{(1)}(v^{(1)}g)(z_0)$$

if (24) is used. Note that as integral operators of the first kind with smooth kernel both the above operators are *compact*. Next, letting

$$\ell_z(z_0) := \mathcal{R}^{(0)}_{\partial S}(w(\cdot, z_0), \Phi_{k_1}(\cdot, z)), \qquad \ell_z \in L^2(\Lambda)$$

we can construct the following first kind ill-posed integral equations

$$\mathbb{R}(\varphi) = \ell_z\ , \qquad \text{for } z \text{ inside } \Lambda \text{ or} \tag{25}$$

$$\mathbb{R}(g) = \ell_z\ , \qquad \text{for } z \text{ inside } \Lambda \tag{26}$$

which we try to solve for $\varphi \in L^2(\Sigma)$ or $g \in L^2(\mathbb{S})$, respectively, of course using regularization techniques, such as Tikhonov regularization. Note that the use of Tikhonov regularization technique is mathematically justified by Theorem 3.2 [8]. For what follows, we need to recall the interior transmission problem corresponding to D

$$\begin{cases} \Delta v_z + k^2 n_1 v_z = 0 & \text{in } D \\ \Delta \varpi_z + k^2 n(x) \varpi_z = 0 & \text{in } D, \\ \varpi_z - v_z = \Phi_{k_1}(\cdot, z) & \text{on } \partial D, \\ \dfrac{\partial \varpi_z}{\partial \nu} - \dfrac{\partial v_z}{\partial \nu} = \dfrac{\partial \Phi_{k_1}(\cdot, z)}{\partial \nu} & \text{on } \partial D. \end{cases} \tag{27}$$

The theorem bellow describes the behavior of the solution φ and g of (25) and (26) respectively, for $z \in D$ and $z \notin D$. To avoid repetition we formulate the result only for (26). Exactly the same can be claimed for the integral equation (25).

Theorem 3.3: *Assume that $k > 0$ is not one of the values excluded in Theorem 3.2 and in addition is not a transmission eigenvalue corresponding to* (27)*. Then*

(1) *For $z \in D$ and given $\epsilon > 0$, there exists a $g_z^\epsilon \in L^2(\mathbb{S})$ such that*

$$\|\mathbb{R}(g_z^\epsilon) - \ell_z\|_{L^2(\Lambda)} < \epsilon$$

and the corresponding Herglotz wave function $v^{(1)}_{g_z^\epsilon}$ given by (24) *converges to v_z in the $L^2(D)$ norm where v_z, ϖ_z is the solution of* (27)*.*

(2) *For fixed $\epsilon > 0$, we have that*

$$\lim_{z\to\partial D} \|v^{(1)}_{g_z^\epsilon}\|_{L^2(D)} = \infty, \qquad \textit{and} \qquad \lim_{z\to\partial D} \|g_z^\epsilon\|_{L^2(\mathbb{S})} = \infty.$$

(3) *For z in the annulus bounded by ∂D and Λ and given $\epsilon > 0$, every $g_z^\epsilon \in L^2(\mathbb{S})$ that satisfies*

$$\|\mathbb{R}(g_z^\epsilon) - \ell_z\|_{L^2(\Lambda)} < \epsilon$$

is such that

$$\lim_{\epsilon\to 0} \|v^{(1)}_{g_z^\epsilon}\|_{L^2(D)} = \infty, \qquad \textit{and} \qquad \lim_{\epsilon\to 0} \|g_z^\epsilon\|_{L^2(\mathbb{S})} = \infty.$$

Proof: Here we provide a sketch of the proof since the main ingredients of the proof are provided by Theorem 3.2 (see [5] for more details). Let v_z, ϖ_z both in $L^2(D)$ be the solution of (27). Integrating by parts leads to

$$R^{(1)}p = k^2 \int_D (n - n_1) w(\cdot, z_0) p dx. \tag{28}$$

If p coincided with v_z, then since $w(\cdot, z_0)$ and ϖ_z satisfy the same equation in D we obtain that $R^{(1)}p = R^{(1)}\Phi_{k_1}(\cdot, z)$. In general, for given $\epsilon > 0$ it is possible to find a $g_z^\epsilon \in L^2(\mathbb{S})$ such that $v^{(1)}_{g_z^\epsilon}$ approximate v_z with respect to the $L^2(D)$-norm with discrepancy ϵ [9]. Furthermore, from the well-posedness of the exterior transmission problem we have that

$$\|\Phi_{k_1}(\cdot, z)\|_{H^1_{loc}(\mathbb{R}^d\setminus\overline{D})} \le \|v_z\|_{L^2(D)}.$$

Hence due to the singularity of the fundamental solution we can conclude that $\|v_z\|_{L^2(D)} \to \infty$ as $z \to \partial D$ and so do $\|v^{(1)}_{g_z^\epsilon}\|_{L^2(D)}$ and $\|g_z^\epsilon\|_{L^2(\mathbb{S})}$. This proves part *(1)* and *(2)* of the theorem.

To prove the last part of the theorem we start by using the Green's formula to obtain

$$\begin{aligned}\mathbb{R}(g_z^\epsilon) - \ell_z = k^2 \int_D (n - n_1) w(\cdot, z_0) v^{(1)}_{g_z^\epsilon} dx + \int_S (n_0 - n_1)\Phi_{k_1}(\cdot, z) v_{z_0} dx \\ + v_{z_0}(z) + \Phi_{k_1}(z_0, z).\end{aligned}$$

Now we let $\epsilon \to 0$ and assume that $(\mathbb{R}(g_z^\epsilon) - \ell_z) \to 0$ and $\|v^{(1)}_{g_z^\epsilon}\|_{L^2(D)}$ remains bounded. Thus $\|v^{(1)}_{g_z^\epsilon}\|_{L^2(D)}$ converges weakly to a function $v \in L^2(D)$ and furthermore we have

$$\Phi_{k_1}(z_0, z) = k^2 \int_D (n - n_1) w(\cdot, z_0) v dx + \int_S (n_0 - n_1)\Phi_{k_1}(\cdot, z) v_{z_0} dx + v_{z_0}(z).$$

Since the right hand side is regular and as a function of z_0 can be uniquely continued for $z_0 \in S \setminus \overline{D}$ as a solution of $\Delta u + k_1^2 u = 0$, we arrive at a contradiction by letting $z_0 \to z$. Hence $\|v^{(1)}_{g_z^\epsilon}\|_{L^2(D)}$ is not bounded as $\epsilon \to 0$ and so is the kernel $\|g_z^\epsilon\|_{L^2(\mathbb{S})}$, which ends the proof of the theorem. □

We end this section with the following remark on the case of buried object under the earth surface.

Remark 3 : The above procedure can be applied to recover buried objects in a two-layered unbound medium, such as under the earth or water surface. Assuming that the receivers are distributed over a much larger surface then the transmitters, we can assume that the total field on the sides and bottom a rectangular region containing the target is very small. Hence, we can apply the first step of the reciprocity gap functional on the open measurement surface to recover the data on a closed interface and then proceed as in the above discussion

4. Numerical Test

In this section we shall present some preliminary examples using synthetic data for the Helmholtz equation in $\mathbb{R}^2$. The forward problem is solved by a finite element solver. The Cauchy data recorded has around 3% noise. We use Tikhonov regularization and the Morozov discrepancy principle for solving the ill-posed integral equations (13) and (26). Our numerical examples are performed only for the case of parametrization by Herglotz wave functions for both steps. Roughly speaking our algorithm can be described as follows. We first solve (13) for g and compute the corresponding Herglotz function $v_g^{(0)}$. Then we compute approximated Cauchy data for w_{z_0}, for $z_0 \in \Lambda$ using (11) where v_{z_0} is replaced by the computed Herglotz function. As a second step, we solve (26) for g and the target is visualized by plotting the level curves of $1/\|g\|_{L^2(\mathbb{S})}$. The exact boundary of the target is also shown as the solid line in the same plot for reference.

4.1. *Non-absorbing medium*

We first consider the case of non-absorbing medium. The domain Ω is a disk whose radius is $r_\Omega = 4$. On $\partial\Omega$ we record the Cauchy data of the total field. Another domain S, inside Ω, is a disk whose radius is $r_s = 3$. Outside S, the refractive index is 1. Inside S, $n = 2$. The refractive index of the target D, embedded in S is set to $n = 10$. Between ∂S and ∂D, we choose another curve Λ, the circumference of a cirlce whose radius is $r_\Lambda = 2$. On Λ we choose uniformly the locations of the phantom sources z_0's. Note that the actual sources x_0 are on some curve C outside Ω. In the implementation, we choose a circle with radius $r_c = 6$. For all examples, we set $k = 3$.

For the first example, we choose D as a triangular target whose vertices are $(0, 1/2)$, $(1/3, -1/3)$, $(-1/3, -1/3)$. We put 20 point sources uniformly on C and calculate the Cauchy data on $\partial\Omega$ using the forward finite element solver. Then we solve the linear integral equations (13) for the phantom sources z_0's on Λ to obtain approximate Herglotz wave functions $v_g^{(0)}$'s. We evaluate the sum of $v_{g_{z_0}}$'s and $\Phi_{k_0}(\cdot, z_0)$'s on the ∂S to obtain the Cauchy data of w_{z_0} needed in (26). Then we choose a sampling domain inside Λ and apply the Reciprocity Gap Functional method on ∂S with the phantom sources on Λ. Fig. 2 shows the reconstruction of the support of the triangular target. The solid line is the exact boundary of the target.

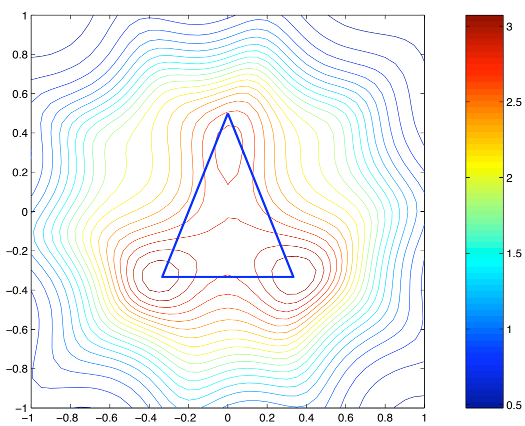

Figure 2. Contour plot of $1/\|g_z\|$ for the triangular target using the multistep method.

For the second example, we choose a circle with radius $r_D = 0.5$ as the target. Fig. 3 shows the reconstruction of the support of the circular target.

For comparison, we also show the reconstructions of the target by the Reciprocity Gap method using the inhomogeneous background Green's function in Fig. 4

4.2. *Absorbing Medium*

Next we consider the case when S is absorbing medium. We shall test the case of the triangular target only. Two refractive indices $n = 2 + 0.02i$ and $n = 2 + 0.2i$ are

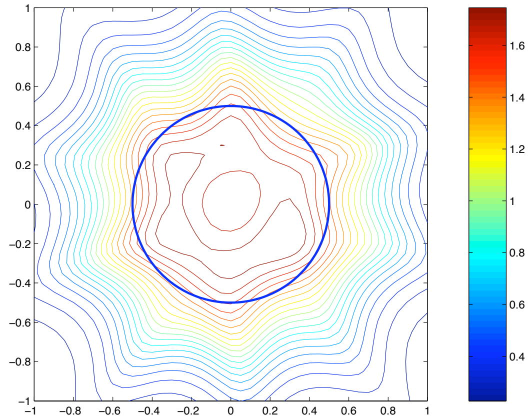

Figure 3. Contour plot of $1/\|h_z\|$ for the circular target.

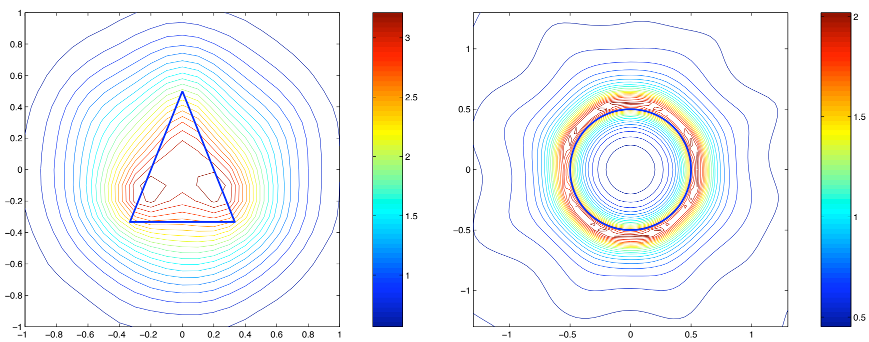

Figure 4. The contour plots of $1/\|g_z\|$ for the targets by the Reciprocity Gap method with the inhomogeneous background Green's function. Left: the triangular target. Right: the circular target.

chosen for S. Everything else keeps the same as the above case of the non-absorbing medium. In Fig. 5, we show the reconstructions of the support of D. It can be seen that the reconstruction is similar to the case of non-absorbing medium when the absorption is comparably small. For larger absorption, we are still able to obtain the correct location and size of the target.

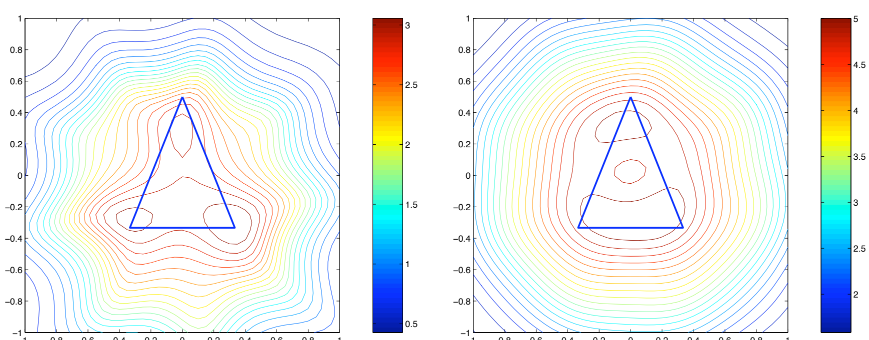

Figure 5. The contour plots of $1/\|g_z\|$ for the triangular target using the multistep method. Left: refractive index of S: $n = 2 + 0.02i$. Right: refractive index of S: $n = 2 + 0.2i$.

Again for comparison, we show the construction of the target by the Reciprocity Gap method using the inhomogeneous background Green's function in Fig. 6

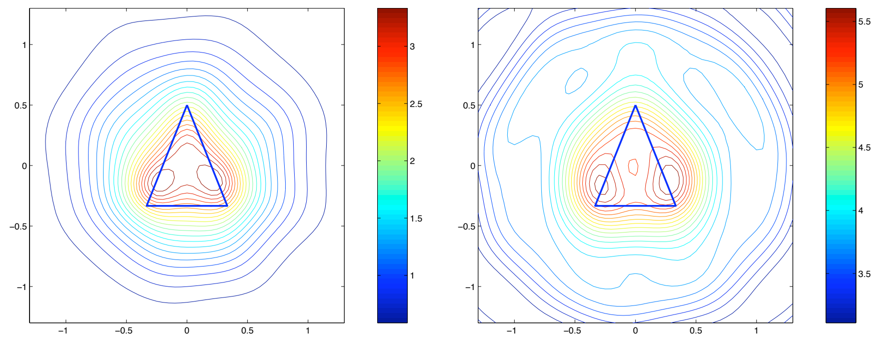

Figure 6. The contour plots of $1/\|h_z\|$ for the triangular target by the Reciprocity Gap method with the correct background Green's function. Left: refractive index of S: $n = 2 + 0.02i$. Right: refractive index of S: $n = 2 + 0.2i$.

4.3. *Buried Target*

Finally, we consider problem of the detection of a target buried in the earth. Unlike the above examples, we are only able to put the point sources above the air-earth interface and the measurement can not be done all round the target as well. Since the wave field attenuates quickly away from the sources and the target inside the earth, one should take measurement on a long line above the air-earth interface as shown in Fig. 7 (see Remark **3**). We set $k = 3$ in the simulation. The 40 point source locations are slightly above the air-earth interface. The target is buried in the earth under the point source locations. The curve Λ of the locations of the phantom point sources is inside the earth between the target and the air-earth interface (see Fig. 7).

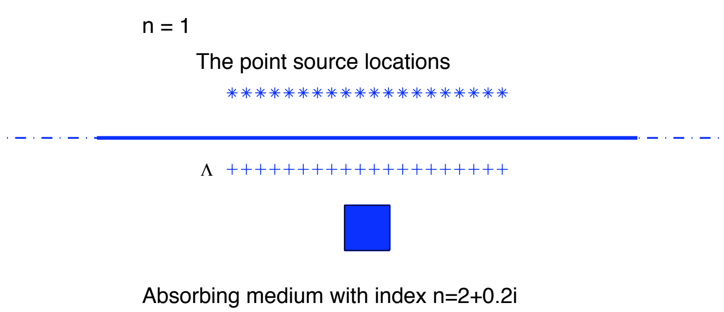

Figure 7. Explicative figure for the buried target. The refractive indexes are chosen to be 1 for the air and $2 + 0.2i$ for the earth. The point source locations are immediately above the air-earth interface. The refractive index of the target is 10. Λ is between the air-earth interface and the buried target. Measurements are taken on the air-earth interface above the target.

We set $n = 1$ for the air, $n = 2 + 0.2i$ for the earth and $n = 10$ for the target. A target, whose boundary is shown as the solid line in Fig. 8, is buried in the earch. The reconstruction is shown in Fig. 8.

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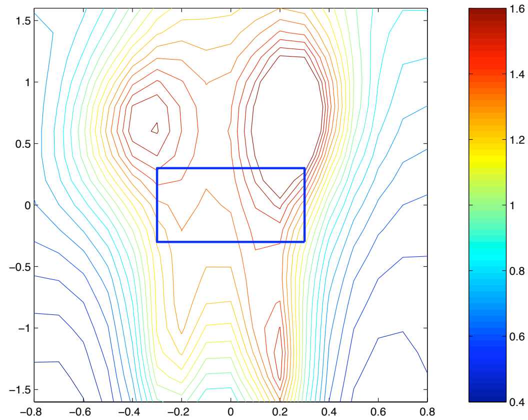

Figure 8. Contour plot of $1/\|g_z\|$ for the buried target using the multistep method.

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