A decomposition method for an interior inverse scattering problem *

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Abstract

We consider an interior inverse scattering problem of reconstructing the shape of a cavity. The measurements are the scattered fields on a curve inside the cavity due to one point source. We employ the decomposition method to reconstruct the cavity and present some convergence result. Numerical examples are provided to show the effectiveness of the method.

1 Introduction

Typical inverse scattering problems are considered as exterior problems in the sense that the scattering objects are illuminated by incident waves from the exterior of the objects. The measurements are also in the exterior of the objects, or even far way from them (far field pattern). Examples of such inverse problems appear in a wide range of applications such as radar techniques, geophysical explorations, non-destructive testing, etc [2]. In this paper, we consider a different class of inverse scattering problems, i.e. the *interior* inverse scattering problems of reconstructing the boundary of a cavity. These problems can arise from non-destructive testing in industrial applications such as the test of the structural integrity of cavities [8]. The term *interior* comes from the fact that the sources (incident waves) and measurements (scattered waves) are both inside the cavity. The goal is to reconstruct the shape of the scattering object. To be precise, we consider a bounded domain $D \subset \mathbb{R}^2$ with the sound soft boundary condition on ∂D. Assuming the measurements are available on some curve C inside D due to one point source, the interior inverse scattering problem we are interested in is to reconstruct ∂D (see Fig. 1).

The interior inverse scattering problem is a fairly new research topic. There are a few recent works. In [8], Jakubik and Potthast use the solutions of the Cauchy problem by potential methods and the range test to study the integrity of the boundary of some

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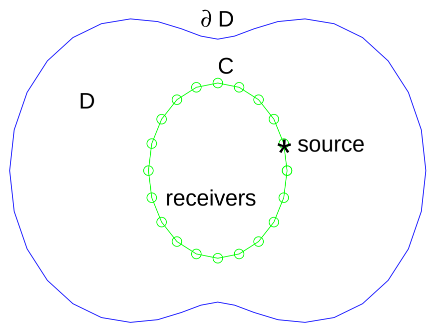

Figure 1: Explicative figure. The point source and receivers are on a curve C inside D.

cavity by acoustic waves. Assuming the measurements on C are available for all point sources on the same curve, Qin and Colton apply the linear sampling method to the above problem in 2D case in [19]. They further extend their method to reconstruct both the shape of the cavity and the surface impedance in [20]. Zeng et. al. [21] consider the interior electromagnetic scattering problem in 3D case using the linear sampling method. Note that the above papers require multistatic data, i.e., the scattered fields on C due to many point sources. For the case of one point source and several measurements, nonlinear integral equations have been used to reconstruct the cavity [18].

In this paper, we study the determination of the shape of the cavity using the decomposition method due to Kirsch and Kress [11] (see also Section 5.4 of [2] and [14, 3]). The original version of the method was for the exterior inverse scattering problems using far field data. It has a close connection to the approach of Colton and Monk [4, 5]. We refer the readers to [12, 14, 3] and the references therein for further discussion of the method and its applications to other types of the inverse scattering problems. Our treatment of the decomposition method for interior inverse scattering problems follows the steps in Section 5.4 of [2].

The rest of the paper is organized as follows. In Section 2, we introduce the interior direct and inverse scattering problems and present recent results of uniqueness theorems in [19, 20, 18]. In Section 3, we discuss the decomposition method for the interior inverse scattering problem which breaks into two parts: the first part deals with the ill-posedness by constructing the scattered field from the measurement on C and the second part deals with the nonlinearity by reconstructing ∂D as the minimization curve of the L^2 norm of the total field. In Section 4, we present some numerical examples.

Finally, in Section 5, we make conclusions and discuss some future works.

2 The interior inverse scattering problem

We first consider the interior scattering problem due to a point source inside the cavity. Let $D \subset \mathbb{R}^2$ be a bounded simply connected domain with C^2 boundary ∂D. Let $k > 0$ be the wave number and z be a point inside D. The direct scattering problem is to find the scattered field u^s such that

$$\triangle u^s + k^2 u^s = 0, \qquad \text{in } D, \tag{2.1a}$$

$$u^s = -\Phi(\cdot, z), \qquad \text{on } \partial D. \tag{2.1b}$$

Here Φ is the fundamental solution to the Helmholtz equation given by

$$\Phi(x, z) = \frac{i}{4} H_0^{(1)}(k|x - z|) \tag{2.2}$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order zero. In the rest of the paper, we assume that $k^2 > 0$ is not a Dirichlet eigenvalue of D. The well-posedness of this scattering problem is well-known [2].

The inverse problem we are interested in is to determine the shape of D from the measurement of the scattered field on some curve C inside D due to a point source $z \in C$. Note that $z \in C$ is not an essential requirement.

We first present a uniqueness result from [19] for multistatic data. Assuming the knowledge of the scattered field $u^s(x, z)$ for all $x, z \in C$, Qin and Colton [19] proved the following theorem.

Theorem 2.1. *(Theorem 2.1 of [19]) If k^2 is not a Dirichlet eigenvalue for the interior of C, then the boundary ∂D is uniquely determined from $u^s(x, z)$ for $x, z \in C$.*

The requirement that k^2 is not a Dirichlet eigenvalue for the interior of C is non-essential since we can always change C. The proof of the above theorem in [19] is based on Schiffer's proof in 1967. Unfortunately, Schiffer's proof cannot be generalized immediately to other boundary conditions. In the following, we present a proof which is based on the approach of Kirsch and Kress for the exterior inverse scattering problem [10].

Proof. Assume that $D_1 \neq D_2$ are two bounded domains and u_1^s and u_2^s satisfy the scattering problem (2.1), respectively. Suppose that $u_1^s(x, z) = u_2^s(x, z)$ on C for all $z \in C$. Let $w = u_1^s - u_2^s$. We have that

$$\triangle w + k^2 w = 0, \qquad \text{in } \dot{C}, \tag{2.3a}$$

$$w = 0, \qquad \text{on } C, \tag{2.3b}$$

where $\dot{C}$ denotes the interior of C. Since k^2 is not a Dirichlet eigenvalue for $\dot{C}$, we have that $w = 0$ in $\dot{C} \cup C$.

Let D_0 be the connected component of $D_1 \cap D_2$ containing $\dot{C}$. By analyticity, $w = 0$ in $\overline{D}_0$, i.e.

$$u_1^s(x,z) = u_2^s(x,z)$$

for all $x \in \overline{D}_0$ and $z \in C$. By the reciprocity relation, we have that

$$u_1^s(z,x) = u_2^s(z,x)$$

for all $x \in \overline{D}_0$ and $z \in C$. Using the same argument as above, we have that

$$u_1^s(x,z) = u_2^s(x,z)$$

for all $x, z \in \overline{D}_0$.

Without loss of generality, there exists $x^* \in \partial D_0$ such that $x^* \in \partial D_1$ and $x^* \notin \partial D_2$. In particular, we set

$$z_n := x^* - \frac{1}{n}\nu(x^*) \in D_0$$

for sufficiently large n. Here $\nu(x^*)$ denotes the unit outward normal of ∂D_1 at x^*. In view of the well-posedness of the cavity problem for scatterer D_2, on one hand, we have that

$$\lim_{n\to\infty} u_2^s(x^*, z_n) = u_2^s(x^*, x^*).$$

On the other hand, we have that

$$\lim_{n\to\infty} u_1^s(x^*, z_n) = \infty$$

because of the boundary condition for u_1^s in terms of the point source located at $z_n \to x^*$ as $n \to \infty$. This is a contradiction and thus $D_1 = D_2$. □

In case of only finitely many incident wave, i.e., $u^s(x,z)$ is known for finitely many point sources $z \in C$, a general uniqueness theorem for the interior scattering problem is an open problem. However, if some a priori information on the size of D is available, the shape of D can be uniquely determined following the idea of Colton and Sleeman [6] for the exterior problems.

Theorem 2.2. *(Theorem 2.1 of [18]) Assume that D_1 and D_2 are two bounded simply connected regions containing C and contained in a disk of radius R, and let*

$$N := \sum_{t_{0l} < kR} 1 + \sum_{t_{nl} < kR, n\neq 0} 2, \tag{2.4}$$

where t_{nl} ($l = 0, 1, \ldots; n = 0, 1, \ldots$) denote the positive zeros of the Bessel functions J_n, i.e., $J_n(t_{nl}) = 0$. Denote by $u_1^s(\cdot, z)$ and $u_2^s(\cdot, z)$ the scattered field corresponding to D_1 and D_2, respectively, due to the point source $\Phi(\cdot, z)$. If $u_1^s(\cdot, z)$ and $u_2^s(\cdot, z)$ coincide on C for $N + 1$ distinct locations $z \in C$ and one fixed wave number k, then $D_1 = D_2$.

As a consequence of the above theorem, Qin and Cakoni [18] also prove the following uniqueness result.

Corollary 2.3. *(Corollary 2.2 of [18]) Assume that D_1 and D_2 are two bounded simply connected regions containing C and contained in a disk of radius R such that $kR < t_0$, where t_0 (≈ 2.40483) is the smallest positive zero of the Bessel function J_0. If the measured data $u^s(\cdot, z)$ on C coincide for one location $z \in C$ and one fixed wave number k, then $D_1 = D_2$.*

3 The decomposition method

In this section, we present the decomposition method to treat the interior inverse scattering problem. In general, the decomposition method can be divided into two steps: at the first step, the scattered field u^s is constructed from the measurements u^s on the curve C; at the second step, the unknown boundary ∂D of the cavity is determined by finding the location where the boundary condition for the total field $u^i + u^s$ is satisfied in a least-squares sense. The first step mainly deals with the ill-posedness of the inverse problem, and the second step deals with the nonlinearity of the problem [3, 11].

Suppose that the scattered field $u^s(x, z)$ is known for all $x \in C$ due to a point source $z \in C$, denoted by u^s_C. Assume that Γ is a curve outside D, i.e., a prior information on the rough size of D is known. We define the single layer potential by

$$(\mathcal{S}\phi)(x) := \int_\Gamma \phi(y)\Phi(x, y)\,\mathrm{d}s(y) \quad x \notin \Gamma. \tag{3.5}$$

Theorem 3.1. *Assume k^2 is not a Dirichlet eigenvalue for the negative Laplacian in the interior of Γ or in the interior of C. Then the operator $\mathcal{S} : L^2(\Gamma) \to L^2(C)$ is compact, injective and has a dense range.*

Proof. The compactness is obvious.

To show that $\mathcal{S}$ is injective, we set

$$(\mathcal{S}g)(x) = \int_\Gamma g(z)\Phi(x, z)\,\mathrm{d}s(z) = 0, \quad x \in C,$$

and define

$$w(x) = \int_\Gamma g(z)\Phi(x, z)\,\mathrm{d}s(z), \quad x \in \mathbb{R}^2 \setminus \Gamma.$$

Then $w(x)$ is the solution of

$$\triangle w + k^2 w = 0, \qquad \text{in } \dot{C}, \tag{3.6a}$$

$$w = 0, \qquad \text{on } C. \tag{3.6b}$$

Since k^2 is not a Dirichlet eigenvalue for the negative Laplacian in $\dot{C}$, we conclude that $w = 0$ in $\dot{C}$. In addition, $\triangle w + k^2 w = 0$ in $\dot{\Gamma}$ implies that $w = 0$ in $\dot{\Gamma} \cup \Gamma$ due to analyticity. Using the property of the single layer potential, we see that $w(x)$ also satisfy

$$\triangle w + k^2 w = 0, \qquad \text{in } \mathbb{R}^2 \setminus (\dot{\Gamma} \cup \Gamma), \tag{3.7a}$$

$$w = 0, \qquad \text{on } \Gamma, \tag{3.7b}$$

and the Sommerfeld radiation condition

$$\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial w}{\partial r}-ikw\right)=0$$

where $r=|x|$. Since the solution of the exterior Dirichlet problem is unique (see [2]), $w=0$ in $\mathbb{R}^2\setminus(\dot{\Gamma}\cup\Gamma)$. For the single layer potential, we have that

$$w_+=w_-,\quad \frac{\partial w_+}{\partial\nu}-\frac{\partial w_-}{\partial\nu}=-g,$$

where ν is the unit outward normal to Γ and $\pm$ denotes the limits as $x\to\Gamma$ from outside and inside. Thus we have $g=0$ and the injectivity is proved.

Next we show that $\mathcal{S}$ has a dense range. For $\psi\in L^2(C)$, we assume that $(\mathcal{S}g,\psi)=0$, i.e.

$$\int_C\int_\Gamma \Phi(x,z)g(z)\,\mathrm{d}s(z)\overline{\psi(x)}\,\mathrm{d}s(x)=0$$

for all $g\in L^2(\Gamma)$. Interchanging the order of integration we have that

$$\int_\Gamma\int_C \Phi(x,z)\overline{\psi(x)}\,\mathrm{d}s(x)g(z)\,\mathrm{d}s(z)=0$$

for all $g\in L^2(\Gamma)$. Thus we have

$$v(z)=\int_C \Phi(x,z)\overline{\psi(x)}\,\mathrm{d}s(x)=0$$

for all $z\in\Gamma$. Define

$$w(z)=\int_C \Phi(x,z)\overline{\psi(x)}\,\mathrm{d}s(x),\quad z\in\mathbb{R}^2\setminus C.$$

Then $w(x)$ satisfies $\triangle w+k^2w=0$ in $\dot{\Gamma}$ and $w(x)=0$ on Γ. Since k^2 is not Dirichlet eigenvalue for the negative Laplacian in $\dot{\Gamma}$, we have $w(x)=0$ in $\dot{\Gamma}$. Note that $\dot{C}\subset\dot{\Gamma}$. Using the jump relation of the single layer potential again, we have $\psi=0$. Hence $\mathcal{S}$ has dense range in $L^2(C)$.

□

Given the scattered field u^s_C, we can set up the ill-posed integral equation

$$\mathcal{S}\phi=u^s|_C. \tag{3.8}$$

To solve (3.8), we employ the classical Tikhonov regularization. In particular, we seek the approximation solution ϕ_α by solving the following regularized problem

$$\alpha\phi_\alpha+\mathcal{S}^*\mathcal{S}\phi_\alpha=\mathcal{S}^*u^s_C \tag{3.9}$$

where α denotes the regularization parameter and $\mathcal{S}^*:L^2(C)\to L^2(\Gamma)$ denotes the adjoint of $\mathcal{S}$. In the rest of the paper, α is chosen according to the Morozov's discrepancy principle. It is well-known that solving (3.9) is equivalent to the minimization of the Tikhonov functional [11, 2]

$$\|\mathcal{S}\phi_\alpha-u^s_C\|^2_{L^2(C)}+\alpha\|\phi_\alpha\|^2_{L^2(\Gamma)}. \tag{3.10}$$

Once we have found ϕ_α, we can compute the approximation $u^s_\alpha := \mathcal{S}\phi_\alpha$ for the scattered field u^s. Then the boundary of D is determined by identifying a curve so that the sound-soft boundary condition is approximately satisfied, i.e.,

$$u^i(x) + \int_\Gamma \phi_\alpha(y)\Phi(x,y)\,\mathrm{d}s(y) \approx 0. \tag{3.11}$$

A simple approach would be just plotting the absolute value of the left hand side of (3.11). One can choose a sampling domain inside Γ and outside C and plot the absolute value of the total field

$$U(x) = \left| u^i + \int_\Gamma \phi_\alpha(y)\Phi(x,y)\,\mathrm{d}s(y) \right|. \tag{3.12}$$

Then the closed curve outside C with rather small value of $U(x)$ can be taken as the reconstruction of ∂D.

It is obvious that measurements due to additional point sources can be added in a straightforward way. Suppose that we have measurements $u^s_{C,j}$ due to point sources $u^i_j = \Phi(x, z_j)$ at z_j, $j = 1, \dots, N$. For each $u^s_{C,j}$, we compute the corresponding regularized solution $\phi_{\alpha,j}$ of (3.9). Note that a fixed regularization parameter can be used if the noise levels are the same. Then we can plot the norm of the sum of the total field

$$U(x) = \sum_{j=1}^N \left| u^i_j + \int_\Gamma \phi_{\alpha,j}(y)\Phi(x,y)\,\mathrm{d}s(y) \right|. \tag{3.13}$$

The reconstruction of ∂D can be decided in the same way as for the case of a single point source.

The decomposition method treats (3.11) as an optimization problem. We select a family of starlike curves as the admissible set for the boundary of D. It is clear that ∂D is between C and Γ. Hence we can find r_1 and r_2 with $r_1 < r_2$ and define

$$V := \{ r \in C^{1,\beta}(\Omega) | 0 < r_1 < r < r_2 \} \tag{3.14}$$

where $\Omega = \{\hat{x} \in \mathbb{R}^2 | |\hat{x}| = 1\}$ and $C^{1,\beta}(\Omega), 0 \le \beta \le 1$ denotes the space of uniformly Hölder continuously differentiable functions. We define

$$A := \{ r(\hat{x})\hat{x} | \hat{x} \in \Omega \}, \quad r \in V. \tag{3.15}$$

The decomposition method seeks an approximation to ∂D by solving the following minimization problem

$$\min_{r \in V} \int_\Omega |(u^s_\alpha + \Phi(\cdot, z)) \circ r|^2\,\mathrm{d}s. \tag{3.16}$$

Combining the above two steps, we can formulate an optimization problem for the interior inverse scattering problem. For a given measurement $u^s_C \in L^2(C)$ and a regularization parameter α, define the functional $\mu : L^2(\Gamma) \times V \to \mathbb{R}$ by

$$\mu(\phi, r; u^s_C, \alpha) = \|\mathcal{S}\phi - u^s_C\|^2_{L^2(C)} + \alpha\|\phi\|^2_{L^2(\Gamma)} + \gamma\|(\mathcal{S}\phi + \Phi(\cdot,z)) \circ r\|^2_{L^2(\Omega)}. \tag{3.17}$$

Here $\gamma > 0$ denotes a coupling parameter. Ideally, the choice of γ would make the first and third terms in (3.17) of the same magnitude. For simplicity, we choose $\gamma = 1$ in the rest of the paper.

The interior inverse scattering problem is to minimize μ, i.e., to seek $(\phi, r) \in L^s(\Gamma) \times V$ such that

$$\mu(\phi, r; u_C^s, \alpha) = M(u_C^s, \alpha) := \inf\{\mu(\psi, q; u_C^s, \alpha) : \psi \in L^2(\Gamma), q \in V\}. \quad (3.18)$$

Theorem 3.2. *The optimization problem* (3.18) *of the interior inverse scattering problem has a solution.*

Proof. Let $(\phi_n, r_n) \in L^2(\Gamma) \times V$ be a minimizing sequence

$$\lim_{n\to\infty} \mu(\phi_n, r_n) = M(u_C^s, \alpha). \quad (3.19)$$

Since V is compact (see Ch. 3 of [2]), we have $r_n \to r$ for some $r \in V$ as $n \to \infty$. From (3.17), we obtain

$$\alpha \|\phi_n\|^2_{L^2(\Gamma)} \leq \mu(\phi_n, r_n) \to M(u_C^s, \alpha).$$

Thus (ϕ_n) is bounded and converges weakly to some $\phi \in L^2(\Gamma)$. Since $\mathcal{S}$ is compact, we have that

$$\mathcal{S}\phi_n \to \mathcal{S}\phi, \quad n \to \infty,$$

and

$$(\mathcal{S}\phi_n) \circ r_n \to (\mathcal{S}\phi) \circ r, \quad n \to \infty.$$

From (3.19), we see that $\|\phi_n\|^2 \to \|\phi\|^2, n \to \infty$. Together with the weak convergence, we have the norm convergence $\phi_n \to \phi, n \to \infty$. By continuity, we have

$$\mu(\phi, r) = \lim_{n\to\infty} \mu(\phi_n, r_n) = M(u_C^s, \alpha)$$

which completes the proof. ∎

Theorem 3.3. *Let u_C^s be the scattered field due to a point source at z on C. If the boundary of D can be represented by some $r \in V$, then*

$$\lim_{\alpha\to 0} M(u_C^s, \alpha) = 0. \quad (3.20)$$

Proof. We note that if k^2 is not a Dirichlet eigenvalue for the interior of Γ the range $\{\mathcal{S}\phi|_{\partial D} : \phi \in L^2(\Gamma)\}$ is dense in $L^2(\partial D)$. Then there exists $\phi \in L^2(\Gamma)$ such that

$$\|(\mathcal{S}\phi + \Phi(\cdot, z)) \circ r\|_{L^2(\Omega)} < \epsilon$$

for any $\epsilon > 0$. Since the solution of the Helmholtz equation continuously depends on the boundary data, there exists a constant c such that

$$\|\mathcal{S}\phi - u_C^s\|_{L^2(C)} \leq c\|(\mathcal{S}\phi - u^s) \circ r\|_{L^2(\Omega)}.$$

From the boundary condition on ∂D we have

$$\mu(\phi, r; u_C^s, \alpha) \leq (1 + c^2)\epsilon^2 + \alpha\|\phi\|^2_{L^2(\Gamma)} \to (1 + c^2)\epsilon^2, \quad \alpha \to 0.$$

The proof is complete since ϵ is arbitrary. ∎

Theorem 3.4. *Let u^s_C be the scattered field on a curve C inside D. Assume that ∂D can be represented by some $r \in V$. Let (α_n) be a null sequence and let (ϕ_n, r_n) be a solution to the minimization problem with regularization parameter α_n. Then there exists a convergent subsequence of (r_n). There is only a finite number of limit points and every limit point represents a surface on which the total field $u^s + u^i$ vanishes.*

Proof. Since V is compact, there exists a convergent subsequence of (r_n), still denoted by (r_n), such that

$$\lim_{n\to\infty} r_n = r^*.$$

Let u^* be the unique solution to the direct scattering problem for the domain with boundary Λ^* given by r^*. Thus we have that

$$(u^* + \Phi(\cdot, z)) \circ r^* = 0 \quad \text{on } \Omega. \tag{3.21}$$

We denote $\mathcal{S}\phi_n$ by u_n for $n = 1, 2, 3, ...$, the solutions of the interior scattering problems with boundary values $\mathcal{S}\phi_n|_{\Lambda_n}$ on Λ_n described by r_n. By Theorem 3.3, we have

$$\| (u_n + \Phi(\cdot, z)) \circ r_n\|_{L^2(\Omega)} \to 0, \quad n \to \infty. \tag{3.22}$$

Since $r_n \to r^*$ as $n \to \infty$, we have that

$$\|u_n - u^*\|_{L^2(\Lambda_n)} \to 0, \quad n \to \infty.$$

This implies that $S\phi_n$ converges uniformly to u^*. On the other hand, Theorem 3.3 implies

$$\|S\phi_n - u^s_C\|_{L^2(C)} \to 0, \quad n \to \infty.$$

Assuming that k^2 is not a Dirichlet eigenvalue for $\dot{C}$, we have that $S\phi_n$ converges uniformly to u^s in $\dot{C}$. Thus we have $u^s = u^*$ by analytic continuation and (3.21) implies that $u^s + u^i$ vanishes on Λ^*.

If there are an infinite number of different limit points, there exists a convergent sub-sequence of these limit points. This implies that there are arbitrary small domains for which $u^s + u^i$ are eigenfunctions for the Laplacian. This is not possible and the proof is complete. ☐

The problem of multiple point sources can be formulated in a straightforward manner. Assuming that $z_1, \ldots, z_n$ are n point sources and $u^{s,1}_C, \ldots, u^{s,n}_C$ are the corresponding scattered fields on C, then the minimization functional is simply

$$\mu(\phi_1, \ldots, \phi_n, r; u^{s,1}_C, \ldots, u^{s,n}_C, \alpha)$$

given by

$$\sum_{i=1}^{n} \left\{ \|\mathcal{S}\phi_i - u^s_C\|^2_{L^2(C)} + \alpha\|\phi_i\|^2_{L^2(\Gamma)} + \|(\mathcal{S}\phi_i + \Phi(\cdot, z)) \circ r\|^2_{L^2(\Omega)} \right\}.$$

Similar to the exterior inverse scattering problem, we can not expect more than sub-sequence convergence due to the fact that we do not have uniqueness either for the interior inverse scattering problem or for the optimization problem (see Section 5.4 of [2] for the discussion of the exterior inverse scattering problem).

4 Numerical examples

We present some numerical examples to verify the effectiveness of the decomposition method. We choose two targets for D. One is a peanut given by

$$x(\theta) = 3\left(\cos^2\theta + 0.25\sin^2\theta\right)\cos\theta, \tag{4.23a}$$

$$y(\theta) = 3\left(\cos^2\theta + 0.25\sin^2\theta\right)\sin\theta, \quad 0 \le \theta \le 2\pi. \tag{4.23b}$$

The other is a kite given by

$$x(\theta) = 2\cos\theta + 1.3\cos 2\theta - 1.3, \tag{4.24a}$$

$$y(\theta) = 3\sin\theta, \qquad 0 \le \theta \le 2\pi. \tag{4.24b}$$

We choose the curve C to be the unit circle and put 40 measurement locations uniformly distributed on C. The point source is located at $(1,0)$ on C. The curve Γ is the circle with radius $r = 6$.

We set the wave number $k = 1$. The direct interior scattering problems are solved by a linear finite element method on a mesh with the mesh size $h \approx \lambda/100$. The scattered fields u^s_C at measurement locations are recorded and 5% random noise is added.

For the inverse problem, we first replace the ill-posed integral equation (3.8) by a finite dimensional approximations. In particular, we employ the trapezoidal rule for the integral with 320 nodes. Tikhonov regularization (3.9) is then applied to obtain a regularized solution ϕ_a of (3.8). The regularization parameter is chosen to be $\alpha = 1.0e-4$ for all examples due to the fact that the noise levels are the same.

4.1 The plot of $U(x)$

We first check the plot of $U(x)$ defined in (3.12) to obtain some idea how the result would look like. We choose a domain $S := [-4,4]\times[-4,4] \setminus \dot{C}$. For each sampling point x in S, we compute $U(x)$ and plot it in S using the Matlab 'image' function.

In Fig. 2, we show the reconstruction for the peanut. Note that the wave length $\lambda = 2\pi$ which is a little larger than the size of the object. There seems to be some other curves outside the target also give small $U(x)$.

For the kite, the setting is the same as the peanut case. The reconstruction is shown in Fig. 3. It can be seen that the two wings of the kite are missing.

For both cases, we see some defects of the plots, i.e., locations not on ∂D where $U(x)$ is rather small. Considering the interior measurements, we could choose the first reasonable closed curve outside C as an approximation of ∂D.

4.2 The optimization result

Next we search in a suitable set of curves of V for the optimization problem. In all examples, we replaced V by trigonometric polynomials of degree 2 as in [11]. The nonlinear minimization problem is solved by a Matlab routine 'fminsearch' which is based on a simplex method [15]. The initial guess is a circle outside C with radius

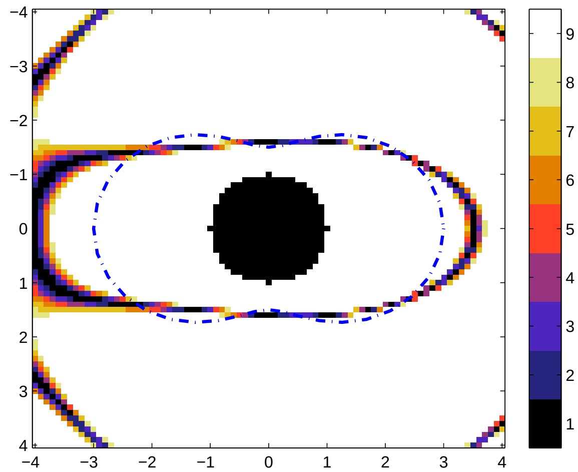

Figure 2: The plot of $U(x)$ for the peanut target when $k = 1$. The dashed line is the exact boundary.

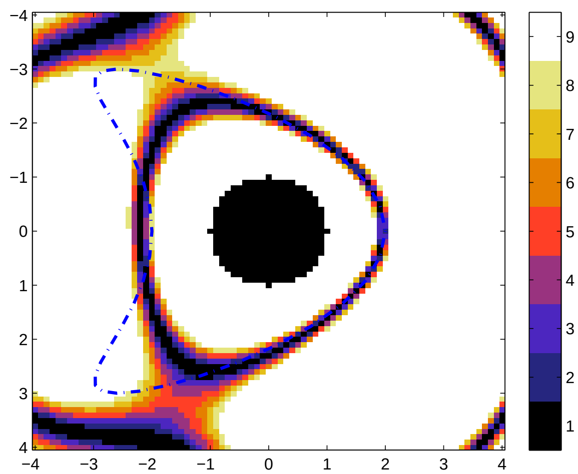

Figure 3: The plot of $U(x)$ for the kite when $k = 1$. The dashed line is the exact boundary.

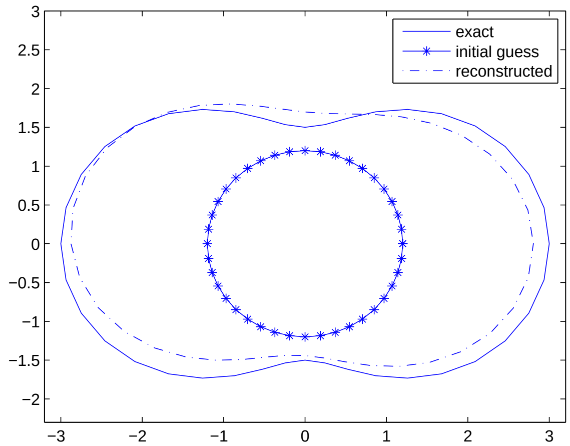

Figure 4: Reconstruction of the peanut. The point source is at $z = (1, 0)$ and receivers are on the unit circle. The initial guess is a circle of radius 1.2.

1.2. We note that the initial guess can be any curve outside C. Since the measurement is inside D the choice of the initial guess is easier than the exterior inverse scattering problem.

In Fig. 4, we show the reconstruction of the peanut. Considering the added noise, the result is satisfactory. In Fig. 5, we show the reconstruction of the kite. Again the two wings of the kite are missing.

5 Conclusions and future work

In this paper, we consider a new category of inverse scattering problems - the interior inverse scattering problem. In contrast to the standard exterior problems, the point source(s) and measurements are inside the cavity. We employ the decomposition method, which has been used for the exterior inverse scattering problems in literature, to construct the cavity boundary.

For the interior problems, the case of k^2 being a Dirichlet eigenvalue of D has to be avoided. This is due to the fact that using a point source at an eigenfrequency to probe the cavity would lead to a resonance state [18]. Physically, the interior inverse scattering problem can be more complicated than the exterior scattering due to the fact that the scattered waves are repeatedly reflected off ∂D [19]. This can also be seen from the numerical examples. The two wings of the kite are difficult to resolve.

Due to the similarity of mathematical models, we believe many existing methods for the exterior inverse scattering problems, e.g., the factorization method [9], the reciprocity gap method [1, 7, 16], the optimization method [10, 12, 17], can be applied to the interior inverse scattering problems. In future, we would like to extend the study to

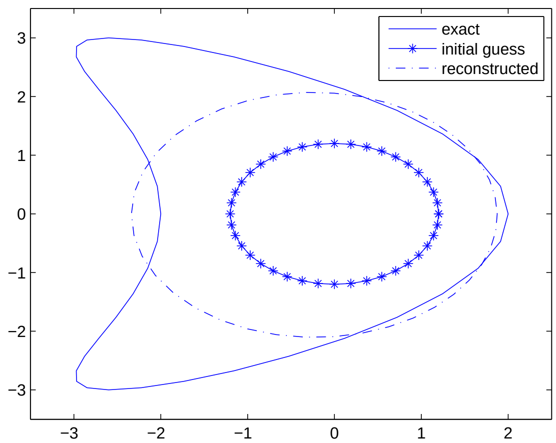

Figure 5: Reconstruction of the kite. The point source is at $z = (1, 0)$ and receivers are on the unit circle. The initial guess is a circle of radius 1.2.

three dimensional problems and other types of boundary conditions. It is also worth to study the Fréchet differentiability of the boundary integral operators such that Newton type methods for the interior inverse scattering problems can be justified.

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