A probing method for the transmission eigenvalue problem

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Abstract

In this paper, we consider an integral eigenvalue problem, which is a reformulation of the transmission eigenvalue problem arising in the inverse scattering theory. The boundary element method is employed for discretization and leads to a generalized matrix eigenvalue problem. We propose a novel method based on the spectrum projection. The method probes a given region on the complex plane using contour integrals and decides if the region contains eigenvalue(s) or not. It is particularly suitable to test if zero is an eigenvalue of the generalized eigenvalue problem, which in turn implies that the associated wavenumber is a transmission eigenvalue. Effectiveness and efficiency of the new method are demonstrated by numerical examples.

1 Introduction

We consider a nonlinear nonselfadjoint transmission eigenvalue problem, which arises in the inverse scattering theory [8]. The problem has attracted quite some attention from numerical analysts [9, 23, 14, 1, 24, 17, 7, 18, 13]. However, computation of both real and complex eigenvalues remains difficult due to the fact that the finite element methods usually end up with large sparse generalized non-Hermitian eigenvalue problems. Traditional methods such as shift and invert Arnoldi are handicapped by the lack of a priori spectrum information.

In this paper, we adopt an integral formulation for the transmission eigenvalue problem. Using boundary element method (BEM), the integral equations are discretized and a generalized eigenvalue problem of dense matrices is obtained. The matrices are significantly smaller than those from finite element methods. If zero is a generalized eigenvalue, the corresponding wavenumber k is a transmission eigenvalue. We propose a probing method based on the spectrum projection using contour integrals. We choose the contour to be a small circle with the origin inside and a numerical quadrature is used to compute the spectrum projection of a random vector. The norm of the projected vector is used as an indicator of whether zero is an eigenvalue or not.

Integral based methods [10, 21, 20, 3] for eigenvalue computation, having their roots in the classical spectral perturbation theory (see, e.g., [16]), become popular in many areas, e.g., electronic structure calculation. These methods are based on eigenprojections using contour integrals of the resolvent [2]. Randomly chosen functions are projected to the generalized eigenspace corresponding

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to the eigenvalues inside a closed contour, which leads to a relative small finite dimension eigenvalue problem. However, estimation on the locations, number of eigenvalues and dimensions of eigenspace are critical for their successes. The method proposed here is more related to the methods used in [17] and [12]. But it has a very distinct feature in the sense that it does not compute the actual eigenvalues.

The rest of the paper is arranged as follows. In Section 2, we introduce the transmission eigenvalue problem and rewrite it using integral operators. In Section 3, we present the probing method based on contour integrals. We present numerical results in Section 4. Discussion and future works are contained in Section 5.

2 The transmission eigenvalue problem

Let $D \subset \mathbb{R}^2$ be an open bounded domain with C^2 boundary Γ. The transmission eigenvalue problem is to find $k \in \mathbb{C}$ such that there exist non-trivial solutions w and v satisfying

$$\Delta w + k^2 n w = 0, \qquad \text{in } D, \tag{1a}$$

$$\Delta v + k^2 v = 0, \qquad \text{in } D, \tag{1b}$$

$$w - v = 0, \qquad \text{on } \Gamma, \tag{1c}$$

$$\frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} = 0, \qquad \text{on } \Gamma, \tag{1d}$$

where ν is the unit outward normal to Γ. The wavenumber k's for which the transmission eigenvalue problem has non-trivial solutions are called transmission eigenvalues. Here n is the index of refraction, which is assumed to be a constant greater than 1 in this paper.

In the following, we describe an integral formulation of the transmission eigenvalue problem following [6] (see also [17]). Let Φ_k be the Green's function given by

$$\Phi_k(x, y) = \frac{i}{4} H_0^{(1)}(k|x - y|),$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order 0. The single and double layer potentials are defined as

$$(SL_k\phi)(x) = \int_{\partial\Omega} \Phi_k(x, y)\phi(x)\, ds(y),$$
$$(DL_k\phi)(x) = \int_{\partial\Omega} \frac{\partial \Phi_k}{\partial \nu(y)}(x, y)\phi(x)\, ds(y),$$

where ϕ is the density function.

Let $(v, w) \in H^1(D) \times H^1(D)$ be a solution to (1). Denote by $k_1 = \sqrt{n}k$ and set

$$\alpha := \frac{\partial v}{\partial \nu}\Big|_\Gamma = \frac{\partial w}{\partial \nu}\Big|_\Gamma \in H^{-1/2}(\Gamma),$$
$$\beta := v|_\Gamma = w|_\Gamma \in H^{1/2}(\Gamma).$$

Then v and w has the following integral representation

$$v = SL_k\alpha - DL_k\beta, \qquad \text{in } D, \tag{2a}$$

$$w = SL_{k_1}\alpha - DL_{k_1}\beta, \qquad \text{in } D. \tag{2b}$$

Let $u := w - v$. Then $u|_\Gamma = 0$ and $\frac{\partial u}{\partial \nu}|_\Gamma = 0$. The boundary conditions of (1) imply that the transmission eigenvalues are k's such that

$$Z(k)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = 0 \tag{3}$$

where

$$Z(k) = \begin{pmatrix} S_{k_1} - S_k & -K_{k_1} + K_k \\ -K'_{k_1} + K'_k & T_{k_1} - T_k \end{pmatrix},$$

and the potentials S_k, K_k, K'_k, T_k are given by

$$(S_k\phi)(x) = \int_\Gamma \Phi_k(x,y)\phi(y)ds(y), \tag{4a}$$

$$(K_k\psi)(x) = \int_\Gamma \frac{\partial \Phi_k}{\partial \nu(y)}(x,y)\phi(y)ds(y), \tag{4b}$$

$$(K'_k\phi)(x) = \int_\Gamma \frac{\partial \Phi_k}{\partial \nu(x)}(x,y)\phi(y)ds(y), \tag{4c}$$

$$(T_k\psi)(x) = \frac{\partial}{\partial \nu(x)}\int_\Gamma \frac{\partial \Phi_k}{\partial \nu(y)}(x,y)\phi(y)ds(y). \tag{4d}$$

It is shown in [6] that

$$Z(k) := H^{-3/2}(\Gamma)\times H^{-1/2}(\Gamma) \to H^{3/2}(\Gamma)\times H^{1/2}(\Gamma)$$

is of Fredholm type with index zero and analytic on $\mathbb{C}\setminus\mathbb{R}^-$.

From (3), k is a transmission eigenvalue if zero is an eigenvalue of $Z(k)$. Unfortunately, $Z(k)$ is compact. The eigenvalues of $Z(k)$ accumulate at zero, which makes it impossible to distinguish zero and other eigenvalues numerically. The workaround proposed in [5] is to consider a generalized eigenvalue problem

$$Z(k)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \lambda B(k)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \tag{5}$$

where $B(k) = Z(ik)$. Since there does not exist purely imaginary transmission eigenvalues [9], the accumulation point is shifted to -1. Then 0 becomes isolated.

Now we describe the boundary element discretization of the potentials and refer the readers to [15, 22] for more details. One discretizes the boundary Γ into element segments. Suppose the computational boundary Γ is discretized into N segments $\Gamma_1, \Gamma_2, ..., \Gamma_N$ by nodes $x_1, x_2, ..., x_N$ and $\tilde{\Gamma} = \cup_{i=1}^N \Gamma_i$. Let $\{\psi_j\}, j = 1, 2, ..., N$, be piecewise constant basis functions and $\{\varphi_j\}, j = 1, 2, ..., N$, be piecewise linear basis functions. We seek an approximate solution α_h and β_h in the form

$$\alpha_h = \sum_{j=1}^N \alpha_j\psi_j, \quad \beta_h = \sum_{j=1}^N \beta_j\varphi_j.$$

We arrive at a linear system

$$\begin{aligned}(V_{k,h}-V_{k_1,h})\vec{\alpha}+(-K_{k,h}+K_{k_1,h})\vec{\beta} &= 0,\\ (K'_{k,h}-K'_{k_1,h})\vec{\alpha}+(W_{k,h}-W_{k_1,h})\vec{\beta} &= 0,\end{aligned}$$

where $\vec{\alpha}=(\alpha_1,...,\alpha_N)^T$, $\vec{\beta}=(\beta_1,...,\beta_N)^T$, and $V_{k,h}, K_{k,h}, K'_{k,h}, W_{k,h}$ are matrices with entries

$$\begin{aligned}V_{k,h}(i,j) &= \int_{\tilde{\Gamma}}(S_k\psi_j)\psi_i ds,\\ K_{k,h}(i,j) &= \int_{\tilde{\Gamma}}(K_k\varphi_j)\psi_i ds,\\ K'_{k,h}(i,j) &= \int_{\tilde{\Gamma}}(K'_k\psi_j)\varphi_i ds,\\ W_{k,h}(i,j) &= \int_{\tilde{\Gamma}}(T_k\varphi_j)\varphi_i ds.\end{aligned}$$

In the above matrices, we can use series expansions of the first kind Hankel function as

$$\begin{aligned}H_0^{(1)}(x) &= \sum_{m=0}^{\infty}\frac{(-1)^m}{(m!)^2}\left(\frac{x}{2}\right)^{2m}+\frac{2i}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{(m!)^2}\left(\frac{x}{2}\right)^{2m}\left(\ln\frac{x}{2}+c_e\right)\\ &- \frac{2i}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{(m!)^2}\left(\frac{x}{2}\right)^{2m}\left(1+\frac{1}{2}+\frac{1}{m}\right),\end{aligned}$$

where c_e is the Euler constant. Thus,

$$H_0^{(1)}(k|x-y|)=\sum_{m=0}^{\infty}\left(C_5(m)+C_6(m)\ln\frac{k}{2}\right)k^{2m}|x-y|^{2m}+C_6(m)\ln|x-y|k^{2m}|x-y|^{2m},$$

where

$$\begin{aligned}C_5(m) &= \frac{(-1)^m}{2^{2m}(m!)^2}\left[1+\frac{2c_e i}{\pi}-\frac{2i}{\pi}\left(1+\frac{1}{2}+\frac{1}{m}\right)\right],\\ C_6(m) &= \frac{(-1)^m i}{2^{2m-1}(m!)^2\pi}.\end{aligned}$$

We also need the following integrals which can be computed exactly.

$$\begin{aligned}Int_7(m) &= \int_{-1}^{1}\int_{-1}^{1}(\xi_1-\xi_2)^{2m}d\xi_2 d\xi_1\\ &= \frac{2^{2m+2}}{(2m+1)(m+1)},\end{aligned}$$

$$\begin{aligned}Int_8(m) &= \int_{-1}^{1}\int_{-1}^{1}(\xi_1-\xi_2)^{2m}\ln|\xi_1-\xi_2|d\xi_2 d\xi_1\\ &= \frac{2^{2m+2}\ln 2}{(2m+1)(m+1)}-\frac{(4m+3)2^{2m+3}}{(2m+1)^2(2m+2)^2},\end{aligned}$$

$$
\begin{aligned}
Int_9(m) &= \int_{-1}^{1}\int_{-1}^{1}(\xi_1-\xi_2)^{2m}\xi_1\xi_2 d\xi_2 d\xi_1 \\
&= \sum_{l=0}^{2m}\frac{(-1)^l C_{2m}^l}{(l+2)(2m+2-l)}[1-(-1)^l]^2,
\end{aligned}
$$

and

$$
\begin{aligned}
Int_{10}(m) &= \int_{-1}^{1}\int_{-1}^{1}(\xi_1-\xi_2)^{2m}\xi_1\xi_2\ln|\xi_1-\xi_2|d\xi_2 d\xi_1 \\
&= \frac{-m2^{2m+2}\ln 2}{(2m+1)(m+1)(m+2)}+\frac{1}{(2m+1)(m+1)}\left[\frac{2^{2m+3}}{2m+3}-\frac{2^{2m+2}}{(m+2)^2}-\frac{2^{2m+1}}{m+1}\right] \\
&+ \frac{1}{2(m+1)^2(2m+1)^2}\sum_{l=0}^{2m+1}C_{2m+1}^l\left[\frac{(2m+1)^2}{l+2}(1-(-1)^l)-\frac{4m+3}{l+3}(1-(-1)^{l+1})\right].
\end{aligned}
$$

Now we consider

$$
\begin{aligned}
V_{k,h}(i,j) &= \int_{\tilde{\Gamma}}(V_k\psi_j)\psi_i ds \\
&= \int_{\tilde{\Gamma}}\int_{\tilde{\Gamma}}\Phi_k(x,y)\psi_j(y)\psi_i(x)ds_y ds_x \\
&= \int_{\Gamma_i}\int_{\Gamma_j}\Phi_k(x,y)\psi_j(y)\psi_i(x)ds_y ds_x.
\end{aligned}
$$

The integral over $\Gamma_i\times\Gamma_j$ can be calculated as

$$
\begin{aligned}
\int_{\Gamma_i}\int_{\Gamma_j}\Phi_k(x,y)\psi_j(y)\psi_i(x)ds_y ds_x &= \frac{i}{4}\int_{\Gamma_i}\int_{\Gamma_j}H_0^{(1)}(k|x-y|)\psi_j(y)\psi_i(x)ds_y ds_x \\
&= \frac{iL_iL_j}{16}\int_{-1}^{1}\int_{-1}^{1}H_0^{(1)}(k|x(\xi_1)-y(\xi_2)|)d\xi_2 d\xi_1,
\end{aligned}
$$

where

$$
\begin{aligned}
x(\xi_1) &= x_i+\frac{1+\xi_1}{2}(x_{i+1}-x_i), \\
y(\xi_2) &= x_j+\frac{1+\xi_2}{2}(x_{j+1}-x_j).
\end{aligned}
$$

When $i\neq j$, it can be calculated by Gaussian quadrature rule. When $i=j$, we have

$$
\begin{aligned}
&\frac{iL_i^2}{16}\int_{-1}^{1}\int_{-1}^{1}H_0^{(1)}(k|x(\xi_1)-y(\xi_2)|)d\xi_2 d\xi_1 \\
=\ &\frac{iL_i^2}{16}\sum_{m=0}^{\infty}\frac{k^{2m}L_i^{2m}}{2^{2m}}\left(C_5(m)+C_6(m)\ln\frac{kL^i}{4}\right)\int_{-1}^{1}\int_{-1}^{1}(\xi_1-\xi_2)^{2m}d\xi_2 d\xi_1 \\
&+\frac{iL_i^2}{16}\sum_{m=0}^{\infty}\frac{k^{2m}L_i^{2m}}{2^{2m}}C_6(m)\int_{-1}^{1}\int_{-1}^{1}(\xi_1-\xi_2)^{2m}\ln|\xi_1-\xi_2|d\xi_2 d\xi_1 \\
=\ &\sum_{m=0}^{\infty}\frac{ik^{2m}L_i^{2m+2}}{2^{2m+4}}\left[\left(C_5(m)+C_6(m)\ln\frac{kL^i}{4}\right)Int_7(m)+C_6(m)Int_8(m)\right].
\end{aligned}
$$

The following regularization formulation is needed to discretize the hyper-singular boundary integral operator

$$W_k\beta(x) = -\frac{d}{ds_x}V_k(\frac{d\beta}{ds})(x) - k^2\nu_x \cdot V_k(\beta\nu)(x). \tag{6}$$

We refer the readers to [11] for details of the discretization.

The above boundary element method leads to the following generalized eigenvalue problem

$$A\mathbf{x} = \lambda B\mathbf{x} \tag{7}$$

where $A, B \in \mathbb{C}^{n\times n}$, $\lambda \in \mathbb{C}$ is a scalar, and $\mathbf{x} \in \mathbb{C}^n$.

To compute transmission eigenvalues, the following method is proposed in [5]. A searching interval for wave numbers is discretized. For each k, the boundary integral operators $Z(k)$ and $Z(ik)$ are discretized to obtain (7). Then all eigenvalues $\lambda_i(k)$ of (7) are computed and arranged such that

$$0 \le |\lambda_1(k)| \le |\lambda_2(k)| \le \dots$$

If k is a transmission eigenvalue, $|\lambda_1|$ is very close to 0 numerically. If one plots the inverse of $|\lambda_1(k)|$ against k, the transmission eigenvalues are located at spikes.

3 The probing method

The method in [5] only uses the smallest eigenvalue. Hence it is not necessary to compute all eigenvalues of (5). In fact, there is no need to know the exact value of λ_1. The only thing we need is that, if k is a transmission eigenvalue, the generalized eigenvalue problem (5) has an isolated eigenvalue close to 0. This motivates us to propose a probing method to test if 0 is an generalized eigenvalue of (5). The method does not compute the actual eigenvalue and only solves a couple of linear systems. The workload is reduced significantly in two dimension and even more in three dimension.

We start to recall some basic results from spectral theory of compact operators [16]. Let $T : \mathcal{X} \to \mathcal{X}$ be a compact operator on a complex Hilbert space $\mathcal{X}$. The resolvent set of T is defined as

$$\rho(T) = \{z \in \mathbb{C} : (z - T)^{-1} \text{ exists as a bounded operator on } \mathcal{X}\}. \tag{8}$$

For any $z \in \rho(T)$, the resolvent operator of T is defined as

$$R_z(T) = (z - T)^{-1}. \tag{9}$$

The spectrum of T is $\sigma(T) = \mathbb{C} \setminus \rho(T)$. We denote the null space of an operator A by $N(A)$. Let α be such that

$$N\left((\lambda - T)^{\alpha}\right) = N\left((\lambda - T)^{\alpha+1}\right).$$

Then $m = \dim N\left((\lambda - T)^{\alpha}\right)$ is called the algebraic multiplicity of λ. The vectors in $N\left((\lambda - T)^{\alpha}\right)$ are called generalized eigenvectors of T corresponding to λ. Geometric multiplicity of λ is defined as $\dim N(\lambda - T)$.

Let γ be a simple closed curve on the complex plane $\mathbb{C}$ lying in $\rho(T)$ which contains m eigenvalues, counting multiplicity, of T: $\lambda_i, i = 1, \dots, m$. We set

$$P = \frac{1}{2\pi i}\int_{\gamma} R_z(T)dz.$$

It is well-known that P is a projection from $\mathcal{X}$ onto the space of generalized eigenfunctions $\mathbf{u}_i, i = 1, \ldots, m$ associated with $\lambda_i, i = 1, \ldots, m$ [16].

Let $\mathbf{f} \in \mathcal{X}$ be randomly chosen. If there are no eigenvalues inside γ, we have that $P\mathbf{f} = 0$. Therefore, $P\mathbf{f}$ can be used to decide if a region contains eigenvalues of T or not.

For the generalized matrix eigenvalue problem (7), the resolvent is

$$R_z(A, B) = (zB - A)^{-1} \tag{10}$$

for z in the resolvent set of the matrix pencil (A, B). The projection onto the generalized eigenspace corresponding to eigenvalues enclosed by γ is given by

$$P_k(A, B) = \frac{1}{2\pi i} \int_\gamma (zB - A)^{-1} dz. \tag{11}$$

We write P_k to emphasize that P depends on the wavenumber k.

The approximation of $P_k\mathbf{f}$ is computed by suitable quadrature rules

$$P_k\mathbf{f} = \frac{1}{2\pi i} \int_\gamma R_z(A, B)\mathbf{f} dz \approx \frac{1}{2\pi i} \sum_{j=1}^{W} \omega_j R_{z_j}(A, B)\mathbf{f} = \frac{1}{2\pi i} \sum_{j=1}^{W} \omega_j \mathbf{x}_j, \tag{12}$$

where w_j are weights and z_j are quadrature points. Here $\mathbf{x}_j$'s are the solutions of the following linear systems

$$(z_j B - A)\mathbf{x}_j = \mathbf{f}, \quad j = 1, \ldots, W. \tag{13}$$

Similar to the continuous case, if there are no eigenvalues inside γ, then $P_k = 0$ and thus $P_k\mathbf{f} = \mathbf{0}$ for all $\mathbf{f} \in \mathbb{C}^n$. Similar to [12], we project the random vector twice for a better result, i.e., we compute $P_k^2\mathbf{f}$.

For a fixed wavenumber k, the algorithm of the probing method is as follows.

Input: a small circle γ center at the origin with radius $r << 1$ and a random $\mathbf{f}$

Output: 0 - k is not a transmission eigenvalue; 1 - k is a transmission eigenvalue

1. Compute $P_k^2\mathbf{f}$ by (12);
2. Decide if γ contains an eigenvalue:
   - No. output 0.
   - Yes. output 1.

4 Numerical Examples

We start with an interval (a, b) of wavenumbers and uniformly divide it into K subintervals. At each wavenumber

$$k_j = a + jh, \quad j = 0, 1, \ldots, K, \; h = \frac{b - a}{K},$$

we employ the boundary element method to discretize the potentials. We choose $N = 32$ and end up with a generalized eigenvalue problem (7) with 64×64 matrices A and B. To test whether 0 is a

Table 1: TEs of a disk with radius $r = 1/2$ and index of refraction $n = 16$.

$m = 0$	1.9880	3.7594	6.5810
$m = 1$	2.6129	4.2954	5.9875
$m = 2$	3.2240	4.9462	6.6083

generalized eigenvalue of (7), we choose γ to be a circle of radius $1/100$. Then we use 16 uniformly distributed quadrature points on γ and evaluate the eigenprojection (12). If at a wavenumber k_j, the projection is of $O(1)$, then k_j is a transmission eigenvalue. For the actual computation, we use a threshold value $\sigma = 1/2$ to decide if k_j is a transmission eigenvalue or not, i.e., k_j is a transmission eigenvalue if $\|P_{k_j}^2 \mathbf{f}\|/\|P_{k_j}\mathbf{f}\| \geq \sigma$ and not otherwise.

Let D be a disk with radius $1/2$. The index of refraction is $n = 16$. In this case, the exact transmission eigenvalues are known [9]. They are the roots of the following

$$d_m = J_1(k/2)J_0(2k) - 4J_0(k/2)J_1(2k), \quad m = 0, \tag{14}$$

$$d_m = J_{m-1}(k/2)J_m(2k) - 4J_m(k/2)J_{m-1}(2k), \quad m = 1, 2, \ldots \tag{15}$$

The actual values are given in Table (1).

We choose the interval to be $(1.5, 3.5)$ and uniformly divide it into 2000 subintervals. At each k_j we compute the projection (12) twice. The probing method finds three eigenvalues in $(1.5, 3.5)$

$$k_1 = 1.988, \quad k_2 = 2.614, \quad k_3 = 3.228,$$

which approximate the exact eigenvalues (the first column of Table (1)) accurately. We also plot the log of $|P^2\mathbf{f}|$ against the wavenumber k in Fig. 1. The method is robust since the eigenvalues can be easily identified.

We repeat the experiment by choosing $n = 9$ and $(a, b) = (3, 5)$. The rest parameters keep the same. The following eigenvalues are obtained

$$k_1 = 3.554, \quad k_2 = 4.360.$$

The log of $|P^2\mathbf{f}|$ against the wavenumber k is shown in Fig. 2.

Finally, we compare the proposed method with the method in [5]. We take $n = 16$ and compute for 2000 wave numbers. The CPU time in second is shown in Table 2. We can see that the proposed method saves more time if the size of the generalized eigenvalue problem is larger. We expect that it has a greater advantage for three dimension problems since the size of the matrices are much larger than two dimension cases.

We also show the log plot of $1/|\lambda_{min}|$ by the method of [5] in Fig. 3. Comparing with Figures 1 and 2, it is clear that the probing method has much narrower span.

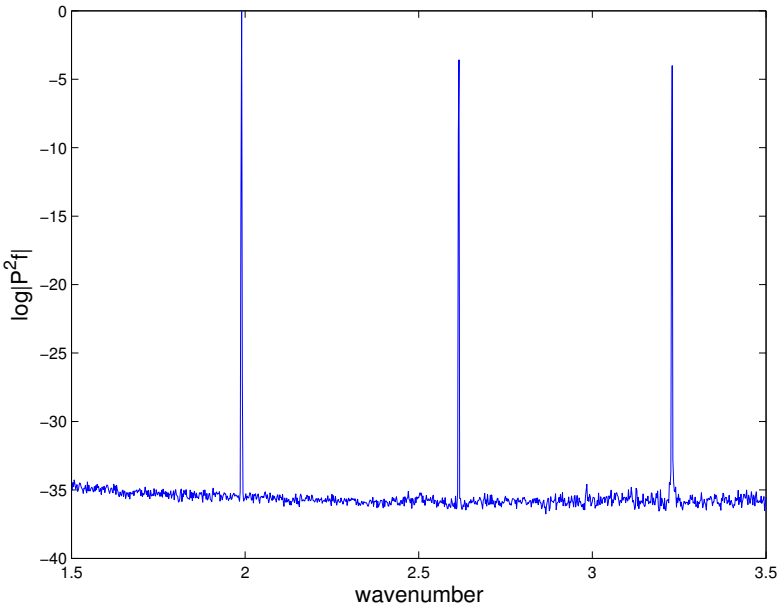

Figure 1: The plot of $\log |P^2\mathbf{f}|$ against the wavenumber k for $n = 16$.

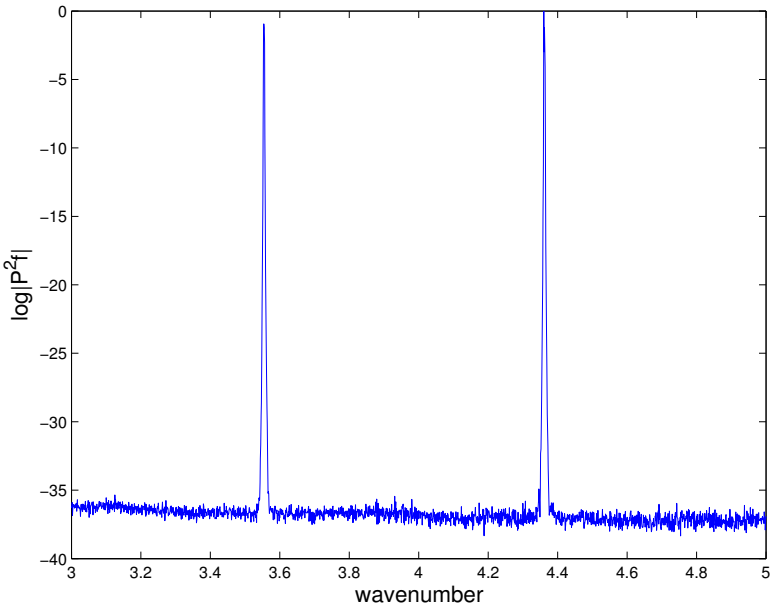

Figure 2: The plot of $\log |P^2\mathbf{f}|$ against the wavenumber for $n = 9$.

Table 2: Comparison. The first column is the size of the matrix problem. The second column is the time used by the proposed method in second. The second column is the time used by the method given in [5]. The fourth column is the ration.

size	probing method	method in [5]	ratio
64×64	1.741340	5.742839	3.30
128×128	5.653961	31.152448	5.51
256×256	25.524530	224.435704	8.79
512×512	130.099433	1822.545973	14.01

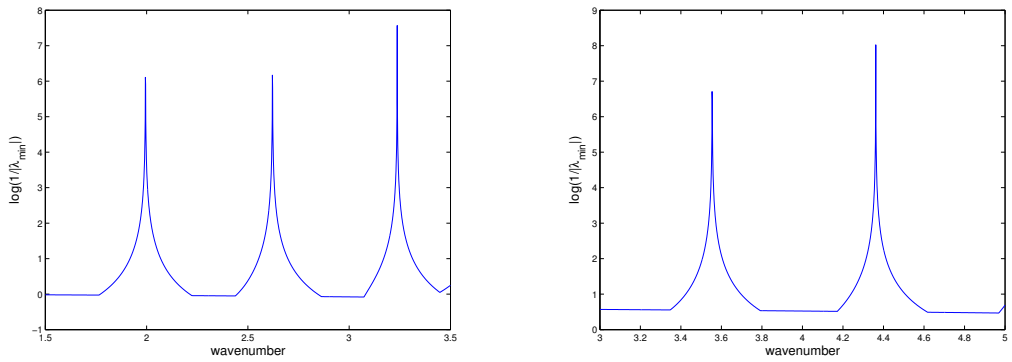

Figure 3: Log plot of $1/|\lambda_{min}|$. Left: $n = 16$. Right: $n = 9$.

5 Conclusions and Future Works

In this paper, we proposed a probing method based on contour integrals for the transmission eigenvalue problems. The method only tests if a given region contains an eigenvalue or not. Comparing to the existing methods, it needs little a prior spectrum information and seems to be more efficient. The method can be viewed as an eigensolver without computing eigenvalues.

Note that one needs to construct two matrices for each wavenumber. It is time consuming if one wants to divide the searching interval into more subintervals to improve accuracy. The overload is much more in three dimension. Currently, we are developing a parallel version of the method using graphics processing units (GPUs).

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