Transmission eigenvalues and non-destructive testing of anisotropic magnetic materials with voids

Isaac Harris[1], **Fioralba Cakoni**[1] **and Jiguang Sun**[2]

[1] Department of Mathematical Sciences, University of Delaware, Newark, Delaware 19716, USA

[2] Department of Mathematical Sciences, Michigan Technological University, Houghton, MI 49931, USA

E-mail: `iharris@udel.edu`, `cakoni@math.udel.edu`, `jiguangs@mtu.edu`

Abstract. In this paper we consider the transmission eigenvalue problem corresponding to the scattering problem for an anisotropic magnetic materials with voids, i.e. subregion with refractive index the same as the background. Under weak assumptions on the material properties, we show that the transmission eigenvalues can be determined from the far field measurements. Then assuming that the contrast on the material properties does not change sign, we prove the existence of at least one transmission eigenvalue for sufficiently small voids. We also show that the first transmission eigenvalue can be used to determining material properties and give qualitative information about the size of the void. Some numerical examples are given to demonstrate the theoretical results.

Keywords: Transmission eigenvalues, anisotropic media with voids, non-destructive testing, inverse scattering problem.

1. Introduction

The non-destructive testing of composite materials using electromagnetic waves is an important problem in engineering. A number of such problems involve complicated materials, in particular anisotropic, hence many methods of reconstructing the matrix refractive index are either unfeasible or computationally expensive. On the other hand for practical purposes it suffices to obtain some partial information on the refractive index in order to evaluate the integrity of the material. The so-called qualitative methods in inverse scattering do just this (see e.g. [4], [20]). In this paper we consider the problem of detecting voids in a known anisotropic dielectric material from electromagnetic measurements in the frequency domain for a range of frequencies. An attempt to reconstruct the voids would involve computing the Green's function for anisotropic media for possibly complicated geometry. On the other hand, our inversion method is based on quantifying the effect that the presence of voids have

on the so-called *transmission eigenvalues*, which are detectable from the scattering data [7] [21], [24]. Transmission eigenvalues have been used to determine material properties of the scattering media starting with [8] for isotropic inhomogeneities and followed by [3] [5], [9], [18] and [25] for more complicated media.

We begin this paper by showing that the real transmission eigenvalues can be seen in the far field measured data following the approach in [7]. We note that a rigorous characterization of the real transmission eigenvalues in terms of the scattering matrix is given in [7] but the analysis there requires restrictive assumptions on the material properties. Next we prove that real transmission eigenvalues exists for anisotropic magnetic dielectric media with voids. The existing results on this question [6], [16] include only the case of non-magnetic material, i.e. when the magnetic permeability of the media is the same as of the background and the approach used in these papers rely heavily on the fact that the contrast is only on one constitutive parameters of the medium. Our approach to proving the existence of transmission eigenvalues follows the formulation introduced in [12] with appropriate modifications to allow for the presence of voids. We conclude the paper with numerical examples showing how the volume, shape and the location of voids affect the transmission eigenvalue and what kind of information we can obtain about voids from the first transmission eigenvalue.

2. Formulation of the problem

We begin by considering electromagnetic waves propagating in an inhomogeneous anisotropic dielectric medium in $\mathbb{R}^3$ with electric permittivity $\epsilon = \epsilon(x)$ and magnetic permeability $\mu = \mu(x)$. For time harmonic electromagnetic waves of the form

$$\mathcal{E}(x,t) = \tilde{E}(x)e^{-i\omega t}, \qquad \mathcal{H}(x,t) = \tilde{H}(x)e^{-i\omega t}$$

with frequency $\omega > 0$, we deduce that the complex valued space dependent parts $\tilde{E}$ and $\tilde{H}$ satisfy

$$\nabla \times \tilde{E} - i\omega\mu(x)\tilde{H} = 0 \quad \text{and} \quad \nabla \times \tilde{H} + i\omega\epsilon(x)\tilde{E} = 0.$$

Now let us suppose that the inhomogeneity occupies an infinitely long cylinder with cross section D having piece-wise smooth boundary ∂D with ν being the unit outward normal to ∂D. We assume that the axis of the cylinder coincides with the z-axis. We further assume that the conductor is imbedded in a non-conducting homogeneous background, i.e. the electric permittivity $\epsilon_0 > 0$ and the magnetic permeability $\mu_0 > 0$ of the background medium. In addition we assume that inside the inhomogeneous media there is a subregion (possibly multiply-connected) with cross section $\overline{D}_1 \subset D$ that has the same electric permittivity and magnetic permeability as the background, i.e. ϵ_0 and μ_0 respectively. For an orthotropic medium we have that the matrices $\mathcal{A}$ and $\mathcal{N}$ are

independent of the z-coordinate and are of the form

$$\mathcal{A} := \frac{\epsilon(x)}{\epsilon_0} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a \end{pmatrix} \qquad \mathcal{N} := \frac{\mu(x)}{\mu_0} = \begin{pmatrix} n_{11} & n_{12} & 0 \\ n_{21} & n_{22} & 0 \\ 0 & 0 & n \end{pmatrix}.$$

Then it is well-known [4] that the only component u of the total magnetic field $\tilde{H} = (0, 0, u)$ polarized perpendicular to the axis of the cylinder satisfies

$$\nabla \cdot A(x)\nabla u + k^2 n(x) u = 0$$

where

$$A := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1},$$

and analogously the scattered field u^s satisfies $\Delta u^s + k^2 u^s = 0$ outside the scatter D. We can now rigorously formulate our scattering problem in $\mathbb{R}^2$. Let $D \subset \mathbb{R}^2$ be a bounded simply connected open set with piece-wise smooth boundary ∂D. Furthermore assume that we have a symmetric matrix valued function $A(x) \in L^\infty(D, \mathbb{R}^{2\times 2})$ that is uniformly positive definite in D and $n(x) \in L^\infty(D)$ such that $n(x) > n_0 > 0$. We are particularly interested in the case were there exists $\overline{D}_1 \subset D$ (possibly multiple connected) with $A(x) = I$ and $n(x) = 1$ for all $x \in D_1$. Let us denote $D_2 = D \setminus \overline{D}_1$ that is the support of the inhomogeneous media without the voids.

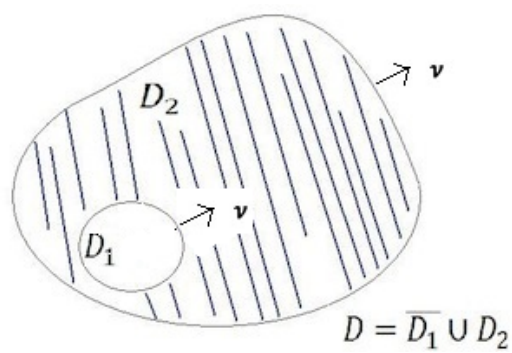

Figure 1. Example of the geometry of a medium with voids

Then the scattering of a plane wave $u^i := e^{ikx\cdot d}$ by this anisotropic inhomogeneous media with voids can be formulated as: find $(u^s, u) \in H^1_{loc}\left(\mathbb{R}^2 \setminus \overline{D}\right) \times H^1(D)$ such that:

$$\nabla \cdot A(x)\nabla u + k^2 n(x) u = 0 \quad \text{in} \quad D \tag{1}$$

$$\Delta u^s + k^2 u^s = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \overline{D} \tag{2}$$

$$u - u^s = e^{ikx\cdot d} \quad \text{in} \quad D \tag{3}$$

$$\frac{\partial u}{\partial \nu_A} - \frac{\partial u^s}{\partial \nu} = \frac{\partial}{\partial \nu} e^{ikx\cdot d} \quad \text{on } \partial D \tag{4}$$

$$\lim_{r\mapsto\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \tag{5}$$

where $\partial u/\partial \nu_A := \nu A \cdot \nabla u$.

It is well-known (see e.g. [4]) that the above system is well-posed. Furthermore, it can be shown that the scattered field u^s assumes the asymptotic behavior

$$u^s(x,d;k) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\theta,\phi,k) + \mathcal{O}(r^{-3/2}) \quad \text{as} \quad r \to \infty$$

uniformly with respect to r, where $r = |x|$, the incident direction $d := (\cos\phi, \sin\phi)$ and the observation direction $\hat{x} := (\cos\theta, \sin\theta)$, $\theta, \phi \in [0,\, 2\pi]$. Here $u_\infty(\theta,\phi)$ is called the far-field pattern of the scattering problem (1)-(5), which is a function of the observation angle for a given incident angle. The far-field patterns for all incident directions d defines the far field operator $F : L^2(0,2\pi) \mapsto L^2(0,2\pi)$ by

$$(Fg)(\theta) := \int_0^{2\pi} u_\infty(\theta,\phi) g(\phi)\, d\phi.$$

It is also well-known (see e.g. [4], Theorem 6.2) that the far-field operator is injective if and only if there does not exist a nontrivial (w,v) solving:

$$\nabla \cdot A\nabla w + k^2 n w = 0 \qquad \text{in } D \tag{6}$$

$$\Delta v + k^2 v = 0 \qquad \text{in } D \tag{7}$$

$$w = v \qquad \text{on } \partial D \tag{8}$$

$$\frac{\partial w}{\partial \nu_A} = \frac{\partial v}{\partial \nu} \qquad \text{on } \partial D \tag{9}$$

such that v takes the form of a Herglotz function

$$v_g(x) := \int_0^{2\pi} e^{ikx\cdot d} g(\phi)\, d\phi, \qquad d := (\cos\phi, \sin\phi).$$

The values of $k \in \mathbb{C}$ for which the *homogeneous interior transmission problem* (6)-(9) has nontrivial solutions are called *transmission eigenvalues*. The goal of this paper is to obtain information about the voids from a knowledge of the real transmission eigenvalues. To this end we first show that real transmission eigenvalues can be determine for the far-field pattern $u_\infty(\theta,\phi)$ for $\theta,\phi \in [0,\, 2\pi]$ (or possibly in a subset of $[0,\, 2\pi]$). Let us introduce the following notation:

$$\inf_{x\in D_2} \inf_{|\xi|=1} \overline{\xi} \cdot A(x)\xi = A_{min} \quad \text{and} \quad \inf_{x\in D_2} n(x) = n_{min}$$

$$\sup_{x\in D_2} \sup_{|\xi|=1} \overline{\xi} \cdot A(x)\xi = A_{max} \quad \text{and} \quad \sup_{x\in D_2} n(x) = n_{max}$$

and

$$\inf_{x\in \mathcal{N}_\delta(\partial D)} \inf_{|\xi|=1} \overline{\xi} \cdot A(x)\xi = A_\star \quad \text{and} \quad \inf_{x\in \mathcal{N}_\delta(\partial D)} n(x) = n_\star$$

$$\sup_{x\in \mathcal{N}_\delta(\partial D)} \sup_{|\xi|=1} \overline{\xi} \cdot A(x)\xi = A^\star \quad \text{and} \quad \sup_{x\in \mathcal{N}_\delta(\partial D)} n(x) = n^\star$$

where $\mathcal{N}_\delta(\partial D)$ is a neighborhood of the boundary. From physical considerations we assume that $A_{min} > 0$, $n_{min} > 0$ and $A_{max} < \infty$, $n_{max} < \infty$.

3. Determination of transmission eigenvalues from scattering data

We show that the real transmission eigenvalues corresponding to our problem can be determined from the far-field measurements following the approach in [7] where the same result is proven for the case of isotropic media with $A = I$. To this end we introduce the far field equation

$$(Fg_z)(\theta) = \Phi_\infty(\hat{x}, z), \qquad z \in D, \quad \hat{x} = (\cos\theta, \sin\theta) \tag{10}$$

where $\Phi_\infty(\hat{x}, z) = \gamma e^{-ik\hat{x}\cdot z}$, $\gamma := e^{i\pi/4}/\sqrt{8\pi k}$ is the far-field pattern of the radiating fundamental solution to the Helmholtz equation in $\mathbb{R}^2$ given by $\Phi(x, y) = \frac{i}{4}H_0^{(1)}\big(k|x-y|\big)$. Let F^δ be the far-field operator corresponding to the noisy measurements $u_\infty^\delta(\theta, \phi)$ satisfying $||u_\infty^\delta(\theta, \phi) - u_\infty(\theta, \phi)||_{L^2} \leq \delta$. We find the Tikhonov regularized solution $g_{z,\delta} := g^\delta_{z,\epsilon(\delta)}$ of the far-field equation defined as the unique minimizer of

$$\|F^\delta g - \Phi_\infty(\cdot, z)\|^2_{L^2[0,\,2\pi]} + \epsilon\|g\|^2_{L^2[0,\,2\pi]}$$

where the regularization parameter $\epsilon := \epsilon(\delta) \to 0$ as $\delta \to 0$. Provided that F has dense range (which is true in general except for the case when the transmission eigenfunction v takes the form of Herglotz function; see Appendix of [7]) the regularized solution $g_{z,\delta}$ is such that

$$\lim_{\delta\to 0} \|F^\delta g_{z,\delta} - \Phi_\infty(\cdot, z)\|_{L^2[0,\,2\pi]} = 0 \tag{11}$$

Following [1] and using the results developed in Section 7.2 in [4] it is possible to show that if k is not a transmission eigenvalue then the Herglotz function $v_{g_{z,\delta}}$ converges in the $H^1(D)$-norm to v where (v, w) solves

$$\nabla \cdot A\nabla w + k^2 nw = 0 \qquad \text{in } D \tag{12}$$

$$\Delta v + k^2 v = 0 \qquad \text{in } D \tag{13}$$

$$w - v = \Phi(\cdot, z) \qquad \text{on } \partial D \tag{14}$$

$$\frac{\partial w}{\partial \nu_A} - \frac{\partial v}{\partial \nu} = \frac{\partial \Phi(\cdot, z)}{\partial \nu} \qquad \text{on } \partial D, \tag{15}$$

provided that the solution exists (in other words (12)-(15) is Fredholm with index zero). Let us recall an equivalent variational formulation for the above interior transmission problem analyzed in [2]. To this end, we define a lifting of the essential boundary data into the domain D. Thus we let $\phi_z \in H^1(D)$ be such that $\phi_z = \Phi(\cdot, z)$ on ∂D and attempt to find a solution of the interior transmission problem where $v = v_0 - \phi_z$, and the pair $(w, v_0) \in X(D) := \{w, v_0 \in H^1(D) \,|\, w - v_0 \in H_0^1(D)\}$ satisfies

$$\mathcal{A}_k\big((w, v_0); (\varphi_1, \varphi_2)\big) = \ell(\varphi_1, \varphi_2) \;\text{ for all }\; (\varphi_1, \varphi_2) \in X(D) \tag{16}$$

where the sesqulinear form $\mathcal{A}_k(\cdot\,;\cdot) : X(D) \times X(D) \mapsto \mathbb{C}$ and the conjugate linear functional $\ell(\cdot) : X(D) \mapsto \mathbb{C}$ are given by

$$\mathcal{A}_k\big((w, v_0); (\varphi_1, \varphi_2)\big) := \int_D A\nabla w \cdot \nabla\overline{\varphi_1} - \nabla v_0 \cdot \nabla\overline{\varphi_2}\, dx - k^2 \int_D nw\overline{\varphi_1} - v_0\overline{\varphi_2}\, dx,$$

$$\ell(\varphi_1,\varphi_2) := \int_{\partial D} \overline{\varphi}_1 \frac{\partial}{\partial \nu}\Phi(x,z)\, ds_x - \int_D \nabla\phi_z \cdot \nabla\overline{\varphi}_2 - k^2\phi_z\overline{\varphi}_2\, dx.$$

It has been proven in [2] that the variational problem (16) satisfies the Fredholm property provided $A_\star > 1$, $n_\star > 1$, or $A^\star < 1$, $n^\star < 1$. In particular, if k is a transmission eigenvalue with transmission eigenfunctions (w_k, v_k) then (16) has a solution if and only if the following solvability condition is satisfied

$$\ell(w_k,v_k) = \int_{\partial D} \overline{w}_k \frac{\partial}{\partial \nu}\Phi(x,z)\, ds_x - \int_D \nabla\phi_z \cdot \nabla\overline{v}_k - k^2\phi_z\overline{v}_k\, dx = 0. \tag{17}$$

Theorem 3.1 *Let k be a real transmission eigenvalue and assume that $A_\star > 1$, $n_\star > 1$, or $A^\star < 1$, $n^\star < 1$. Then for almost every $z \in D$, $||v_{g_{z,\delta}}||_{H^1(D)}$ can not be bounded as $\delta \to 0$, where $g_{z,\delta}$ satisfies (11).*

Proof. Assume there is a set of positive measure such that $||v_{g_{z,\delta}}||_{H^1(D)}$ is bounded. Hence, a subsequence $v_{g_{z,\delta_n}}$ converges weakly to a $v \in H^1(D)$ satisfying $\Delta v + k^2 v = 0$ in D. Since $Fg_{z,\delta} \to \Phi_\infty(\cdot, z)$, Rellich's lemma implies that $u^s = \Phi(\cdot, z)$ on $\mathbb{R}^2 \setminus \overline{D}$, where u^s is the scattered field with the far-field pattern $Fg_{z,\delta}$. Now, the corresponding total field w in D and v satisfy the interior transmission problem (12)-(15), which gives that there is a solution to the variational problem $\mathcal{A}_k\big((w, v_0); (\varphi_1, \varphi_2)\big) = \ell(\varphi_1, \varphi_2)$. Using integration by parts on the Fredholm solvability condition (17) and using that $\phi_z = \Phi(\cdot, z)$ on ∂D and $\Delta v_k + k^2 v_k = 0$ in D, we have that

$$\int_{\partial D} \overline{w}_k \frac{\partial}{\partial \nu}\Phi(x,z) - \frac{\partial \overline{v}_k}{\partial \nu}\Phi(x,z)\, ds_x = 0$$

Notice that since $w_k = v_k$ on ∂D, Green's representation theorem and the unique continuation principle implies that $v_k = 0$ in D. So v_k has zero Cauchy data on ∂D, which implies that $w_k = 0$ and $\frac{\partial w_k}{\partial \nu_A} = 0$ on ∂D, whence $w_k = 0$ in D. Therefore $(w_k, v_k) = (0, 0)$ which contradicts the fact that (w_k, v_k) are eigenfunctions. □

Remark 3.1 *Note that the proof of Theorem 3.1 carries over to any case of inhomogeneous media for which the corresponding interior transmission problem satisfies the Fredholm property. In particular this holds if $A_{min} > 1$ or $A_{max} < 1$, and k^2 is not a Dirichlet eigenvalue for each component of voids D_1.*

Remark 3.2 *From [2] and [19] we also have the following discreteness result. Assume that either $A_\star > 1$ and $n_\star > 1$, or $A^\star < 1$ and $n^\star < 1$, or $A_{min} > 1$ or $A_{max} < 1$ hold. Then the set of transmission eigenvalues is at most discrete. The real positive transmission eigenvalues can accumulate only at $+\infty$.*

The above analysis indicates that when plotting the $||v_{g_{z,\delta}}||_{H^1(D)}$ against k, where $g_{z,\delta}$ is the Tikhonov regularized solution to the far field equation, the transmission eigenvalues will appear as sharp peaks in the graph. In Section 5 we present numerical examples that show the viability of this approach to determine real transmission eigenvalues form far-field data.

4. Existence of transmission eigenvalues

In this section we consider the existence of transmission eigenvalues for anisotropic magnetic dielectric media with voids. We apply similar analysis as in [12] which must be modified to account for the presence of voids, that are subregions D_1 where $A = I$ and $n = 1$.

At this point, we consider the case $(A_{min} - 1) > 0$ and $(n_{max} - 1) < 0$, or $(A_{max} - 1) < 0$ and $(n_{min} - 1) > 0$. Our goal is to prove the existence of real transmission eigenvalues, hence we assume that $k^2 \geq 0$. To this end, we formulate the transmission eigenvalue problem (6)-(9) as a problem for the difference $u := v - w \in H_0^1(D)$. By subtracting the partial differential equations and boundary conditions for v and w we have that the boundary value problem for v and u is given by

$$\nabla \cdot A\nabla u + k^2 n u = \nabla \cdot (A - I)\nabla v + k^2(n-1)v \qquad \text{in } D \tag{18}$$

$$\frac{\partial u}{\partial \nu_A} = \frac{\partial v}{\partial \nu_A} - \frac{\partial v}{\partial \nu} \qquad \text{on } \partial D \tag{19}$$

Notice that from $\partial w^+/\partial_A \nu = \partial w^-/\partial \nu$ and the continuity of $\partial v^+/\partial \nu = \partial v^-/\partial \nu$ across ∂D_1 we have that

$$\frac{\partial u^+}{\partial \nu_A} - \frac{\partial u^-}{\partial \nu} = \frac{\partial v^+}{\partial \nu_A} - \frac{\partial v^+}{\partial \nu} \tag{20}$$

where the superscripts $+$ and $-$ indicate approaching the boundary from outside and inside D_1 respectively. Next we need to assume that k^2 is not a Dirichlet eigenvalue for $-\Delta$ in D_1 and define the interior Dirichlet to Neumann mapping $T_k : H^{1/2}(\partial D_1) \to H^{-1/2}(\partial D_1)$ by

$$T_k : u\big|_{\partial D} \mapsto \frac{\partial u}{\partial \nu}\Big|_{\partial D} \qquad \text{where} \quad \Delta u + k^2 u = 0, \qquad \text{in } D_1. \tag{21}$$

(If D_1 has multiple simply connected components then we define the Dirichlet to Neumann operator component wise). Then for a given $u \in H_0^1(D)$ satisfying the Helmholtz equation inside D_1, we see (18)-(20) as a Neumann boundary value problem for v which can be written in an equivalent variational form as follows

$$\int_{D_2} (A - I)\nabla v \cdot \nabla \overline{\varphi} - k^2(n-1)v\overline{\varphi}\, dx = \int_{D_2} A\nabla u \cdot \nabla \overline{\varphi} - k^2 n u \overline{\varphi}\, dx + \int_{\partial D_1} \overline{\varphi}\, T_k u\, ds \qquad \forall \varphi \in H^1(D_2) \tag{22}$$

We use the variational formulation to define a bounded linear operator that maps $u \in H_0^1(D) \mapsto v_u \in H^1(D_2)$. To this end let us define the bounded sesqulinear form and the bounded conjugate linear functional from the variational formulation as:

$$\mathcal{B}_k(v, \varphi) := \int_{D_2} (A - I)\nabla v \cdot \nabla \overline{\varphi} - k^2(n-1)v\overline{\varphi}\, dx,$$

$$f_u(\varphi) := \int_{D_2} A\nabla u \cdot \nabla\overline{\varphi} - k^2 nu\overline{\varphi}\, dx + \int_{\partial D_1} \overline{\varphi}\, T_k u\, ds.$$

and consider the variational problem of finding $v \in H^1(D_2)$ such that $\mathcal{B}_k(v,\varphi) = f_u(\varphi)$ for all $\varphi \in H^1(D_2)$. We split the solution $v = \widehat{v} + c$ where c is a constant and $\widehat{v} \in \widehat{H}^1(D_2) := \{\widehat{v} \in H^1(D_2) \mid \int_{D_2}(n-1)\widehat{v}dx = 0\}$ equipped with the $H^1(D_2)$ inner-product. It can be shown that functions in $\widehat{H}^1(D_2)$ satisfies the Poincaré inequality, that is $||\widehat{v}||^2_{L^2(D_2)} \leq C_p||\nabla\widehat{v}||^2_{L^2(D_2)}$ for all $\widehat{v} \in \widehat{H}^1(D_2)$. Now letting $\varphi = 1$ for $k^2 \neq 0$ we have that

$$k^2 \int_{D_2} (n-1)v\, dx = k^2 \int_{D_2} nu\, dx + \int_{\partial D_1} \frac{\partial u}{\partial \nu}\, ds = k^2 \int_{D_2} nu\, dx - k^2 \int_{D_1} u\, dx$$

where the latter equality holds due to the fact that u solves the Helmholtz equation in D_1. Using this along with $v = \widehat{v} + c$ we have that $c = \frac{1}{\int_{D_2}(n-1)\,dx}\left(\int_{D_2} nu\, dx - \int_{D_1} u\, dx\right)$. If $k^2 = 0$ we require c to still be defined as in the non-zero case. Now we show the variational problem is well posed in the space $\widehat{H}^1(D_2)$ by proving that $\pm\mathcal{B}_k(\widehat{v},\widehat{\varphi})$ is $\widehat{H}^1(D_2)$-coercive, when $A_{min}-1 > 0$ and $n_{max}-1 < 0$, or $A_{max}-1 < 0$ and $n_{min}-1 > 0$ respectively. If $A_{min} - 1 > 0$ and $n_{max} - 1 < 0$

$$\begin{aligned}
\mathcal{B}_k(\widehat{v},\widehat{v}) &= \int_{D_2} (A-I)\nabla\widehat{v}\cdot\nabla\overline{\widehat{v}} - k^2(n-1)|\widehat{v}|^2\, dx \\
&\geq \int_{D_2} (A_{min}-1)\nabla\widehat{v}\cdot\nabla\overline{\widehat{v}} + k^2(1-n_{max})|\widehat{v}|^2\, dx \\
&\geq (A_{min}-1)||\nabla\widehat{v}||^2_{L^2(D_2)} \geq C||\widehat{v}||^2_{H^1(D_2)}
\end{aligned}$$

where we have used the Poincaré inequality. Similarly we can show that if $A_{max}-1 < 0$ and $n_{min}-1 > 0$, $-\mathcal{B}_k(\cdot,\cdot)$ is coercive. Having $v_u \in H^1(D_2)$ defined in the annulus D_2 for any $u \in H^1_0(D)$. Since the transmission eigenfunction v solves the Helmholtz equation in the domain D, we insure that v_u can be extended to a solution of the Helmholtz equation in D. Using the Riesz representation theorem we can now define $\mathbb{L}_k u$ by

$$(\mathbb{L}_k u, \varphi)_{H^1(D_2)} = \int_{D_2} \nabla v_u \cdot \nabla\overline{\varphi} - k^2 v_u\overline{\varphi}\, dx + \int_{\partial D_1} \overline{\varphi}\, T_k v_u\, ds \quad \forall\varphi \in H^1_0(D_2, \partial D). \tag{23}$$

where

$$H^1_0(D_2,\partial D) = \left\{u \in H^1(D_2) :\ u = 0 \text{ on } \partial D, \qquad D_2 = D \setminus \overline{D}_1\right\}.$$

Notice that the mapping $k \mapsto \mathbb{L}_k$ is continuous for $k \in \mathbb{R}$ and k^2 not a Dirichlet eigenvalue. We can now connect the kernel of the operator $\mathbb{L}_k : H^1_0(D_2,\partial D) \to H^1_0(D_2,\partial D)$ to the set of transmission eigenfunctions.

Theorem 4.1 *Assume that k^2 is not a Dirichlet eigenvalue for $-\Delta$ in D_1 and assume that either $(A_{min}-1) > 0$ and $(n_{max}-1) < 0$, or $(A_{max}-1) < 0$ and $(n_{min}-1) > 0$.*

If $w, v \in H^1(D)$ solves (6)-(9) then $u|_{D_2} = w - v \in H^1_0(D_2, \partial D)$ is such that $\mathbb{L}_k u = 0$. Conversely, if $\mathbb{L}_k u = 0$ for $u \in H^1_0(D_2, \partial D)$ then v_u and u can be extended to solution $v, u \in H^1(D)$ of Helmholtz equation in D_1 and the pair $(u + v, v)$ solves (6)-(9).

Proof. The first part of the Theorem is by construction. Obviously $\mathbb{L}_k u = 0$ since v_u satisfies the Helmholtz equation in D. Conversely, let $\mathbb{L}_k u = 0$ and define $v := v_u \in H^1(D_2)$ as above and in D_1 by

$$\Delta v + k^2 v = 0 \qquad \text{in } D_1, \qquad v = v_u^+ \qquad \text{on } \partial D_1. \tag{24}$$

Since $\mathbb{L}_k u = 0$, (23) implies that $v \in H^1(D)$ and satisfies the Helmholtz equation in D. Furthermore, extending u in D_1 by

$$\Delta u + k^2 u = 0 \qquad \text{in } D_1, \qquad u = u^+ \qquad \text{on } \partial D_1,$$

then (22) implies that $(u + v, v)$ solves (6)-(9). □

The following lemma states some properties of the operator $\mathbb{L}_k$.

Lemma 4.1 *Assume that $k^2 \geq 0$ is not a Dirichlet eigenvalue for $-\Delta$ in D_1 and assume that either $(A_{min} - 1) > 0$ and $(n_{max} - 1) < 0$, or $(A_{max} - 1) < 0$ and $(n_{min} - 1) > 0$.*

(i) The operator $\mathbb{L}_k : H^1_0(D_2, \partial D) \mapsto H^1_0(D_2, \partial D)$ is self-adjoint.

(ii) The operator $\mathbb{L}_0$ or $-\mathbb{L}_0$ is coercive when $(A_{min} - 1) > 0$ or $(A_{max} - 1) < 0$, respectively.

(iii) $\mathbb{L}_k - \mathbb{L}_0$ is a compact.

Proof. (i) Let u_1 and u_2 be given in $H^1_0(D_2, \partial D)$ and consider v_1 and v_2 in $H^1(D_2)$ satisfying (22) extended inside D as solutions to the Helmholtz equation by (24). Thus, for these functions we have

$$\int_{D_2} (A - I)\nabla v_i \cdot \nabla \overline{\varphi} - k^2(n-1) v_i \overline{\varphi}\, dx = \int_{D_2} A\nabla u_i \cdot \nabla \overline{\varphi} - k^2 n u_i \overline{\varphi}\, dx + \int_{\partial D_1} \overline{\varphi}\, T_k u_i\, ds \qquad \forall \varphi \in H^1(D) \tag{25}$$

By the definition of $\mathbb{L}_k$ we have that

$$\begin{aligned}(\mathbb{L}_k u_1, u_2)_{H^1(D_2)} &= \int_{D_2} \nabla v_1 \cdot \nabla \overline{u_2} - k^2 v_1 \overline{u_2}\, dx + \int_{\partial D_1} \overline{u_2}\, T_k v_1\, ds \\ &= -\int_{D_2} (A - I)\nabla v_1 \cdot \nabla \overline{u_2} - k^2(n-1) v_1 \overline{u_2}\, dx \\ &\quad + \int_{D_2} A\nabla v_1 \cdot \nabla \overline{u_2} - k^2 n v_1 \overline{u_2}\, dx + \int_{\partial D_1} \overline{u_2}\, T_k v_1\, ds \end{aligned} \tag{26}$$

Taking $i = 2$ and $\varphi = v_1$ and then $i = 1$ and $\varphi = u_2$ in (25) we obtain

$$(\mathbb{L}_k u_1, u_2)_{H^1(D_2)} = -\int_{D_2} A\nabla u_1 \cdot \nabla \overline{u_2} - k^2 n u_1 \overline{u_2}\, dx - \int_{\partial D_1} \overline{u_2}\, T_k u_1\, ds \tag{27}$$

$$+ \int_{D_2} (A - I)\nabla v_1 \cdot \nabla \overline{v_2} - k^2(n-1) v_1 \overline{v_2}\, dx - \int_{\partial D_1} v_1 T_k \overline{u_2} - \overline{u_2}\, T_k v_1\, ds$$

$$= -\int_{D_2} A\nabla u_1 \cdot \nabla \overline{u_2} - k^2 n u_1 \overline{u_2}\, dx - \int_{D_1} \nabla u_1\, \nabla \overline{u_2} - k^2 u_1 \overline{u_2}\, dx$$

$$+ \int_{D_2} (A - I)\nabla v_1 \cdot \nabla \overline{v_2} - k^2(n-1) v_1 \overline{v_2}\, dx = (u_1, \mathbb{L}_k u_2)_{H^1(D_2)}$$

which proves that $\mathbb{L}_k$ is selfadjoint.

(ii) To prove that $\pm\mathbb{L}_0$ is coercive we first assume that $A_{min} - 1 > 0$ and therefore we consider the operator $\mathbb{L}_0$. Letting $v - u = w$ we have that

$$(\mathbb{L}_0 u, u)_{H^1(D_2)} = \int_{D_2} \nabla v \cdot \nabla \overline{u}\, dx + \int_{\partial D_1} T_0 v\, \overline{u}\, ds$$

$$= \int_{D_2} |\nabla u|^2\, dx + \int_{D_2} \nabla w \cdot \nabla \overline{u}\, dx + \int_{\partial D_1} T_0 u\, \overline{u}\, ds + \int_{\partial D_1} T_0 w\, \overline{u}\, ds.$$

Using (22) for $k^2 = 0$ and $\varphi = w$ and taking the conjugate, we see that

$$\int_{D_2} (A - I)\nabla w \cdot \nabla \overline{v}\, dx = \int_{D_2} A\nabla w \cdot \nabla \overline{u}\, dx + \int_{\partial D_1} w\, T_0 \overline{u}\, ds$$

Now once again using that $v = u + w$ we see that

$$\int_{D} \nabla w \cdot \nabla \overline{u}\, dx = \int_{D_2} (A - I)\nabla w \cdot \nabla \overline{w}\, dx - \int_{\partial D_1} w\, T_0 \overline{u}\, ds.$$

Now using that $A_{min} - 1 > 0$ we have that

$$\int_{D_2} (A - I)\nabla w \cdot \nabla \overline{w}\, dx \geq (A_{min} - 1) \int_{D_2} \nabla w \cdot \nabla \overline{w}\, dx \geq 0$$

Therefore

$$(\mathbb{L}_0 u, u)_{H^1(D_2)} \geq \int_{D_2} |\nabla u|^2\, dx + \int_{\partial D_1} \overline{u}\, T_0 u\, ds \geq \int_{D_2} |\nabla u|^2\, dx$$

proving the coercivity due to the zero boundary condition on ∂D.
Next, assume that $A_{max} - 1 < 0$, therefore considering the operator $-\mathbb{L}_0$. From (27) we have that

$$(-\mathbb{L}_0 u, u)_{H^1(D_2)} = -\int_{D_2} (A - I)\nabla v \cdot \nabla \overline{v}\, dx + \int_{D_1} |\nabla u|^2\, dx + \int_{D_2} A\nabla u \cdot \nabla \overline{u}\, dx$$

Now since $A_{max} - 1 < 0$ we have that:

$$-\int_{D_2} (A - I)\nabla v \cdot \nabla \overline{v}\, dx \geq (1 - A_{max}) \int_{D_2} |\nabla v|^2\, dx \geq 0.$$

Therefore $(-\mathbb{L}_0 u, u)_{H^1(D_2)} \geq A_{min} \int_{D_2} |\nabla u|^2\, dx$ proving coercivity in this case.

(iii) We now show the compactness of $\mathbb{L}_k - \mathbb{L}_0$. Assume that the sequence $u^j \rightharpoonup 0$ in $H_0^1(D_2, \partial D)$ and therefore we have the existence of $v_k^j \rightharpoonup 0$ and $v_0^j \rightharpoonup 0$ in $H^1(D_2)$, corresponding to solutions of (25). Since zero and k^2 are not Dirichlet eigenvalues, we have that their extension as solution to the Helmholtz equation inside D_1 converge weakly to 0 in D. From the Rellich's embedding theorem, a subsequence of the aforementioned sequences, still denoted by v_k^j and v_0^j converge strongly to zero in $L^2(D)$. Letting $\tilde{v}^j := v_k^j - v_0^j$ and using (25) with the sequences v_k^j and v_0^j we have that

$$\int_{D_2} (A - I)\nabla \tilde{v}^j \cdot \nabla \overline{\varphi}\, dx = k^2 \int_{D_2} (n-1) v_k^j \overline{\varphi} - n u^j \overline{\varphi}\, dx + k^2 \int_{D_1} u^j \overline{\varphi}\, dx \quad \forall\, \varphi \in H^1(D).$$

Letting $\varphi = \tilde{v}^j$ and for either $A - I$ positive or negative definite we obtain that $\tilde{v}^j \to 0$ in $H^1(D_2)$. Now we have that

$$\Delta \tilde{v}^j = -k^2 v_k^j \ \text{ in } D_1 \ \text{ and } \ \tilde{v}^j = v_k^j - v_0^j \ \text{ on } \ \partial D_1.$$

Therefore

$$||\tilde{v}^j||_{H^1(D_1)} \leq C \left(||v_k^j - v_0^j||_{H^1(D_2)} + ||v_k^j||_{L^2(D)} \right) \to 0$$

where we have used the trace theorem on ∂D_1. Now

$$\Big((\mathbb{L}_k - \mathbb{L}_0) u^j, \varphi \Big)_{H^1(D_2)} = \int_{D_2} \nabla \tilde{v}^j \cdot \nabla \overline{\varphi} - k^2 v_k^j \overline{\varphi}\, dx + \int_{\partial D_1} \overline{\varphi} \left(T_k v_k^j - T_0 v_0^j \right)\, ds.$$

Finally, by the definition of the Dirichlet to Neumann operator and using Cauchy-Schwartz inequality we have that

$$||(\mathbb{L}_k - \mathbb{L}_0) u^j||_{H^1(D_2)} \leq C \Big(||\tilde{v}^j||_{H^1(D)} + ||v_k^j||_{L^2(D)} \Big)$$

proving the claim since the right hand side tends to zero. □

Notice that the second part of this theorem says that for $k = 0$ the operator $\pm \mathbb{L}_k$ is positive. We now prove that $\pm \mathbb{L}_k$ is positive for a range of values, which gives a lower bound on the transmission eigenvalues.

Theorem 4.2 *Let $\lambda_1(D)$ be the first Dirichlet eigenvalue of $-\Delta$ in D and let k^2 be a real transmission eigenvalue. Then*

(i) if $(A_{min} - 1) > 0$ and $(n_{max} - 1) < 0$, then we have that $k^2 \geq \lambda_1(D)$,

(ii) if $(A_{max}-1)<0$ *and* $(n_{min}-1)>0$*, then we have that* $k^2 \geq \dfrac{A_{min}}{n_{max}}\lambda_1(D)$.

Proof (i) Assume that $A_{min}-1>0$ and $n_{max}-1<0$, we have that if u is the difference of eigenfunctions then $(\mathbb{L}_k u, u)_{H^1(D_2)}=0$. So by the definition of $\mathbb{L}_k$ and by using $v=u+w$ we have that

$$\begin{aligned}(\mathbb{L}_k u, u)_{H^1(D_2)} &= \int_{D_2} \nabla v\cdot\nabla\overline{u} - k^2 v\overline{u}\,dx + \int_{\partial D_1} \overline{u}T_k v\,ds = \int_D \nabla v\cdot\nabla\overline{u} - k^2 v\overline{u}\,dx\\ &= \int_D |\nabla u|^2 - k^2|u|^2\,dx + \int_D \nabla w\cdot\nabla\overline{u} - k^2 w\overline{u}\,dx.\end{aligned}$$

Now we use the variational form (22) for $\varphi = w$ which gives

$$\begin{aligned}\int_{D_2}(A-I)\nabla v\cdot\nabla\overline{w} - k^2(n-1)v\overline{w}\,dx - \int_{D_2} A\nabla u\cdot\nabla\overline{w} - k^2 nu\overline{w}\,dx\\ = \int_{D_1}\nabla u\cdot\nabla\overline{w} - k^2 u\overline{w}\,dx.\end{aligned}$$

On the left hand side we once again use that $v=w+u$ and combine the integrals involving both u and w giving that

$$\int_D \nabla u\cdot\nabla\overline{w} - k^2 u\overline{w}\,dx = \int_{D_2}(A-I)\nabla w\cdot\nabla\overline{w} - k^2(n-1)|w|^2\,dx.$$

Now we look at $(\mathbb{L}_k u, u)_{H^1(D_2)}$ and use the fact that under the assumptions on the coefficients that

$$\int_{D_2}(A-I)\nabla w\cdot\nabla\overline{w} - k^2(n-1)|w|^2\,dx \geq 0.$$

Therefore

$$\begin{aligned}(\mathbb{L}_k u, u)_{H^1(D_2)} &= \int_D |\nabla u|^2 - k^2|u|^2\,dx + \int_{D_2}(A-I)\nabla w\cdot\nabla\overline{w} - k^2(n-1)|w|^2\,dx\\ &\geq \int_D |\nabla u|^2 - k^2|u|^2\,dx \geq \left[\lambda_1(D)-k^2\right]\int_D |u|^2\,dx. \qquad (28)\end{aligned}$$

So if $(\lambda_1(D)-k^2)>0$, we have that $(\mathbb{L}_k u, u)_{H^1(D_2)}>0$ which contradicts the fact that $\mathbb{L}_k u = 0$ which implies that all real transmission eigenvalues must satisfy $k^2 \geq \lambda_1(D)$.

(ii) Assuming now that $A_{max}-1<0$ and $n_{min}-1>0$, we have that if u is the difference of eigenfunctions then $(-\mathbb{L}_k u, u)_{H^1(D_2)}=0$. But from (27) we have that

$$\begin{aligned}(-\mathbb{L}_k u, u)_{H^1(D_2)} = &-\int_{D_2}(A-I)\nabla v\cdot\nabla\overline{v} - k^2(n-1)|v|^2\,dx + \int_{D_1}\nabla|u|^2 - k^2|u|^2\,dx\\ &+ \int_{D_2} A\nabla u\cdot\nabla\overline{u} - k^2 n|u|^2\,dx.\end{aligned}$$

Notice that under the assumptions on the coefficients we have that

$$-\int_{D_2}(A-I)\nabla v\cdot\nabla\overline{v}-k^2(n-1)|v|^2\,dx\geq 0.$$

Therefore

$$\begin{aligned}(-\mathbb{L}_k u,u)_{H^1(D_2)}&\geq\int_{D_1}|\nabla u|^2-k^2|u|^2\,dx+\int_{D_2}A\nabla u\cdot\nabla\overline{u}-k^2n|u|^2\,dx\\&\geq\int_{D_1}|\nabla u|^2-k^2|u|^2\,dx+A_{min}\int_{D_2}|\nabla u|\,dx-k^2n_{max}\int_{D_2}|u|^2\,dx\\&\geq A_{min}\int_{D}|\nabla u|\,dx-k^2n_{max}\int_{D}|u|^2\,dx\\&\geq\Big[A_{min}\lambda_1(D)-k^2n_{max}\Big]\int_{D}|u|^2\,dx.\end{aligned}$$

So if $(A_{min}\lambda_1(D)-k^2n_{max})>0$, we have that $(-\mathbb{L}_k u,u)_{H^1(D_2)}>0$ which contradicts the fact that $\mathbb{L}_k u=0$, which implies all real transmission eigenvalues satisfy $k^2\geq\frac{A_{min}}{n_{max}}\lambda_1(D)$. $\square$

The previous result shows that the operator $\pm\mathbb{L}_k$ is positive for a range of k values. We next show that the operator is non-positive for some k on a subset of $H_0^1(D_2,\partial D)$.

Lemma 4.2 *Provided that the measure of each component of the void D_1 is sufficiently small, there exists a $k>0$ such that $\mathbb{L}_k$, or $-\mathbb{L}_k$ for $(A_{min}-1)>0$ and $(n_{max}-1)<0$, or $(A_{max}-1)<0$ and $(n_{min}-1)>0$ respectively, is non-positive on a subspace of $H_0^1(D_2,\partial D)$.*

Proof. Assume that $(A_{min}-1)>0$ and $(n_{max}-1)<0$, and look at the operator $\mathbb{L}_k$. We denote by B_r the ball of radius r. Let R and ϵ be positive constants such that $\overline{B_R}\subset D$, $\overline{D_1}\subset B_\epsilon$ and $R>\epsilon$. By using separation of variables one can see that there exists transmission eigenvalues for the system (see Section 5)

$$\begin{aligned}&\Delta\hat{w}+\tau^2\hat{w}=0 && \text{in } B_\epsilon\\&\nabla\cdot A_{min}\nabla\hat{w}+\tau^2n_{max}\hat{w}=0 && \text{in } B_R\setminus\overline{B_\epsilon}\\&\Delta\hat{v}+\tau^2\hat{v}=0 && \text{in } B_R\\&\hat{w}^-=\hat{w}^+\text{ and }\frac{\partial\hat{w}^-}{\partial\nu}=\frac{\partial\hat{w}^+}{\partial\nu_{A_{min}}} && \text{on } \partial B_\epsilon\\&\hat{w}=\hat{v}\text{ and }\frac{\partial\hat{w}}{\partial\nu_{A_{min}}}=\frac{\partial\hat{v}}{\partial\nu} && \text{on } \partial B_R\end{aligned}\tag{29}$$

Now recall we can only define $\mathbb{L}_k$ when k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D_1, so we denote the first eigenvalue as $\lambda_1(D_1)$. Since $\lambda_1(D_1)\to\infty$ as $|D_1|\to 0^+$ we can insure that there is at least one transmission eigenvalue of the form $\tau^2=k^2(B_R,B_\epsilon,A_{min},n_{max})<\lambda_1(D_1)$ provided that the measure of each components of D_1

is sufficiently small. So let $\hat{u}$ be the difference of these eigenfunctions with eigenvalue τ^2 giving that from (28)

$$\int_{B_R} |\nabla \hat{u}|^2 - \tau^2 |\hat{u}|^2 \, dx + \int_{B_R \setminus \overline{B_\epsilon}} (A_{min} - 1) \nabla \hat{w} \cdot \nabla \overline{\hat{w}} - \tau^2 (n_{max} - 1) |\hat{w}|^2 \, dx = 0.$$

Therefore $\hat{u} \in H_0^1(B_R)$, so let the extension by zero of $\hat{u}$ to the whole domain D be denoted $\widetilde{u}$. Now since $\overline{D_1} \subset B_\epsilon$ we have that $\Delta \widetilde{u} + \tau^2 \widetilde{u} = 0$ in D_1. Since $A_{min} - 1 > 0$ and $n_{max} - 1 < 0$, we can construct nontrivial $\widetilde{v} \in H^1(D_2)$ that solve (22) with coefficients A, n in the domain D with void D_1 and let $\widetilde{w} = \widetilde{v} - \widetilde{u}$. Hence from (22) and using that $\widetilde{w} = \widetilde{v} - \widetilde{u}$ we have that:

$$\begin{aligned} \int_{D \setminus \overline{D_1}} (A - I) \nabla \widetilde{w} \cdot \nabla \overline{\varphi} - \tau^2 (n-1) \widetilde{w} \overline{\varphi} \, dx &= \int_{B_R} \nabla \hat{u} \cdot \nabla \overline{\varphi} - \tau^2 \hat{u} \overline{\varphi} \, dx \\ &= \int_{B_R \setminus \overline{B_\epsilon}} (A_{min} - 1) \nabla \hat{w} \cdot \nabla \overline{\varphi} - \tau^2 (n_{max} - 1) \hat{w} \overline{\varphi} \, dx. \end{aligned} \tag{30}$$

Therefore for $\varphi = \tilde{w}$ using (30) and the Cauchy-Schwartz inequality we have that

$$\int_{D \setminus \overline{D_1}} (A - I) \nabla \widetilde{w} \cdot \nabla \overline{\widetilde{w}} - \tau^2 (n-1) \widetilde{w} \overline{\widetilde{w}} \, dx = \int_{B_R \setminus \overline{B_\epsilon}} (A_{min} - 1) \nabla \hat{w} \cdot \nabla \overline{\widetilde{w}} - \tau^2 (n_{max} - 1) \hat{w} \overline{\widetilde{w}} \, dx \leq$$

$$\left[\int_{B_R \setminus \overline{B_\epsilon}} (A_{min} - 1) |\nabla \hat{w}|^2 - \tau^2 (n_{max} - 1) |\hat{w}|^2 \, dx \right]^{\frac{1}{2}} \left[\int_{B_R \setminus \overline{B_\epsilon}} (A_{min} - 1) |\nabla \widetilde{w}|^2 - \tau^2 (n_{max} - 1) |\widetilde{w}|^2 \, dx \right]^{\frac{1}{2}}$$

and using (30) with $\varphi = \tilde{w}$ once more we obtain

$$\int_{D \setminus \overline{D_1}} (A - I) \nabla \widetilde{w} \cdot \nabla \overline{\widetilde{w}} - \tau^2 (n-1) |\widetilde{w}|^2 \, dx \leq \int_{B_R \setminus \overline{B_\epsilon}} (A_{min} - 1) |\nabla \hat{w}|^2 - \tau^2 (n_{max} - 1) |\hat{w}|^2 \, dx.$$

Now we use the definition (28) for the operator $\mathbb{L}_\tau$ with the functions $\widetilde{u}$ and $\widetilde{w}$ to conclude

$$\begin{aligned} (\mathbb{L}_\tau \widetilde{u}, \widetilde{u})_{H^1(D_2)} &= \int_D |\nabla \widetilde{u}|^2 - \tau^2 |\widetilde{u}|^2 \, dx + \int_{D \setminus \overline{D_1}} (A - I) \nabla \widetilde{w} \cdot \nabla \overline{\widetilde{w}} - \tau^2 (n-1) |\widetilde{w}|^2 \, dx \\ &\leq \int_{B_R} |\nabla \hat{u}|^2 - \tau^2 |\hat{u}|^2 \, dx + \int_{B_R \setminus \overline{B_\epsilon}} (A_{min} - 1) \nabla \hat{w} \cdot \nabla \overline{\hat{w}} - \tau^2 (n_{max} - 1) |\hat{w}|^2 \, dx \\ &= 0. \end{aligned}$$

So the operator $\mathbb{L}_\tau$ is non-positive on this one dimensional subspace.

Alternatively, we can construct a finite dimensional subspace of $H_0^1(D_2, \partial D)$ where $\mathbb{L}_\tau$ is non-positive by considering small balls $B_\delta \subset D_2$. In this case let κ, $\hat{w}$ and $\hat{v}$ are the first transmission eigenvalue and corresponding eigenfunctions of the system

$$\begin{aligned} &\nabla \cdot A_{min} \nabla \hat{w} + \kappa^2 n_{max} \hat{w} = 0 \ \text{ in } \ B_\delta \\ &\Delta \hat{v} + \kappa^2 \hat{v} = 0 \qquad\qquad \text{ in } \ B_\delta \\ &\hat{w} = \hat{v} \text{ and } \frac{\partial \hat{w}}{\partial \nu_{A_{min}}} = \frac{\partial \hat{v}}{\partial \nu} \qquad \text{ on } \partial B_\delta. \end{aligned} \tag{31}$$

Now provided that the measure of each component of D_1 is small enough such that κ^2 is smaller than the first corresponding Dirichlet eigenvalue for $-\Delta$, we can use $\hat{u} = \hat{v} - \hat{w} \in H_0^1(B_\delta)$, and its extension by zero $\tilde{u}$ to the whole domain and the corresponding $\tilde{w}$ and $\tilde{v}$ exactly as above to show that $\mathbb{L}_\tau$ is non-positive in an m-dimensional subspace of $H_0^1(D_2, \partial D)$ where m is the number of balls of radius δ included in D_2.

The same result can be proven for $-\mathbb{L}_k$ exactly in a similar way for the case when $A_{max} - 1 < 0$ and $n_{min} - 1 > 0$ where everywhere A_{max} is replaced by A_{min} and n_{min} is replaced by n_{max}. $\square$

To prove now the existence of transmission eigenvalues we use the following theorem

Theorem 4.3 *Let $\mathbb{L}_k : H_0^1(D_2, \partial D) \mapsto H_0^1(D_2, \partial D)$ be as defined above. If*

(i) there exists $k_{min} \geq 0$ such that $\theta \mathbb{L}_{k_{min}}$ is positive on $H_0^1(D_2, \partial D)$

(ii) there exists $k_{max} < \lambda_1(D_1)$ such that $\theta \mathbb{L}_{k_{max}}$ is non-positive on a m-dimensional subspace of $H_0^1(D_2, \partial D)$

then there exists m transmission eigenvalues in $[k_{min}, k_{max}]$, where $\theta = 1$ or $\theta = -1$ provided $A_{min} - 1 > 0$ and $n_{max} - 1 < 0$, or $A_{max} - 1 < 0$ and $n_{min} - 1 > 0$, respectively.

For the proof of this theorem see Theorem 2.6 [12] (see also Theorem 4.7 in [11] or Chapter 6 in [4]). In particular the result can be obtained by using min-max condition for the auxiliary eigenvalue problem for the self adjoint compact operator $\mathbb{I} - \lambda(k)(\theta \mathbb{L}_0)^{-1/2} \theta (\mathbb{L}_k - \mathbb{L}_0)(\theta \mathbb{L}_0)^{-1/2}$.

Now combining Lemma 4.2 and Theorem 4.3 we can prove the following result.

Theorem 4.4 *Assume that either $A_{min} - 1 > 0$ and $n_{max} - 1 < 0$, or $A_{max} - 1 < 0$ and $n_{min} - 1 > 0$. If the first transmission eigenvalue $\tau > 0$ of (29) is smaller than the first Dirichlet eigenvalue for each of the components of D_1, then there exists one transmission eigenvalue in the interval $(0, \tau)$. If the first transmission eigenvalue $\kappa > 0$ of (31) is smaller than the first Dirichlet eigenvalue for each of the components of D_1 then there exits $m := m(\delta)$ transmission eigenvalue (counting their multiplicity) in the interval $(0, \kappa)$, where m is the number of balls of radius $\delta > 0$ that can fit in D_2.*

Note that the number $m(\delta)$ depends on the size of each components of void D_1 and also on the number of voids and their locations.

Remark 4.1 If $A_{min} - 1 > 0$ and $n_{max} - 1 > 0$, or $A_{max} - 1 < 0$ and $n_{min} - 1 < 0$ it is now obvious how to modify the approach of [12] to prove the existence of transmission eigenvalue. In this case in addition to assuming that the voids are small enough it is necessary to assume that $|n-1|$ is also small. We omit here the details in order to avoid repetition.

As a by-product of Theorem 4.2 and Theorem 4.3 we have that the first transmission eigenvalue satisfies the following upper and lower bounds.

Theorem 4.5 *Let $k_1^2(D, D_1, A, n)$, be the first transmission eigenvalue of the given media with voids and the measure of each component of D_1 is small enough (as discussed above). Then the following inequalities hold:*

(i) If $(A_{min} - 1) > 0$ and $(n_{max} - 1) < 0$, then

$$\lambda_1(D) \leq k_1^2(D, D_1, A, n) \leq \min\left\{k_1^2(B_R, B_\epsilon, A_{min}, n_{max}), k_1^2(B_\delta, A_{min}, n_{max})\right\}$$

$k_1^2(B_R, B_\epsilon, A_{min}, n_{max})$ and $k_1^2(B_\delta, A_{min}, n_{max})$ are the first transmission eigenvalue corresponding to (29) and the first transmission eigenvalue corresponding to (31), respectively.

(ii) If $(A_{max} - 1) < 0$ and $(n_{min} - 1) > 0$, then

$$\frac{A_{min}}{n_{max}}\lambda_1(D) \leq k_1^2(D, D_1, A, n) \leq \min\left\{(k^2(B_R, B_\epsilon, A_{max}, n_{min}), k^2(B_\delta, A_{max}, n_{min})\right\}$$

where $k_1^2(B_R, B_\epsilon, A_{max}, n_{min})$ and $k_1^2(B_\delta, A_{min}, n_{max})$ the first transmission eigenvalue corresponding to (29) and the first transmission eigenvalue corresponding to (31), respectively, with A_{max} replaced by A_{min} and n_{min} replaced by n_{max}.

Here $\lambda_1(D_1)$ is the first Dirichlet eigenvalue of $-\Delta$ in D_1.

We conclude this section by proving a monotonicity result for the first transmission eigenvalue with respect to the size of D_1, which can be useful in identifying voids in known anisotropic material as discussed in Section 5. We remark that it is possible to obtain monotonicity results for the first eigenvalue in terms of the material properties, but we do not present them here since the goal of this paper is to detect voids using transmission eigenvalue (see [19] for additional monotonicity results). For given A and n satisfying either $A_{min} - 1 > 0$ and $n_{max} - 1 < 0$, or $A_{max} - 1 < 0$ and $n_{min} - 1 > 0$ we have the following monotonicity results.

Theorem 4.6 *Let $D_1 \subseteq D_1'$. Then $k_1(D_1) \leq k_1(D_1')$ where $k_1(\Omega)$ is the first transmission eigenvalue corresponding to void Ω.*

Proof Assume that $(A_{min} - 1) > 0$ and $(n_{max} - 1) < 0$, and that $\tilde{v}$ and $\tilde{w}$ are the transmission eigenfunctions corresponding to the transmission eigenvalue $k_1(D_1') = \tilde{k}$.

Now let $\tilde{u} = \tilde{v} - \tilde{w}$, therefore we have the existence of $v \in H^1(D \setminus \overline{D}_1)$ that solves (22) and define $w = v - \tilde{u}$. Therefore we have from (28) that

$$\int_D |\nabla \tilde{u}|^2 - \tilde{k}^2 |\tilde{u}|^2 \, dx + \int_{D \setminus \overline{D}_1'} (A - I)\nabla \tilde{w} \cdot \nabla \overline{\tilde{w}} - \tilde{k}^2 (n-1)|\tilde{w}|^2 \, dx = 0.$$

By the definition of w and (22) we obtain that

$$\int_{D \setminus \overline{D}_1} (A - I)\nabla w \cdot \nabla \overline{\varphi} - \tilde{k}^2 (n-1) w \overline{\varphi} \, dx = \int_D \nabla \tilde{u} \cdot \nabla \overline{\varphi} - \tilde{k}^2 \tilde{u} \overline{\varphi} \, dx \tag{32}$$

$$= \int_{D \setminus \overline{D}_1'} (A - I)\nabla \tilde{w} \cdot \nabla \overline{\varphi} - \tilde{k}^2 (n-1) \tilde{w} \overline{\varphi} \, dx.$$

Therefore letting $\varphi = w$ in (32) and using the Cauchy-Schwartz inequality (in the same way as the equations below (30)) along with $D_1 \subseteq D_1'$ we have that

$$\int_{D \setminus \overline{D}_1} (A - I)\nabla w \cdot \nabla \overline{w} - \tilde{k}^2 (n-1)|w|^2 \, dx \leq \int_{D \setminus \overline{D}_1'} (A - I)|\nabla \tilde{w}|^2 - \tilde{k}^2 (n-1)|\tilde{w}|^2 \, dx.$$

Now we use the definition (28) for the operator $\mathbb{L}_{\tilde{k}}$ with the functions $\tilde{u}$ to conclude that

$$\begin{aligned}
(\mathbb{L}_{\tilde{k}} \tilde{u}, \tilde{u})_{H^1(D \setminus \overline{D}_1)} &= \int_D |\nabla \tilde{u}|^2 - \tau^2 |\tilde{u}|^2 \, dx + \int_{D \setminus \overline{D_1}} (A - I)\nabla w \cdot \nabla \overline{w} - \tilde{k}^2 (n-1)|w|^2 \, dx \\
&\leq \int_D |\nabla \tilde{u}|^2 - \tilde{k}^2 |\tilde{u}|^2 \, dx + \int_{D \setminus \overline{D}_1'} (A - I)\nabla \tilde{w} \cdot \nabla \overline{\tilde{w}} - \tilde{k}^2 (n-1)|\tilde{w}|^2 \, dx \\
&= 0
\end{aligned}$$

where $\mathbb{L}_{\tilde{k}}$ is the operator corresponding to the problem with void D_1. Since $\mathbb{L}_{\tilde{k}}$ is nonpositive on the subspace spanned by $\tilde{u}$ it means that there is an eigenvalue corresponding to D_1 in $(0, \tilde{k}]$. Therefore the first transmission eigenvalue $k_1(D_1)$ must satisfy $k_1(D_1) \in (0, \tilde{k}]$ which proves the claim. A similar argument holds for when $A_{max} - 1 < 0$ and $n_{min} - 1 > 0$, by looking at the operator $-\mathbb{L}_{\tilde{k}}$. □

5. Numerical Validation

In this section, we show some numerical examples to show that the first transmission eigenvalue can give information about the voids. We shall address the following issues.

(i) We check if the transmission eigenvalues can be determined from scattering data for for the case on anisotropic magnetic materials with voids based on the discussion of Section 2 (see e.g. [24] for near field data). We confirm that the eigenvalues

determined from the far-field data are actually the transmission eigenvalue. In particular, we consider a special case when the scattering object and the void are concentric disks in which case the transmission eigenvalues can be obtained analytically. For general geometry we compute the transmission eigenvalues using a continous finite element method with eigenvalues searching technique described in [15] (see also [23] and [25]).

(ii) We numerically study how the size, location and geometry of voids affects the first transmission eigenvalue.

(iii) We numerically study the inverse problem of estimating the size of the void(s) using the first transmission eigenvalue. Numerical results indicate that qualitative information can be obtained on the size of the void(s).

Theorem 3.1 suggests that if we solve the far-field equation and plot the L^2 norm of the solution g against a range of k values, at a transmission eigenvalue (TEV) the norm of g "blows up", which should look like a spike in the graph. Below is the numerical procedure with simulated far-field data:

(a) Solve the direct problem using a cubic finite element method with a perfectly matched layer, for a range of k values.

(b) Evaluate an approximate u_∞ with 1% random noise added (unless otherwise stated).

(c) Using the approximated u_∞ to solve the far field equation $Fg_z = \Phi_\infty(\cdot, z)$ for 25 random locations of z in D by a Tikhonov-Morozov regularization strategy.

(d) Plot $||g||_{L^2(0,2\pi)}$ averaged over z versus k

In the following calculations we use N different incident direction ϕ_j and N observation directions θ_i that are uniformly spaced in $[0, 2\pi)$ where $N = 30$ throughout the rest of the paper unless otherwise specified. As explained above the simulated far-field pattern $u_\infty(\theta_i, \phi_j)$ is obtained from solving the direct problem. This would lead to a discretized far field equation with $N \times N$ matrix. We then solve the discretized far-field equation for 25 randomly distributed points in the domain D. Once we have solved this linear systems for $\vec{g}$ which has components $g_i \approx g(\theta_i)$, we plot the average approximation of $||\vec{g}||_{\ell^2}$ over a range of k values.

5.1. Comparison with exact transmission eigenvalues

We will now consider a TEV problem with constant coefficients. For this we assume that $A = \alpha I$ for some constant $\alpha > 0$ and let n be constant such that $n > 0$. Furthermore assume that $D = B_R$ and $D_1 = B_\epsilon$ where $0 < \epsilon < R$. Under these assumptions the

transmission eigenvalue problem reads: find nontrivial v, w such that:

$$\Delta w + k^2 w = 0 \qquad \text{in } B_\epsilon \tag{33}$$

$$\alpha \Delta w + k^2 n w = 0 \qquad \text{in } B_R \setminus \overline{B_\epsilon} \tag{34}$$

$$\Delta v + k^2 v = 0 \qquad \text{in } B_R \tag{35}$$

$$w^- = w^+ \text{ and } \frac{\partial w^-}{\partial r} = \alpha \frac{\partial w^+}{\partial r} \text{ on } \partial B_\epsilon \tag{36}$$

$$w = v \text{ and } \alpha \frac{\partial w}{\partial r} = \frac{\partial v}{\partial r} \qquad \text{on } \partial B_R \tag{37}$$

It can be shown that trying to find transmission eigenfunctions of the form $w(r,\theta) = w_m(r)e^{im\theta}$ and $v(r,\theta) = v_m(r)e^{im\theta}$ with $m \in \mathbb{Z}$ gives that the transmission eigenvalues are given by the roots of $d_m(k)$, where $d_m(k)$ is defined as:

$$d_m(k) := \det \begin{pmatrix} \mathrm{J}_m(k\epsilon) & -\mathrm{J}_m(k\sqrt{\frac{n}{\alpha}}\epsilon) & -\mathrm{Y}_m(k\sqrt{\frac{n}{\alpha}}\epsilon) & 0 \\ -\mathrm{J}'_m(k\epsilon) & \sqrt{n\alpha}\,\mathrm{J}'_m(k\sqrt{\frac{n}{\alpha}}\epsilon) & \sqrt{n\alpha}\,\mathrm{Y}'_m(k\sqrt{\frac{n}{\alpha}}\epsilon) & 0 \\ 0 & \mathrm{J}_m(k\sqrt{\frac{n}{\alpha}}R) & \mathrm{Y}_m(k\sqrt{\frac{n}{\alpha}}R) & -\mathrm{J}_m(kR) \\ 0 & -\sqrt{n\alpha}\,\mathrm{J}'_m(k\sqrt{\frac{n}{\alpha}}R) & -\sqrt{n\alpha}\,\mathrm{Y}'_m(k\sqrt{\frac{n}{\alpha}}R) & \mathrm{J}'_m(kR) \end{pmatrix}$$

where $\mathrm{J}_p(t)$ and $\mathrm{Y}_p(t)$ are the Bessel functions of the first and second kind. To see if the solution of the far-field equation will capture the transmission eigenvalues we apply the method discussed above to (33)-(37). We expect to see spikes in the average norm of g at the known TEVs that are the roots of $d_m(k)$. In Figure 2 we plot the average $||g||_{L^2(0,2\pi)}$ for the parameters $\alpha = 1/5$, $n = 1$ and $R = 1$ with $\epsilon = 0.1$. Using Newton's method with a centered finite difference approximation for the derivative we can compute the first two roots of $d_0(k)$ given by $k \approx 2.48$, 5.27.

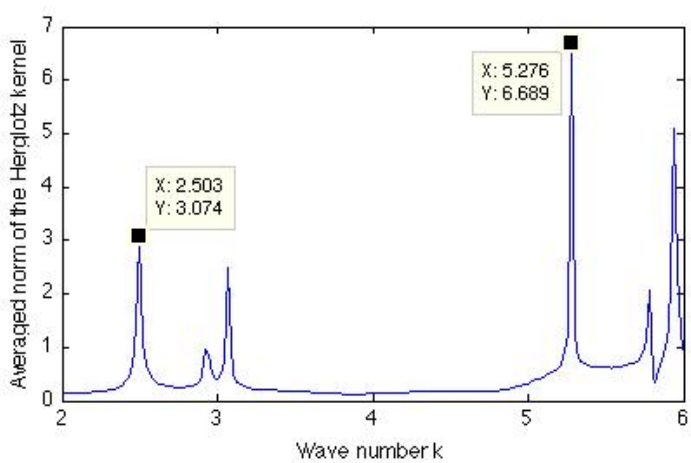

Figure 2. Notice that there are spikes at $k = 2.50$, 5.27 on the graph while the first two roots of $d_0(k)$ are 2.48, 5.27. The other spikes in the graph corresponds to roots for $d_m(k)$ where $m \neq 0$

We let $A_\alpha = \alpha I$, $n = 1$ and compare the roots of $d_0(k)$ to the spikes in the graph for $||g||_{L^2(0,2\pi)}$ for various values of α and ϵ, where we let the outer radius $R = 1$. These results are shown in Table 1. The values agree very well

Table 1. Root Finding v.s. Far-Field Eqation

α	ϵ	1st root of d_0	spike in the $\|\|g\|\|_{L^2(0,2\pi)}$
1/2	0.01	7.99	7.80
1/4	0.1	2.91	2.92
1/10	0.05	1.67	1.68

Next we consider non-circular domains and compare the transmission eigenvalues determined from the far field scattering data based on solving the Far-Field Equation (FFE) against those computed directly using a Finite Element Method (FEM). We now compare the reconstructed TEVs using the FFE with the FEM. We fix $A = \text{Diag}(5, 6)$ and $n = 2$ for the rest of the paper. The direct computation by the FEM in table 2 is done by a continuous FEM using the linear Lagrange elements with the mesh size $h \approx 0.01$. The results in table 2 are for domains with out the presents of a void.

Table 2. Comparison of FFE Computation v.s. FEM Calculations

Method	Domain	1st TEV	2nd TEV
FFE	square (2×2)	1.84	6.60
FEM	square (2×2)	1.84	6.63
FFE	circle $(R = 1)$	1.98	7.23
FEM	circle $(R = 1)$	1.98	7.13

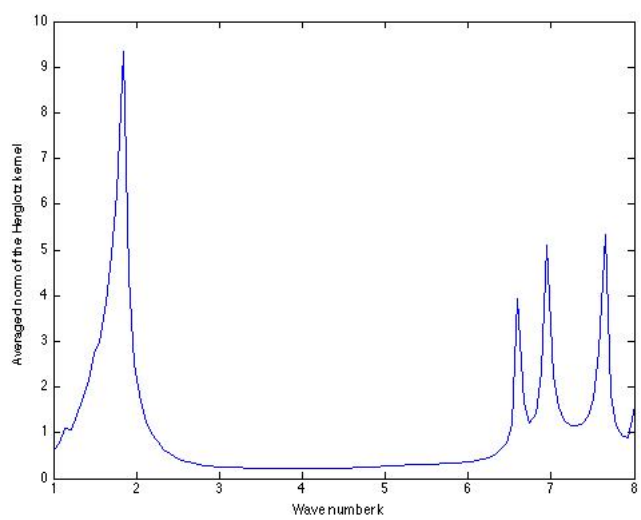

Figure 3. The plot of the average $||g||_{L^2(0,2\pi)}$ for the square (2×2) with no void: $A = \text{Diag}(5, 6)$ and $n = 2$

We now look at the question of partial aperture in using the far field data to compute the transmission eigenvalues. Partial aperture is where the angles ϕ and θ are

not distributed over the entire interval $[0, 2\pi)$, but rather some interval. So in table 3 we use $N = 20$ angles distributed uniformly over $[0, \pi)$. It is known that the smaller the aperture the more unstable the far field equation method is for reconstructing that TEVs. To test this we decrease the amount of random noise added to the calculations to see if the first spike computed with partial aperture data coincides with our first FEM computed transmission eigenvalue for sufficiently small noise. The results are shown in table 3.

Noise	1st spike
10^{-3}	1.84
10^{-6}	1.91
10^{-9}	1.98

Table 3. Limited aperture for Disk with $R = 1$, $A =$ Diag$(5, 6)$, $n = 2$. Note that from Table FEM give k=1.98.

5.2. Determination of void area

We now consider the inverse problem of determining information about the void D_1 form the first transmission eigenvalue. For fixed A and n Theorem 4.6 shows that the first transmission eigenvalue depends monotonically increasing on the size of the void. Indeed, Figure 4 is a plot that shows the monotonicity of the first root of $d_0(k)$ with respect to the size of the circular void ϵ where $\alpha = 1/5$, $n = 1$ and $R = 1$.

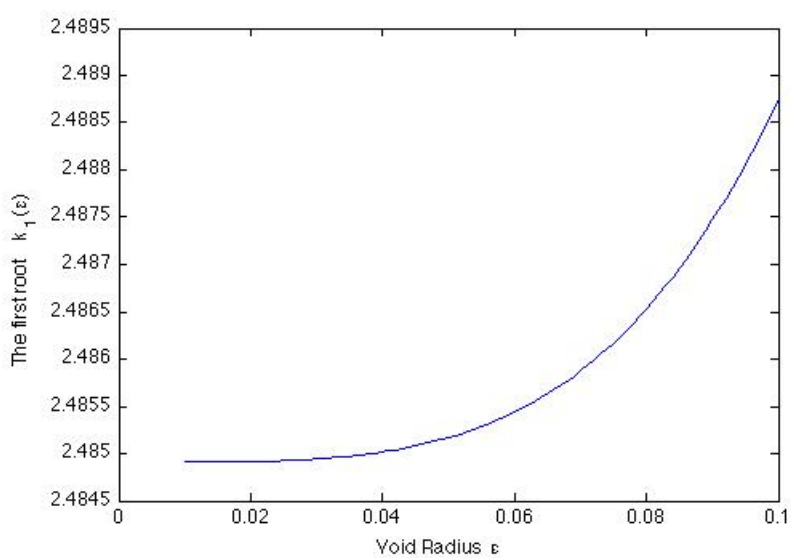

Figure 4. graph of $k_1(\epsilon)$ v.s. $[\epsilon_{min}, \epsilon_{max}]$ to show the monotonicity of the first TEV with respect to the size of the void

This monotonicity dependence is also confirmed by FEM calculations. The results

are shown in Table 4 for a disk of radius 1 and a 2×2 square $[-1,1] \times [-1,1]$ with a void of the form $B_\epsilon(0,0)$ for various ϵ.

Table 4. First TEV for various void sizes computed by the FEM

ϵ	0.2	0.19	0.18	0.17	0.16	0.15	0.14	0.13	0.12	0.11	0.1
Circle	9.53	9.27	9.02	8.77	8.54	8.31	8.08	7.86	7.64	7.43	7.22
Square	7.76	7.57	7.39	7.21	7.04	6.87	6.70	6.53	6.37	6.21	6.05

We also numerically investigated the dependence of the first transmission eigenvalue in terms of the location of the void. In particular we fixed the media $A_a = aI$, $n = 1$ with the support D being the unit ball centered at the origin, in which we considered a small circular void of radius $\epsilon = 0.1$ that is centered at (x_1, x_2). In Table 5 we see little to no difference if the location of the void is changed, where the first transmission eigenvalue is computed computed by solving the far field equation.

Table 5. Dependence of first transmission eigenvalue on void's position

location	(0, 0)	(0.6, 0)	(0.3, 0.7)	(-0.2, 0.4)	(0.6, 0.6)
$A_{1/4}$; k_1	2.90	2.92	2.92	2.96	2.92
$A_{1/9}$; k_1	1.77	1.80	1.78	1.80	1.78

The monotonicity property could be used to obtain information about the volume of the void D_1. Given the first transmission eigenvalue for fixed given material properties, we wish to find information about the size of the void. Hence, we consider the inverse problem of finding the (additive) area of a void(s) from the first transmission eigenvalue and again fix $A = \text{Diag}(5,6)$, $n = 2$. To do so we find an ϵ^* such that a void of the form $B_{\epsilon^*}(0,0)$ satisfies $k_1(\text{void(s)}) \approx k_1(B_{\epsilon^*}(0,0))$. Using this idea we try to reconstruct the area of multiple voids by using the first transmission eigenvalue computed by the FEM. We put two circular voids in the domains considered above. The voids both have radii 0.1 and be centered at $(0,0)$ and $(0.5, 0.5)$ respectively. We compute the first transmission eigenvalue in each case, then find the area of a single void of the shape of a disk that has the same first transmission eigenvalue. Note that the total area of the two voids is approximately 0.0630. The area of the single void $B_{\epsilon^*}(0,0)$ is 0.0607 for the unit disk and 0.0775 for the square. These calculations are presented in Figure 5. In Table 6 we show the results for the area of a void calculated based on the first transmission eigenvalue and on the assumption that (incorrectly) is a disc centered at the origin. These calculations give numerical evidence that the first TEVs can be used to gain qualitative information about the size of the void(s). In those calculations we used the "exact" transmission eigenvalue computed by the FEM. For this to be useful for industrial applications one of course one need to compute the transmission eigenvalue based on the scattering data.

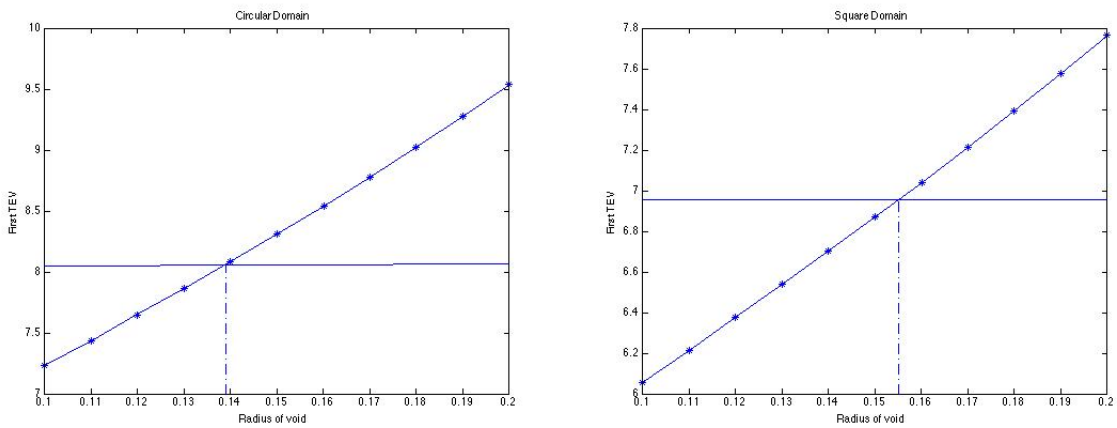

Figure 5. The first transmission eigenvalue of the two domains with a single void v.s the radius of the void. The horizontal lines are the k_1's for the two domains with two voids. The vertical dotted lines are the approximated values of ϵ^* such that a void of the form $B_{\epsilon^*}(0,0)$ gives the same transmission eigenvalue approximately, i.e. $k_1(\text{void(s)}) \approx k_1(B_{\epsilon^*}(0,0))$.

Table 6. Qualitative Reconstruction of Area from FF-measurements

D	D_1	$\lvert B_{\epsilon^*}(0,0)\rvert$	$\lvert D_1\rvert$
Disk $R=1$	Disk $r=0.1$	0.0328	0.0314
	Square	0.0303	0.0300
$[-1,1]\times[-1,1]$	Ellipse	0.0613	0.0628
	Square	0.0749	0.1256

The numerical experiments presented here are preliminary. It is desirable for instance to find a way to use more transmission eigenvalues in order to obtain addition information about voids (see e.g. [13] for a formula that connects perturbation of eigenvalues to the location and physical information of small non-voids inhomogeneities).

Acknowledgments

The research of I. H. is supported in part by the NSF Grant DMS-1106972 and the University of Delaware Graduate Fellowship. The research of F. C. is supported in part by the Air Force Office of Scientific Research Grant FA9550-11-1-0189 and NSF Grant DMS-1106972. The research of J. S is supported in part by the NSF under grant DMS-1016092 and the US Army Research Laboratory and the US Army Research Office under cooperative agreement number W911NF-11-2-0046.

References

[1] T. Arens, Why linear sampling method works, *Inverse Problems* **20** (2004), 163-173 (2004).

[2] AS. Bonnet-BenDhia, L. Chesnel, H. Haddar On the use of t-coercivity to study the interior transmission eigenvalue problem. C. R. Acad. Sci., Ser. I 340: 647-651 (2011).

[3] F. Cakoni, M. Cayoren and D. Colton, Transmission eigenvalues and the nondestructive testing of dielectrics, *Inverse Problems* **24** 066016, (2008).

[4] F. Cakoni and D. Colton, *A Qualitative Approach to Inverse Scattering Theory* Springer, Berlin 2014.

[5] F. Cakoni, D. Colton and H. Haddar, The computation of lower bounds for the norm of the index of refraction in an anisotropic media from far field data *J. Integral Eqns Appl.* **21** 203227, (2008).

[6] F. Cakoni, D. Colton, and H. Haddar, The interior transmission problem for regions with cavities. *SIAM J. Math. Anal.*, 42(1):145–162, (2010).

[7] F. Cakoni, D. Colton, and H. Haddar, On the determination of Dirichlet or transmission eigenvalues from far field data. *C. R. Acad. Sci. Paris*, 348:379–383, (2010).

[8] F. Cakoni, D. Colton, and P. Monk, On the use of transmission eigenvalues to estimate the index of refraction from far field data. *Inverse Problems*, 23:507–522, 2007.

[9] F. Cakoni, D. Colton, P. Monk and J. Sun, The inverse electromagnetic scattering problem for anisotropic media, *Inverse Problems*, **26** 074004 (2010).

[10] F. Cakoni, D. Gintides, and H. Haddar, The existence of an infinite discrete set of transmission eigenvalues. *SIAM J. Math. Anal.*, 42:237–255, (2010).

[11] F. Cakoni and H. Haddar, Transmission Eigenvalues in Inverse Scattering Theory *Inverse Problems and Applications, Inside Out* 60, MSRI Publications 2013.

[12] F. Cakoni and A. Kirsch, On the interior transmission eigenvalue problem. Int. Jour. Comp. Sci. Math. 3:142–167, (2010).

[13] F. Cakoni and S. Moskow, Asymptotic Expansions for Transmission Eigenvalues for Media with Small Inhomogeneities, *Inverse Problems*, **29**, 104014 (2013).

[14] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory.* Springer, New York, 3nd edition, 2013.

[15] D. Colton, P. Monk and J. Sun, *Analytical and computational methods for transmission eigenvalues*, Inverse Problems, Vol. 26, 045011 (2010).

[16] A. Cossonniere and H. Haddar, The electromagnetic interior transmission problem for regions with cavities. *SIAM J. Math. Anal.*, 43(4):1698–1715, (2011).

[17] D. Colton, L.Päivärinta, and J. Sylvester. The interior transmission problem. *Inverse Problems and Imaging*, 1:13–28, (2007).

[18] G. Giovanni and H. Haddar, Computing estimates on material properties from transmission eigenvalues. *Inverse Problems*, **28** paper 055009 (2012)

[19] I. Harris, Non-destructive testing of anisotropic materials, Ph.D. Thesis, University of Delaware.

[20] A. Kirsch A and N. Grinberg, *The Factorization Method for Inverse Problems.* Oxford University Press, Oxford 2008.

[21] A. Kirsch and A. Lechleiter, The insideoutside duality for scattering problems by inhomogeneous media, *Inverse Problems* 29, paper 104011, (2013).

[22] L.Päivärinta and J. Sylvester. Transmission eigenvalues. *SIAM J. Math. Anal*, 40:738–753, (2008).

[23] J. Sun, Iterative methods for transmission eigenvalues, *SIAM J. Numer. Anal.* 49 no. 5, 18601874 (2011).

[24] J. Sun, Estimation of transmission eigenvalues and the index of refraction from Cauchy data, *Inverse Problems*, **27** (2011), 015009.

[25] J. Sun and L. Xu, Computation of Maxwell's transmission eigenvalues and its applications in inverse medium problems, 29, paper 104013 (2013).