Computation of Maxwell's transmission eigenvalues and its applications in inverse medium problems

Jiguang Sun[1] **and Liwei Xu**[2,3]

[1] Department of Mathematical Sciences, Michigan Technological University, U.S.A.

E-mail: `jiguangs@mtu.edu`

[2] College of Mathematics and Statistics, Chongqing University, Chongqing, 401331, P.R. China

[3] Institute of Scientific and Engineering Computing, Chongqing University, Chongqing, 400044, P.R. China

E-mail: `xul@cqu.edu.cn`

Abstract. We present an iterative method to compute the Maxwell's transmission eigenvalue problem which has importance in non-destructive testing of anisotropic materials. The transmission eigenvalue problem is first written as a quad-curl eigenvalue problem. Then we show that the real transmission eigenvalues are the roots of a non-linear function whose value is the generalized eigenvalue of a related self-adjoint quad-curl eigenvalue problem which is computed using a mixed finite element method. A secant method is used to compute the roots of the non-linear function. Numerical examples are presented to validate the method. Moreover, the method is employed to study the dependence of the transmission eigenvalue on the anisotropy and to reconstruct the index of refraction of an inhomogeneous medium.

1. Introduction

The transmission eigenvalue problem is a new research topic in the area of inverse scattering theory due to its importance in non-destructive testing of anisotropic materials. It is well-known that some non-iterative methods, such as the linear sampling method, attempt to avoid these eigenvalues for the reconstruction of inhomogeneous non-absorbing media [3, 5, 11]. Recent results show that transmission eigenvalues can be estimated from the scattering data [6, 4, 26, 28]. This interesting fact leads to new methods to reconstruct the shape and physical properties of the scattering objects. We refer the readers to [14, 19, 9] and the references therein for the existence theories, applications, and reconstruction of transmission eigenvalues.

The transmission eigenvalue problem is non-selfadjoint and non-elliptic. It is not covered by any standard theory of partial differential equations. Due to this fact, the study of transmission eigenvalues is an interesting, but challenging topic. Effective numerical methods are needed since they can be used in optimization methods to estimate the index of refraction using transmission eigenvalues estimated from the scattering data. Furthermore, theoretical results are partial and numerical evidence might lead theorists in the correct direction.

The first numerical treatment of the transmission eigenvalue problem appears in [13] where three finite element methods are proposed for the Helmholtz transmission eigenvalues. In [18], a mixed finite element method and its Matlab implementation are given. This approach is also used in [23] for the Maxwell's transmission eigenvalue problem. Recently, an efficient Fourier-spectral-element method is proposed to treat spherically stratified media [2]. Since the problem is non-selfadjoint, direct discretization by the finite element method leads to non-Hermitian generalized eigenvalue problems or quadratic eigenvalue problems, which are difficult in numerical linear algebra. We refer to the readers to [18, 23, 17] for some attempts to overcome this difficulty. Numerical methods for the corresponding interior transmission problem are also considered by some authors, see Hsiao et al. [16] and Wu and Chen [29].

Based on the existence theory for transmission eigenvalues [25, 8], two iterative methods are proposed in [27], which contains an error estimate. The transmission eigenvalue problem is first written as a fourth order problem. Then it is shown that the transmission eigenvalues are the roots of an algebraic equation. The value of this equation is the eigenvalue of a fourth order, self-adjoint, and positive definite problem which is solved by the Argyris element. Thus instead of a non-selfadjoint non-elliptic problem, we solve a series of self-adjoint positive definite problems. Consequently, the non-Hermitian matrix eigenvalue problem is avoided.

In this paper, we extend the method for Helmholtz transmission eigenvalues in [27] to the Maxwell's case. In particular, we are interested in the smallest transmission eigenvalue, which can be stably estimated from the scattering data and is useful in the reconstruction of the index of refraction. We first write the problem as a quad-curl problem and show that the transmission eigenvalues are the roots of an algebraic equation related to a series of self-adjoint positive definite quad-curl eigenvalue problem. Since it is difficult to build conforming finite elements for the quad-curl problem, we employ a mixed method for the quad-curl eigenvalue problem similar to [22]. Finally, we propose an iterative method to search the roots of the algebraic equation which are transmission eigenvalues and show some numerical examples. In addition, we give some numerical evidence on how the transmission eigenvalues reflect changes in the anisotropy

It is well-known that the smallest transmission eigenvalue can be stably determined from the scattering data (far field or near field) [6, 4, 26, 28, 15]. Thus they have been used to reconstruct the index of refraction. In [26], an optimization method is proposed to reconstruct the index of refraction using the estimated transmission eigenvalues from scattering data. An interesting approach which reconstructs the index of refraction as the smallest eigenvalue of a generalized eigenvalue problem was proposed in [15]. Spherically symmetric case using multiple transmission eigenvalues is considered in [1]. The transmission eigenvalues are also used to reconstruct anisotropies for the Maxwell's equations [6]. In this paper, we employ the proposed numerical method to estimate the index of refraction for the Maxwell's equations.

The rest of the paper is organized as follows. We first introduce the Maxwell's transmission eigenvalue problem in Section 2. It is then written as a quad-curl eigenvalue problem. Following [25, 8], we show that the transmission eigenvalues are in fact the roots of an algebraic equation related to a series of self-adjoint positive definite eigenvalue problems. In Section 3, we propose a mixed finite element method for the quad-curl eigenvalue problem and develop a secant method to compute the roots of the algebraic equations. Numerical examples are provided in Section 4. In Section 5, we study how the transmission eigenvalues reflect changes in the anisotropy numerically. Then we employ an optimization method to estimate the index of refraction from the smallest transmission eigenvalue. Finally, we make conclusions and discuss some future works in Section 6.

2. The Maxwell's transmission eigenvalue problem

Let $D \subset \mathbb{R}^3$ be a bounded simply connected domain with piece-wise smooth boundary ∂D. We denote by $(\cdot, \cdot)$ the $L^2(D)^3$ scalar product and define the following Hilbert spaces

$$
\begin{aligned}
&H(\mathrm{curl}, D) := \{\mathbf{u} \in L^2(D)^3 : \nabla \times \mathbf{u} \in L^2(D)^3\},\\
&H_0(\mathrm{curl}, D) := \{\mathbf{u} \in H(\mathrm{curl}, D) : \mathbf{u} \times \boldsymbol{\nu} = 0 \quad \text{on } \partial D\},
\end{aligned}
$$

where $\boldsymbol{\nu}$ is the unit outward normal vector to ∂D. The scalar product defined on $H(\mathrm{curl}, D)$ is given by

$$(\mathbf{u}, \mathbf{v})_{\mathrm{curl}} = (\mathbf{u}, \mathbf{v}) + (\nabla \times \mathbf{u}, \nabla \times \mathbf{v}).$$

Following [8], we also define the Hilbert spaces

$$
\begin{aligned}
&H(\mathrm{curl}^2, D) := \{\mathbf{u} \in H(\mathrm{curl}, D) : \mathrm{curl}\mathbf{u} \in H(\mathrm{curl}, D)\},\\
&H_0(\mathrm{curl}^2, D) := \{\mathbf{u} \in H_0(\mathrm{curl}, D) : \mathrm{curl}\mathbf{u} \in H_0(\mathrm{curl}, D)\}
\end{aligned}
$$

equipped with the scalar product

$$(\mathbf{u},\mathbf{v})_{H(\mathrm{curl}^2,D)} = (\mathbf{u},\mathbf{v}) + (\nabla\times\mathbf{u},\nabla\times\mathbf{v})_{\mathrm{curl}}$$

and the induced norm $\|\cdot\|_{H(\mathrm{curl}^2,D)}$.

Let $N(\mathbf{x})$ be the index of refraction given by a 3×3 matrix value function defined on D such that $N(\mathbf{x})\in L^\infty(D,\mathbb{R}^{3\times 3})$.

Definition 2.1. *A real matrix field N is said to be bounded positive definite on D if $N\in L^\infty(D,\mathbb{R}^{3\times 3})$ and if there exists a constant $\gamma>0$ such that*

$$\bar{\boldsymbol{\xi}}\cdot N\boldsymbol{\xi}\geq\gamma|\boldsymbol{\xi}|^2,\quad \forall\boldsymbol{\xi}\in\mathbb{C}^3\quad a.e.\quad in\quad D.$$

In the rest of the paper, we assume that N, N^{-1}, and $(N-I)^{-1}$ are bounded positive definite real matrix fields on D. The case for which $(I-N)^{-1}$ is bounded positive definite can be treated in the same way [8].

We first introduce the scattering problem for inhomogeneous media. Let the incident plane wave be given by

$$E^i(\mathbf{x},\mathbf{d},\mathbf{p}) = \frac{i}{k}\nabla\times\nabla\times\,\mathbf{p}\,e^{ik\mathbf{x}\cdot\mathbf{d}},\quad H^i(\mathbf{x},\mathbf{d},\mathbf{p}) = \nabla\times\,\mathbf{p}\,e^{ik\mathbf{x}\cdot\mathbf{d}}$$

where $\mathbf{d}\in\mathbb{R}^3$ is the direction of propagation of the wave, k is the wavenumber, and the vector $\mathbf{p}$ is the polarization. The scattering problem by the anisotropic inhomogeneous medium is to find the interior electric and magnetic fields E,H and the scattered electric and magnetic field E^s,H^s satisfying [11, 5]

$$\mathrm{curl}\,E^s - ikH^s = 0,\quad \text{in}\quad \mathbb{R}^3\setminus D, \tag{1}$$

$$\mathrm{curl}\,H^s + ikE^s = 0,\quad \text{in}\quad \mathbb{R}^3\setminus D, \tag{2}$$

$$\mathrm{curl}\,E - ikH = 0,\quad \text{in}\quad D, \tag{3}$$

$$\mathrm{curl}\,H + ikN(\mathbf{x})H = 0,\quad \text{in}\quad D, \tag{4}$$

$$\nu\times(E^s+E^i) - \nu\times E = 0,\quad \text{on}\quad \partial D, \tag{5}$$

$$\nu\times(H^s+H^i) - \nu\times H = 0,\quad \text{on}\quad \partial D, \tag{6}$$

together with the Silver-Müller radiation condition

$$\lim_{r\to\infty}(H^s\times\mathbf{x} - rE^s) = 0 \tag{7}$$

where $r=|\mathbf{x}|$. We refer the readers to [5] for the well-posedness of (1). The scattered fields have the following asymptotic behavior

$$E^s(\mathbf{x},\mathbf{d},\mathbf{p}) = \frac{e^{ikr}}{r}E_\infty(\hat{\mathbf{x}},\mathbf{d},\mathbf{p}) + O\left(\frac{1}{r^2}\right),\quad r\to\infty, \tag{8}$$

$$H^s(\mathbf{x},\mathbf{d},\mathbf{p}) = \frac{e^{ikr}}{r}\hat{\mathbf{x}}\times E_\infty(\hat{\mathbf{x}},\mathbf{d},\mathbf{p}) + O\left(\frac{1}{r^2}\right),\quad r\to\infty, \tag{9}$$

where $\hat{\mathbf{x}}=\mathbf{x}/r$ and E_∞ is called the electric far field pattern [11]. Given E_∞, the far field operator $F: L^2_t(\Omega)\to L^2_t(\Omega)$ is defined as

$$(F\mathbf{g})(\hat{\mathbf{x}}) := \int_\Omega E_\infty(\hat{\mathbf{x}},\mathbf{d},\mathbf{g}(\mathbf{d}))\,\mathrm{d}\mathbf{s} \tag{10}$$

where $\Omega = \{\hat{\mathbf{x}} \in \mathbb{R}^3; |\hat{\mathbf{x}}| = 1\}$ and $L^2_t(\Omega) := \{\mathbf{u} \in (L^2(\Omega))^3 : \boldsymbol{\nu} \cdot \mathbf{u} = 0 \text{ on } \Omega\}$. The far field operator F has fundamental importance in the non-iterative type of methods, for example, the linear sampling method (see Section 3.3 of [5]). It is known that F has a dense range provided that k does not belong to a special set of wavenumbers. This special set of wavenumbers are transmission eigenvalues which we define next.

The transmission eigenvalue problem for the anisotropic Maxwell's equations in terms of electric fields is to find $k^2 \in \mathbb{C}$, $\mathbf{E} \in L^2(D)^3$, $\mathbf{E}_0 \in L^2(D)^3$, and $\mathbf{E} - \mathbf{E}_0 \in H_0(\text{curl}^2, D)$ such that

$$
\begin{aligned}
&\nabla \times \nabla \times \mathbf{E} - k^2 N \mathbf{E} = 0 && \text{in} \quad D, && (11)\\
&\nabla \times \nabla \times \mathbf{E}_0 - k^2 \mathbf{E}_0 = 0 && \text{in} \quad D, && (12)\\
&\mathbf{E} \times \boldsymbol{\nu} = \mathbf{E}_0 \times \boldsymbol{\nu} && \text{on} \quad \partial D, && (13)\\
&(\nabla \times \mathbf{E}) \times \boldsymbol{\nu} = (\nabla \times \mathbf{E}_0) \times \boldsymbol{\nu} && \text{on} \quad \partial D. && (14)
\end{aligned}
$$

The values $k^2 \in \mathbb{C}$ such that the above equation has non-trivial solutions $\mathbf{E}$ and $\mathbf{E}_0$ are called Maxwell's transmission eigenvalues. Note that there also exist complex transmission eigenvalues [20]. In this paper, we only consider the computation of the smallest real transmission eigenvalue.

We first rewrite the transmission eigenvalue problem as a quad-curl problem. Subtracting (12) from (11), we obtain that

$$
\nabla \times \nabla \times (\mathbf{E} - \mathbf{E}_0) - k^2(\mathbf{E} - \mathbf{E}_0) = k^2(N - I)\mathbf{E}.
$$

Solving for $\mathbf{E}$, we have

$$
\mathbf{E} = \frac{1}{k^2}(N - I)^{-1}(\nabla \times \nabla - k^2)(\mathbf{E} - \mathbf{E}_0).
$$

Applying $\nabla \times \nabla \times -k^2 N$ to the above equation and using (11), we obtain

$$
(\nabla \times \nabla \times -k^2 N)(N - I)^{-1}(\nabla \times \nabla \times -k^2)(\mathbf{E} - \mathbf{E}_0) = 0.
$$

Setting $\tau := k^2$ and $\mathbf{u} = \mathbf{E} - \mathbf{E}_0$, we obtain a variation formulation for the transmission eigenvalue problem: find $\tau \in \mathbb{C}$ and $\mathbf{u} \in H_0(\text{curl}^2, D)$ such that

$$
\mathcal{A}_\tau(\mathbf{u}, \mathbf{v}) - \tau \mathcal{B}(\mathbf{u}, \mathbf{v}) = 0 \quad \forall \, \mathbf{v} \in H_0(\text{curl}^2, D) \tag{15}
$$

where

$$
\mathcal{A}_\tau(\mathbf{u}, \mathbf{v}) = \big((N - I)^{-1}(\nabla \times \nabla \times \mathbf{u} - \tau \mathbf{u}), (\nabla \times \nabla \times \mathbf{v} - \tau \mathbf{v})\big) + \tau^2(\mathbf{u}, \mathbf{v}) \tag{16}
$$

and

$$
\mathcal{B}(\mathbf{u}, \mathbf{v}) = (\nabla \times \mathbf{u}, \nabla \times \mathbf{v}). \tag{17}
$$

The eigenvalue problem (15) is a non-selfadjoint quad-curl problem. It is shown in [25, 8] that, if $(N - I)^{-1}$ is a bounded positive definite matrix field on D, $\mathcal{A}_\tau$ is a coercive Hermitian sesquilinear form on $H_0(\text{curl}^2, D) \times H_0(\text{curl}^2, D)$. Furthermore, the sesquilinear form $\mathcal{B}$ is Hermitian and non-negative. This leads us to consider the auxiliary eigenvalue problem for fixed τ

$$
\mathcal{A}_\tau(\mathbf{u}, \mathbf{v}) - \lambda(\tau) \mathcal{B}(\mathbf{u}, \mathbf{v}) = 0 \quad \forall \, \mathbf{v} \in H_0(\text{curl}^2, D). \tag{18}
$$

Note that the generalized eigenvalue $\lambda(\tau)$ depends on τ since $\mathcal{A}_\tau$ depends on τ. Then the smallest transmission eigenvalue is the first positive root of the function

$$f(\tau) := \lambda(\tau) - \tau \tag{19}$$

where $\lambda(\tau)$ is the smallest generalized eigenvalue of (18).

In the following, we present some analysis of the above generalized eigenvalue problem which motivates the iterative method we shall describe later. We need the space of functions with square-integrable divergence $H(\mathrm{div}, D)$ defined by

$$H(\mathrm{div}, D) = \{\mathbf{u} \in L^2(D)^3 \mid \mathrm{div}\,\mathbf{u} \in L^2(D)^3\}$$

equipped with the scalar product

$$(\mathbf{u}, \mathbf{v})_{H(\mathrm{div},D)} = (\mathbf{u}, \mathbf{v}) + (\mathrm{div}\,\mathbf{u}, \mathrm{div}\,\mathbf{v})$$

and the corresponding norm $\|\cdot\|_{H(\mathrm{div},D)}$. We also define

$$V := \left\{\mathbf{u} \in H_0(\mathrm{curl}^2; D) \cap H(\mathrm{div}; D) \mid \mathrm{div}\,\mathbf{u} = 0\right\}. \tag{20}$$

We first consider the quad-curl eigenvalue problem of finding $\kappa \in \mathbb{R}$ and $\mathbf{u} \in V$ such that

$$(\nabla \times \nabla \times \mathbf{u}, \nabla \times \nabla \times \mathbf{u}) = \kappa(\mathbf{u}, \mathbf{v}), \quad \forall \mathbf{v} \in V. \tag{21}$$

There exits an infinite discrete set of quad-curl eigenvalues $\kappa_j > 0, j = 1, 2, \ldots$ and corresponding eigenfunctions $\mathbf{u}_j \in V$, $\mathbf{u}_j \neq \mathbf{0}$ such that (21) is satisfied and $0 < \kappa_1 \leq \kappa_2 \leq \ldots$. Furthermore

$$\lim_{j\to\infty} \kappa_j = \infty.$$

The eigenfunctions $(\mathbf{u}_j, \mathbf{u}_l)_{L^2(D)^3} = 0$ if $j \neq l$ (see [22]).

We denote by $\kappa_1 > 0$ the smallest quad-curl eigenvalue and λ_0 the smallest Laplacian eigenvalue for D [21]. We define

$$\Theta := 4\left(\frac{\kappa_1^{1/2}}{\lambda_0} + \frac{\kappa_1}{\lambda_0^2}\right)$$

and let $0 \leq \eta_1(\mathbf{x}) \leq \eta_2(\mathbf{x}) \leq \eta_3(\mathbf{x})$ be the eigenvalues of the index of refraction $N(\mathbf{x})$. Note that $\eta_3(\mathbf{x})$ coincides with $\|N(\mathbf{x})\|_2$ and is given by

$$\eta_3(\mathbf{x}) = \sup_{\|\boldsymbol{\xi}\|=1} \bar{\boldsymbol{\xi}} \cdot N(\mathbf{x})\boldsymbol{\xi}.$$

The smallest eigenvalue $\eta_1(\mathbf{x})$ is given by

$$\eta_1(\mathbf{x}) = \inf_{\|\boldsymbol{\xi}\|=1} \bar{\boldsymbol{\xi}} \cdot N(\mathbf{x})\boldsymbol{\xi}.$$

We denote

$$N^* = \sup_D \eta_3(\mathbf{x}), \qquad N_* = \inf_D \eta_1(\mathbf{x}).$$

In the following, we show that, under certain condition of $N(\mathbf{x})$, transmission eigenvalues exist as the roots of $f(\tau) := \lambda(\tau) - \tau$ where $\lambda(\tau)$ is the smallest generalized eigenvalue of (18). We assume that

$$1 + \Theta \leq N_* \leq \bar{\boldsymbol{\xi}} \cdot N(\mathbf{x})\boldsymbol{\xi} \leq N^* < \infty, \quad \|\boldsymbol{\xi}\| = 1, \tag{22}$$

which also implies

$$0 < \frac{1}{N^* - 1} \leq \bar{\boldsymbol{\xi}} \cdot (N - I)^{-1}\boldsymbol{\xi} \leq \frac{1}{N_* - 1} < \infty.$$

Under the above assumption, it is shown in [8] that if

$$0 < \tau_0 < \frac{\lambda_0}{\sup_D \|N\|_2}, \tag{23}$$

then $f(\tau_0) = \lambda(\tau_0) - \tau_0 > 0$. If

$$\tau_1 = \frac{\lambda_0 - 2M\kappa_1^{1/2}}{2 + M}, \quad M = \frac{1}{N_* - 1} \tag{24}$$

then $f(\tau_1) = \lambda(\tau_1) - \tau_1 < 0$. Since $f(\tau)$ is a continuous function of τ, there exist a τ^* such that $f(\tau^*) = 0$, which implies that τ is the smallest transmission eigenvalue.

In summary, to compute the transmission eigenvalue, we first compute τ_0 and τ_1 defined in (23) and (24), respectively. Then we can employ an iterative method, such as the bisection method, to search the root of $f(\tau)$. Note that for the bisection method, at each step, we need to compute a generalized quad-curl problem which is not efficient for 3D problems. Thus, in the next section, we will propose a secant method which converges much faster.

The condition (22) on $N(\mathbf{x})$ is rather strict. We use it for simplicity of presentation. It can be relaxed significantly. The existence of transmission eigenvalues only requires $\|N(\mathbf{x})\|_2 \geq \alpha > 1$ for some positive α, for example, see Theorem 3.3 of [6].

In the inverse scattering theory, one often cares about the smallest transmission eigenvalue. When $N = N_0 I$ for some $N_0 > 1$, much more can be said about $f(\tau)$. The following lemma is proved in [6].

Lemma 2.2. *Let μ_1 be a continuous function which maps the index of refraction N_0 to the smallest transmission eigenvalue. Moreover, denoting $\tau := k^2$, if $f(\tau, N_0) := \mu_1(N_0\tau^2) - (N_0 + 1)\tau$, then $\frac{\partial f}{\partial \tau} < 0$ when $\tau < \frac{N_0+1}{2N_0}\lambda_0(D)$ where $\lambda_0(D)$ is the first Dirichlet eigenvalue of the negative Laplacian in D.*

3. Computation of transmission eigenvalues

We have shown that the transmission eigenvalues are the roots of (19). Since the value of $f(\tau)$ depends on the smallest generalized eigenvalue of (18), we first construct a mixed finite element method for the auxiliary eigenvalue problem (18).

It is easy to see that the variational formulation (18) corresponds to the following partial differential equation

$$(\nabla \times \nabla \times -\tau)(N - I)^{-1}(\nabla \times \nabla \times -\tau)\mathbf{w} + \tau^2\mathbf{w} = \lambda\nabla \times \nabla \times \mathbf{w}. \tag{25}$$

This is a quad-curl eigenvalue problem. Similarly to the approach in [22], we first introduce a mixed formulation for (25). Let

$$\begin{aligned}
&\mathbf{u} = \mathbf{w}, \\
&\mathbf{v} = (N - I)^{-1}(\nabla \times \nabla \times -\tau)\mathbf{u}.
\end{aligned}$$

Thus we obtain the following mixed form for (25):

$$(\nabla \times \nabla \times -\tau)\mathbf{v} + \tau^2\mathbf{u} = \lambda\nabla \times \nabla \times \mathbf{u}, \tag{26}$$

$$(\nabla \times \nabla \times -\tau)\mathbf{u} = (N - I)\mathbf{v}. \tag{27}$$

The weak formulation of the above problem can be stated as: find

$$(\lambda, \mathbf{u}, \mathbf{v}) \in (\mathbb{R}, H_0(\mathrm{curl}, D), H(\mathrm{curl}, D))$$

such that

$$(\nabla \times \mathbf{v}, \nabla \times \boldsymbol{\xi}) - \tau(\mathbf{v}, \boldsymbol{\xi}) + \tau^2(\mathbf{u}, \boldsymbol{\xi}) = \lambda(\nabla \times \mathbf{u}, \nabla \times \boldsymbol{\xi}), \tag{28}$$

$$(\nabla \times \mathbf{u}, \nabla \times \boldsymbol{\phi}) - \tau(\mathbf{u}, \boldsymbol{\phi}) = ((N - I)\mathbf{v}, \boldsymbol{\phi}), \tag{29}$$

for all $\boldsymbol{\xi} \in H_0(\mathrm{curl}, D)$ and $\boldsymbol{\phi} \in H(\mathrm{curl}, D)$.

Now we use the curl conforming edge elements of Nédélec [24, 21] for discretization. Following [21], let P_n be the space of polynomials of maximum total degree n and $\tilde{P}_n$ the space of homogeneous polynomials of degree n. We define

$$\begin{aligned}
&S_n = \{\mathbf{p} \in (\tilde{P}_n)^3 \mid \mathbf{x} \cdot \mathbf{p}(\mathbf{x}) = 0 \;\forall \mathbf{x} \in \mathbb{R}^3\}, \\
&R_n = (P_{n-1})^3 \oplus S_n.
\end{aligned}$$

The curl-conforming edge element space [21] is defined as

$$X_h^+ = \{\mathbf{v} \in H(\mathrm{curl}; D) \mid \mathbf{v}|_K \in R_n, \quad \forall K \in \mathcal{T}_h\}$$

where $\mathcal{T}_h$ is a tetrahedral mesh for D. The degrees of freedom are associated with the edge e, faces f and the volume of an element $K \in \mathcal{T}_h$. Letting $\boldsymbol{\tau}$ denote a unit vector parallel to e and $\boldsymbol{\nu}$ denote the unit outward normal to f, the degrees of freedom of edge element are given by

$$\begin{aligned}
M_e(\mathbf{u}) &= \left\{ \int_e \mathbf{u} \cdot \boldsymbol{\tau} q \,\mathrm{d}s \quad \text{for all } q \in P_{n-1}(e) \text{ for each edge } e \text{ of } K \right\}, \\
M_f(\mathbf{u}) &= \left\{ \int_f \mathbf{u} \times \boldsymbol{\nu} \cdot \mathbf{g} \,\mathrm{d}A \quad \text{for all } \mathbf{g} \in (P_{n-2}(f))^2 \text{ for each face } f \text{ of } K \right\}, \\
M_K(\mathbf{u}) &= \left\{ \int_K \mathbf{u} \cdot \mathbf{g} \,\mathrm{d}\mathbf{x} \quad \text{for all } \mathbf{g} \in (P_{n-3}(K))^3 \right\}.
\end{aligned}$$

The linear edge element when $n = 1$ is given by

$$R_1 = \left\{ \mathbf{u}(\mathbf{x}) = \mathbf{a} + \mathbf{b} \times \mathbf{x}, \quad \text{where } \mathbf{a}, \mathbf{b} \in \mathbb{C}^3 \right\}.$$

The six degrees of freedom are determined from the moments $\int_e \mathbf{u} \cdot \boldsymbol{\tau} \mathrm{d}s$ on the six edges of K. The $H_0(\mathrm{curl}; D)$ conforming edge element space is simply given by

$$X_h = \{\mathbf{u}_h \in X_h^+ \mid \boldsymbol{\nu} \times \mathbf{u}_h = 0 \quad \text{on } \partial D\} \tag{30}$$

which can be easily obtained by taking the degrees of freedom associated with edges or faces on ∂D to vanish [21].

In the following implementation, for simplicity, we employ the linear edge element. Let

$$\begin{aligned} S_h &= \text{ the space of lowest order edge element on } D, \\ S_h^0 &= S_h \cap H_0(\text{curl}, D) \\ &= \text{ the subspace of functions in } S_h \text{ that have vanishing DoF on } \partial D, \end{aligned}$$

where DoF stands for degree of freedom. Let $\psi_1, \dots, \psi_K$ be a basis for S_h^0 and $\psi_1, \dots, \psi_K, \psi_{K+1}, \dots, \psi_T$ be a basis for S_h. Let $\mathbf{u}_h = \sum_{i=1}^K u_i \psi_i$ and $\mathbf{v}_h = \sum_{i=1}^T u_i \psi_i$. Furthermore, let $\vec{\mathbf{u}} = (u_1, \dots, u_K)^T$ and $\vec{\mathbf{v}} = (v_1, \dots, v_T)^T$. Then the matrix form corresponding to the above problem is

$$S_{K\times T}\vec{\mathbf{v}} - \tau M_{K\times T}\vec{\mathbf{v}} + \tau^2 M_{K\times K}\vec{\mathbf{u}} = \lambda_h S_{K\times K}\vec{\mathbf{u}}, \tag{31}$$

$$S_{T\times K}\vec{\mathbf{u}} - \tau M_{T\times K}\vec{\mathbf{u}} = M_{T\times T}^{N-I}\vec{\mathbf{v}}, \tag{32}$$

where

Matrix	Dimension	Definition
$S_{K\times K}$	$K \times K$	$S_{K\times K}^{i,j} = (\nabla \times \psi_i, \nabla \times \psi_j), 1 \le i \le K, 1 \le j \le K$
$S_{K\times T}$	$K \times T$	$S_{K\times T}^{i,j} = (\nabla \times \psi_i, \nabla \times \psi_j), 1 \le i \le K, 1 \le j \le T$
$S_{T\times K}$	$T \times K$	$S_{T\times T}^{i,j} = (\nabla \times \psi_i, \nabla \times \psi_j), 1 \le i \le T, 1 \le j \le K$
$M_{K\times K}$	$K \times T$	$M_{K\times T}^{i,j} = (\psi_i, \psi_j), 1 \le i \le K, 1 \le j \le K$
$M_{K\times T}$	$K \times T$	$M_{K\times T}^{i,j} = (\psi_i, \psi_j), 1 \le i \le K, 1 \le j \le T$
$M_{T\times K}$	$T \times K$	$M_{T\times K}^{i,j} = (\psi_i, \psi_j), 1 \le i \le T, 1 \le j \le K$
$M_{T\times T}^{N-I}$	$T \times T$	$(M_{T\times T}^{N-I})^{i,j} = ((N-I)\psi_i, \psi_j), 1 \le i \le T, 1 \le j \le T$

From (32) we obtain

$$\vec{\mathbf{v}} = \left(M_{T\times T}^{N-I}\right)^{-1} (S_{T\times K} - \tau M_{T\times K})\, \vec{\mathbf{u}}.$$

Substituting $\vec{\mathbf{v}}$ in (31), we obtain a generalized matrix eigenvalue problem

$$A\vec{\mathbf{u}} = \lambda S_{K\times K}\vec{\mathbf{u}} \tag{33}$$

where

$$A = \Big((S_{K\times T} - \tau M_{K\times T}) \left(M_{T\times T}^{N-I}\right)^{-1} (S_{T\times K} - \tau M_{T\times K}) + \tau^2 M_{K\times K} \Big).$$

Now we are in the position to present a secant method to compute the smallest transmission eigenvalue k_1.

AlgorithmS: (Secant Method): $\tau^* = \mathit{secantTE}(x_0, x_1, N(\mathbf{x}), \mathit{tol}, \mathit{maxit})$

generate a regular tetrahedra mesh for D

set $it = 1$ and $\delta = \text{abs}(x_1 - x_0)$

compute the smallest generalized eigenvalue λ_A of (18) for $\tau = x_0$

compute the smallest generalized eigenvalue λ_B of (18) for $\tau = x_1$

while $\delta > \mathit{tol}$ and $it < \mathit{maxit}$

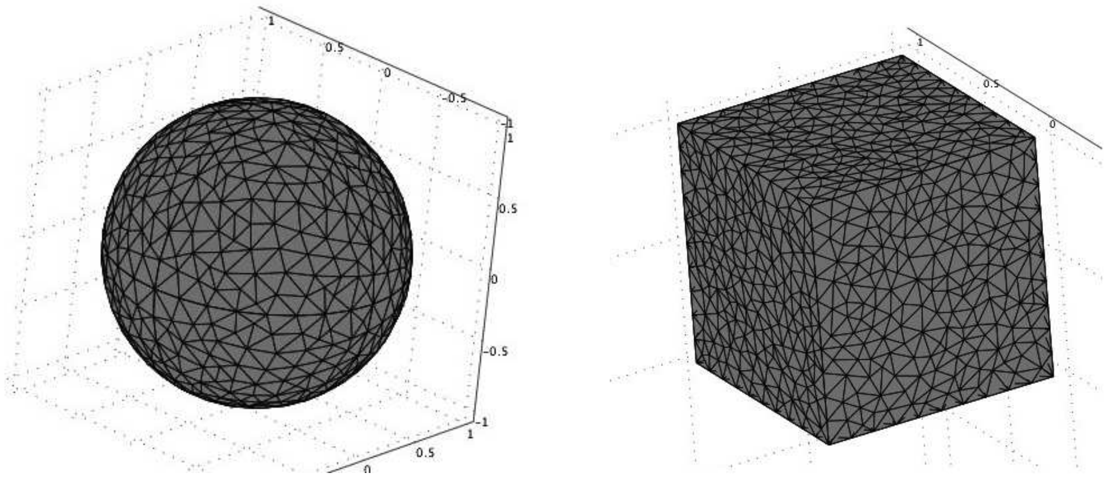

Figure 1. The unit ball and the unit cube with sample meshes.

$\tau = x_1 - \lambda_B \frac{x_1 - x_0}{\lambda_B - \lambda_A}$

compute the smallest eigenvalue λ_τ of $A\mathbf{x} = \lambda B\mathbf{x}$

$\delta = \text{abs}(\lambda_\tau - \tau)$

$x_0 = x_1, x_1 = \tau, \lambda_A = \lambda_B, \lambda_B = \lambda_\tau, it = it + 1.$

end

Here x_0 and x_1 are initial values which are chosen close to zero and $x_0 < x_1 < \frac{\lambda_0}{\sup_D \|N\|_2}$. This is due to the fact that $f(\tau)$ is positive in an interval I right to zero. The parameters *maxit* and *tol* are the maximum number of iterations and precision, respectively.

4. Numerical examples

In this section, we show some examples to compute the smallest transmission eigenvalue. We choose two domains: D_1 the unit ball centered at the origin, and D_2 the unit cube given by $[0,1] \times [0,1] \times [0,1]$ (see Fig. 1).

We only consider $\|N(\mathbf{x})\|_2 \geq \alpha > 1, \alpha > 0$ since the case of $0 < \|N(\mathbf{x})\|_2 \leq 1 - \beta, \beta > 0$ can be treated similarly. We test three different cases for the index of refraction $N(\mathbf{x})$ corresponding to isotropic medium with constant index of refraction, anisotropic medium with constant index of refraction, and anisotropic medium with non-constant index of refraction given by

$$N^1 := \begin{pmatrix} 16 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 16 \end{pmatrix}, N^2 := \begin{pmatrix} 16 & 1 & 0 \\ 1 & 16 & 0 \\ 0 & 0 & 14 \end{pmatrix}, N^3 := \begin{pmatrix} 16 & x & y \\ x & 16 & z \\ y & z & 14 \end{pmatrix}.$$

The eigenvalues of N^1 are 16 with multiplicity 3. The eigenvalues of N^2 are $14, 15, 17$. For the case of N^3, simple calculation shows that $N_* = 13.5698$ and $N^* = 17$ for the unit ball and $N_* = 13.2679$ and $N^* = 17.5616$ for the unit cube.

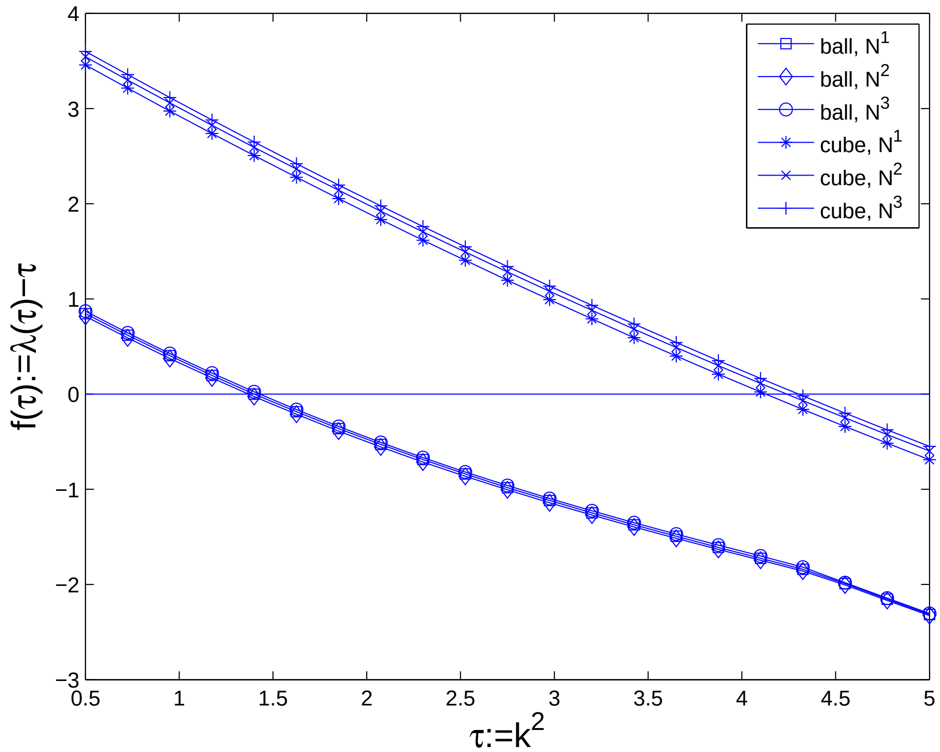

Figure 2. The plot of $f(\tau) = \lambda(\tau) - \tau$ v.s. τ for two domains D_1 and D_2 with N^1, N^2, and N^3.

We first generate tetrahedra meshes for D_1 and D_2, respectively. Due to the restriction of the computation power available, the meshes size is roughly $h \approx 0.2$ which is rather coarse. For each τ, we compute the smallest eigenvalue of the generalized eigenvalue problem (18) using the mixed finite element. In Fig. 2, we plot $f(\tau) = \lambda(\tau) - \tau$ v.s. τ. For two domains D_1 and D_2 with N^1, N^2, and N^3, we see that the function $f(\tau)$ is a monotonically decreasing function which is consistent with Lemma 2.2.

We choose x_0 and x_1 such that

$$x_0 < x_1 < \frac{\lambda_0}{\sup_D \|N\|_2}.$$

In fact, one can choose small values for x_0 and x_1 to avoid computing λ_0 and $\sup_D \|N\|_2$. In the following computation, we simply set $x_0 = 0.1$ and $x_1 = 0.2$. We show the result in Table. 1. The transmission eigenvalues are consistent with those in [23]. It can be seen that the algorithm is very efficient since the computation needs only a few iterations.

For the unit ball with the index of refraction $N = N_0 I$, the transmission eigenvalues k's are known exactly which are given by the roots of

$$\left| \begin{array}{cc} j_n(k\rho) & j_n(k\sqrt{N_0}\rho) \\ \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho j_n(k\rho)\right) & \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho j_n(k\sqrt{N_0}\rho)\right) \end{array} \right|_{\rho=1} = 0, \quad n \geq 1 \tag{34}$$

Table 1. The computed smallest Maxwell's transmission eigenvalues of the unit ball and the unit cube together with the number of iterations used in the secant method.

domain	k_1	number of iterations
unit ball N^1	1.1837	4
unit ball N^2	1.1702	4
unit ball N^3	1.1952	4
unit cube N^1	2.0595	4
unit cube N^2	2.0411	4
unit cube N^3	2.0527	4

and

$$\left| \begin{array}{cc} \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho j_n(k\rho)\right) & \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho j_n(k\sqrt{N_0}\rho)\right) \\ k^2 j_n(k\rho) & k^2 N_0 j_n(k\sqrt{N_0}\rho) \end{array} \right|_{\rho=1} = 0, \quad n \geq 1 \tag{35}$$

where j_n's are the spherical Bessel functions. For $N_0 = 16$, the exact smallest transmission eigenvalue is 1.1654 [23]. To check the convergence, we use the secant method to compute the smallest transmission eigenvalues on a series of meshes with degrees of freedom 273, 497, 1043, 2846 and 9711. Then we calculate the error for each mesh. In Fig. 3, we show the convergence rate of the computed smallest transmission eigenvalue. Note that in 3D, the mesh size h is proportional to the inverse of the cubic root of the degrees of freedom. Although we carry out the computation on a desktop using Matlab, 9711 is still a rather small matrix. In future, we plan to seek for a better solver for the quad-curl eigenvalue problem such that larger 3D problems can be treated.

5. Applications in inverse scattering

5.1. Transmission eigenvalues v.s. anisotropy

Using the above method, we give some numerical evidence to show how the transmission eigenvalues reflect changes in the anisotropy. The unit ball is used as the model domain.

For the first example, we set the index of refraction

$$N = \mathrm{diag}(16, 15, x) \tag{36}$$

with x changing from 12 to 14. In Fig. 4, we show the first transmission eigenvalues $\tau_1 := k_1^2$ v.s. $x := N(3,3)$. The result shows that the first transmission eigenvalue decreases as the smallest eigenvalue of N increases when the other eigenvalues keep the same.

Next we set

$$N = \mathrm{diag}(16, 15, x) \tag{37}$$

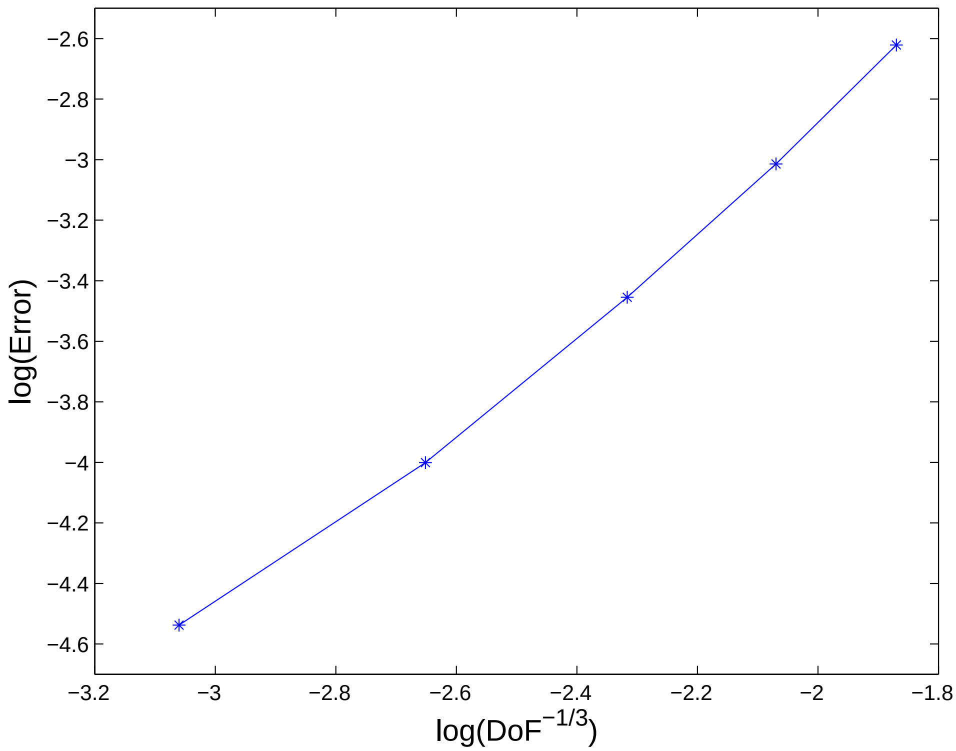

Figure 3. The plot of the error for the the computed smallest transmission eigenvalue for the unit ball for $N = 16I$.

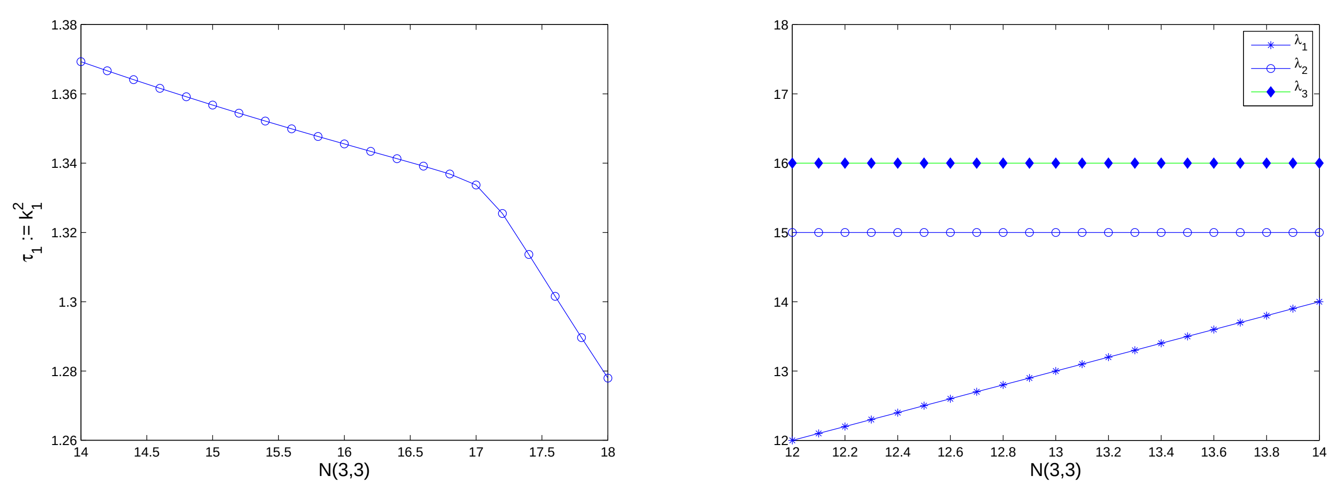

Figure 4. Left: The smallest transmission eigenvalue v.s. $x = N(3,3)$. Right: The eigenvalues of N defined in (36) for different x.

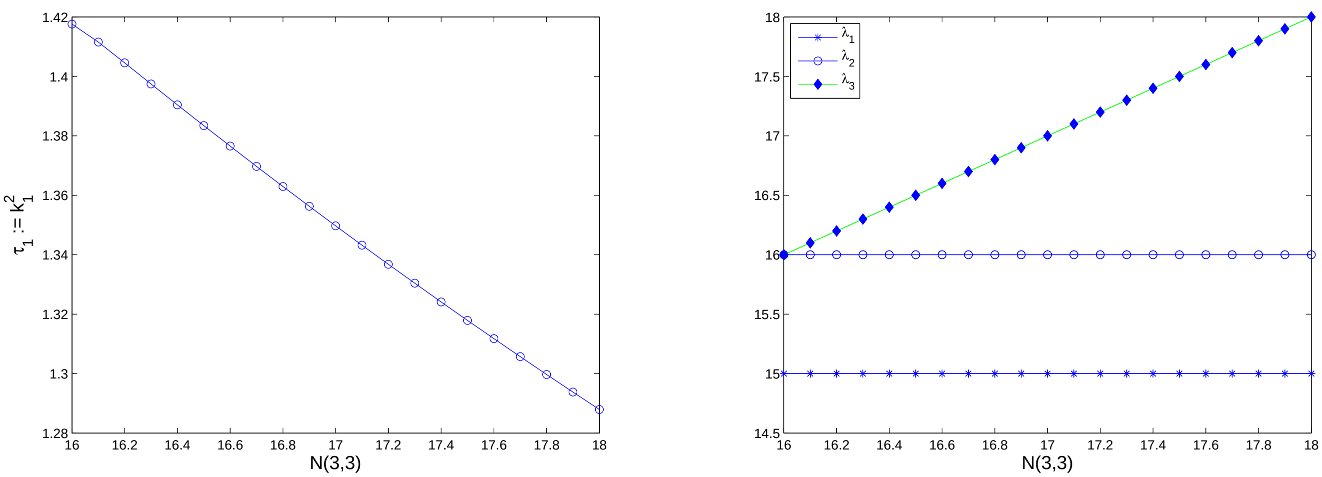

Figure 5. Left: The smallest transmission eigenvalue v.s. $x = N(3,3)$. Right: The eigenvalues of N defined in (37) for different x.

with x changing from 16 to 18. Fig. 7 shows the smallest transmission eigenvalues $\tau_1 := k_1^2$ v.s. $x := N(3,3)$. The result is similar to Fig. 4, i.e., the smallest transmission eigenvalue decreases as the largest eigenvalue of N increases when the other eigenvalues of N keep the same.

For the third example, we set

$$N = \operatorname{diag}(16, 15, x) \tag{38}$$

with x changing from 15 to 16. Again, in Fig. 7, we see that smallest transmission eigenvalue decreases as the second eigenvalue of N increases when the other eigenvalues of N keep the same.

The above numerical examples indicate that, in general, the smallest transmission eigenvalue gets smaller as the eigenvalues of the index of refraction gets larger. Numerical study also shows that the smallest transmission eigenvalue gets larger as the eigenvalues of the index of refraction gets smaller. Since the results are similar, we do not show them for simplicity.

In the following, we consider some more complicated examples. We first choose the unit ball and set

$$N := \begin{pmatrix} 16 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{pmatrix}, \tag{39}$$

with $x := N(2,2)$ changing from 15 to 10 and $y := N(3,3)$ changing from 17 to 22. The smallest transmission eigenvalues and the eigenvalues of the index of refraction N is shown in Fig. 7.

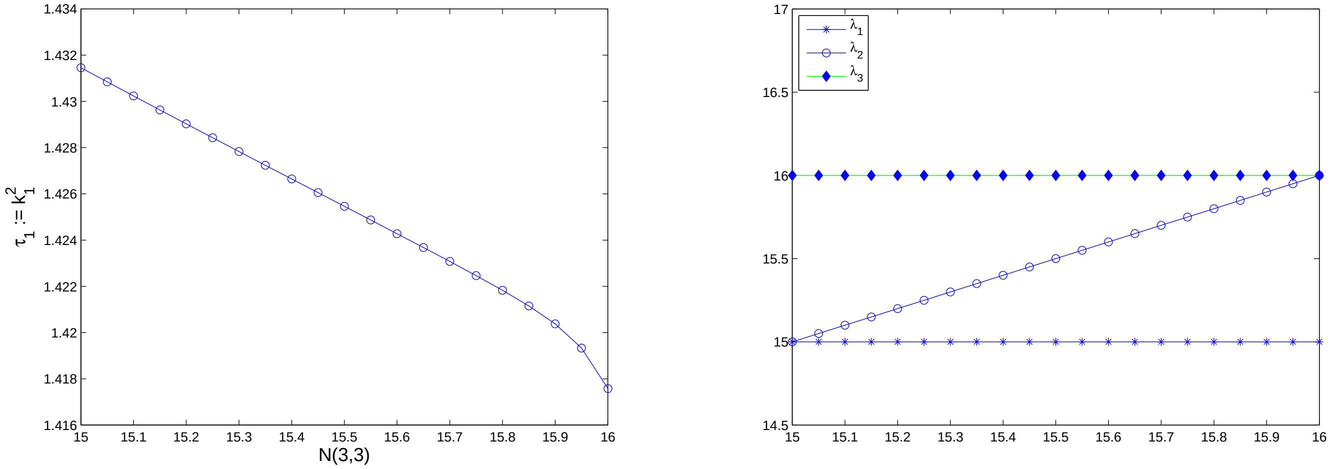

Figure 6. Left: The first transmission eigenvalue v.s. $x = N(3,3)$. Right: The eigenvalues of N defined in (38) for different x.

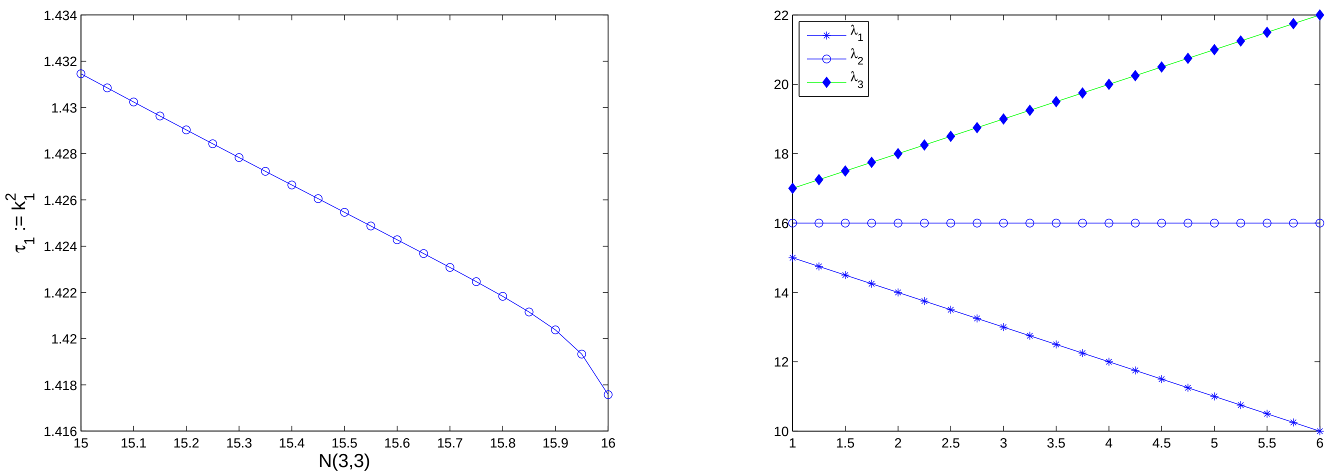

Figure 7. Left: The smallest transmission eigenvalue. Right: The eigenvalues of $N(\mathbf{x})$ defined in (39) for different x.

For the second example, we set

$$N := \begin{pmatrix} 16 & x & z \\ x & 16 & y \\ z & y & 16 \end{pmatrix} \tag{40}$$

with x changing from 1 to 6, y changing from 6 to 1, and z changing from 2 to 4. The first transmission eigenvalue and the eigenvalues of $N(\mathbf{x})$ defined in (40) are shown in Fig. 8.

For above two examples, the behavior of the smallest transmission eigenvalue is difficult to predict when the index of refraction is some what complicate. No obvious

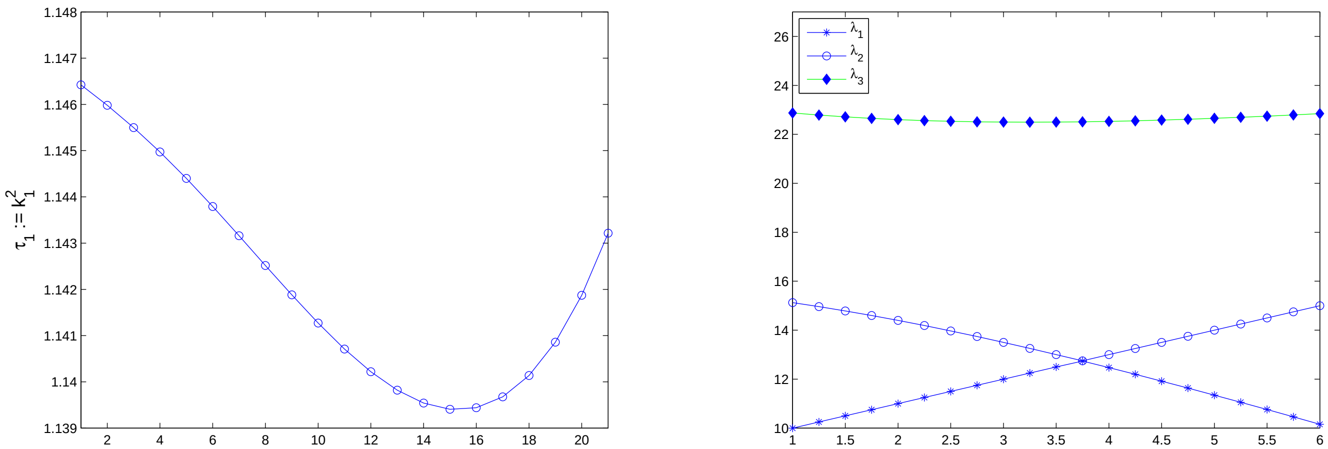

Figure 8. Left: The smallest transmission eigenvalue. Right: The eigenvalues of N defined in (40) for varying x, y, z.

conclusion can be made about how the smallest transmission eigenvalue reflects changes in the anisotropy.

Finally, we present an example with rough data without any comment. We set

$$N := \begin{pmatrix} 16 & x & z \\ x & 16 & y \\ z & y & 16 \end{pmatrix} \tag{41}$$

with x, y, z being randomly generated numbers given by

$$x, y, z = (\text{rand}(1, 21) - 0.5) \times 2.$$

The smallest transmission eigenvalue and the eigenvalues of of $N(\mathbf{x})$ defined in (41) are shown in Fig. 9.

5.2. Estimation of the index of refraction

We consider the inverse scattering problem to reconstruct the index of refraction from the transmission eigenvalues. Suppose that the smallest transmission eigenvalue and the shape of the scatterer D is available, we would like to obtain information of N using an optimization method based on the above numerical method. In particular, given the estimated smallest transmission eigenvalue k_1^δ, we would like to find N_0 such that the smallest transmission eigenvalue of D coincides with k_1^δ. Here δ is the noise level.

Let $\mu_D : \mathbb{R} \to \mathbb{R}$ which maps a given index of refraction N_0 ($N = N_0 I$) to the smallest transmission eigenvalue of D, i.e.,

$$\mu_D(N_0) = k_1(D). \tag{42}$$

We seek a constant N_e minimizing the difference between $\mu_D(N_0)$ and k_1^δ, i.e.,

$$N_e = \text{argmin}_{N_0} |\mu_D(N_0) - k_1^\delta|. \tag{43}$$

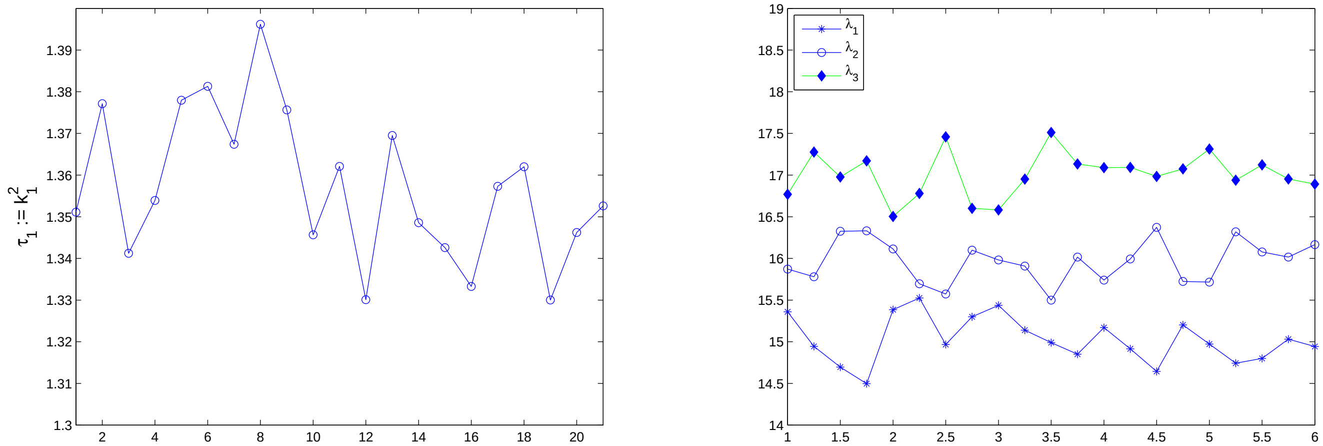

Figure 9. Left: The smallest transmission eigenvalue. Right: The eigenvalues of N defined in (41).

To this end, we quote the following theorem from [6] to justify the optimization method we will give shortly.

Theorem 5.1. *(Lemma 1 of [6]) Let $\mu_D(N_0)$ be defined in (42) which maps the index of refraction $N = N_0 I, N_0 > 1$ to the smallest transmission eigenvalue k_1. Then μ_D is monotonically decreasing and continuous on $(0, \frac{N_0+1}{2N_0}\lambda_0(D))$. Furthermore,*

$$\mu_D(N_0) \to \lambda_0(D), \quad \text{as } N_0 \to \infty, \quad \text{and} \quad \mu_D(N_0) \to \infty, \quad \text{as } N_0 \to 1$$

where $\lambda_0(D)$ is the first Dirichlet eigenvalue of the negative Laplacian in D.

In Fig. 10, we plot the smallest transmission eigenvalue k_1 v.s. the index of refraction N_0. It is clear that k_1 is a monotonically decreasing function of N_0.

Since μ_D is a continuous function of N_0, we can look for N_0 such that the computed smallest transmission eigenvalue coincides with k_1^δ using the following **AlgorithmN**. At each step, the smallest transmission eigenvalue is computed using the proposed iterative method in Section 3. Now we present the optimization algorithm to estimate the index of refraction.

AlgorithmN $N_e = algorithmN(D, k_1^\delta, tol)$

generate a regular tetrahedra mesh for D

estimate a suitable interval a and b

compute k_1^a and k_1^b using the secant method

while abs$(a-b) > tol$

    $c = (a+b)/2$ and compute k_1^c using the secant method

    if $|k_1^c - k_1^\delta| < |k_1^a - k_1^\delta|$

        $a = c$

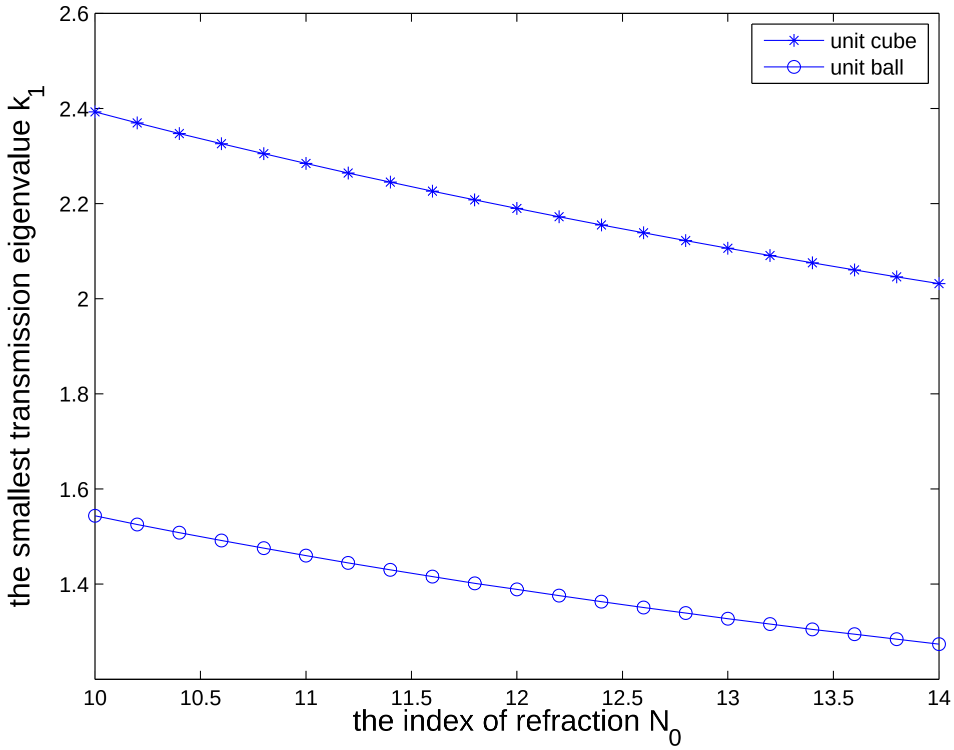

Figure 10. The plot of the smallest transmission eigenvalue k_1 v.s. the index of refraction N_0 for the unit ball and the unit cube.

```
        else
                b=c
        end
    end
```

$N_e = c$

In the following, we assume that the smallest transmission eigenvalue k_1^δ is obtained from the scattering data and seek an isotropic $N_e I$ with k_1^δ being the smallest transmission eigenvalue.

We first consider the unit ball. Let $k_1^\delta = 1.17$ corresponding to the case of N^2. We use **AlgorithmN** and find that $N_e = 14.66$. Then we let $k_1^\delta = 1.19$ corresponding the case of N^3. We obtain the optimal index of refraction $N_e = 15.19$.

Next we consider the unit square. Let $k_1^\delta = 2.04$ corresponding N^2. We get $N_e = 15.38$. Then we let $k_1^\delta = 2.05$ corresponding N^3. We obtain the optimal index of refraction $N_e = 15.51$. We show the result in Table 2 where we also show the smallest and largest eigenvalues of the exact index of refraction. We see that for all the cases, the reconstructed index of refraction N_e are close to the eigenvalues of $N(x)$. In fact, they are between N_* and N^* for all examples.

Table 2. The reconstructed index of refraction N_e.

domain	N_e	N_*	N^*
unit ball N^2	14.66	14	17
unit ball N^3	15.19	13.57	17
unit cube N^2	15.38	14	17
unit cube N^3	15.51	13.27	17.56

6. Conclusions and future work

In this paper, we propose a secant method to compute the smallest transmission eigenvalue. We change the problem into a series of self-adjoint positive definite generalized eigenvalue problem. Thus we avoid computing a non-Hermitian matrix eigenvalue problem which is very difficult for large sparse matrices. Then a mixed finite element is proposed to solve the generalized eigenvalue problems. We show some examples for the computation of the smallest transmission eigenvalues and numerical evidence on how the transmission eigenvalues reflect changes in the anisotropy. Finally, we use the secant method in an optimization method to estimate the index of refraction based on the smallest transmission eigenvalues obtained from scattering data.

Here we only compute the smallest transmission eigenvalue. The method can compute other real transmission eigenvalues as well. For example, to compute the second smallest transmission eigenvalue, one only needs to compute the second smallest generalized eigenvalue of (18) and plug it in $f(\tau)$ (see [8]).

Numerical computation of transmission eigenvalues is a challenging topic due to the fact that the problem is non-standard and non-selfadjoint. The method proposed here can only treat isotropic media. For anisotropic media, a fourth order formulation as (15) is not possible [10]. Although a curl-conforming finite element is proposed in [23], there is no error estimate. Furthermore, the resulting eigenvalue problem is large, sparse and non-Hermitian which itself is a difficult problem in numerical linear algebra as we mentioned above.

Since we are using the linear edge element, we would expect that the convergence order is 2. However, we have the convergence order of roughly 1.7. This may be due to the fact that the meshes are rather coarse. The smallest size of the mesh we use is approximately 0.2. The error analysis of the mixed finite element method for the generalized eigenvalue problem (18) will be carried out in future. Furthermore, an effective ways to solve the discrete matrix eigenvalue problem resulting from the mixed method is desirable such that we can treat larger problems.

We use the smallest transmission eigenvalue to obtain a constant estimation N_eI for a matrix valued index of refraction. In fact, more than one transmission eigenvalue can be obtained from the scattering data. Thus an interesting problem is how to use multiple transmission eigenvalues to get more information of the index of refraction.

This is an inverse spectrum problem of the determination of the index of refraction from transmission eigenvalue(s). We refers the readers to [1, 12] for some discussion for the spherically stratified media.

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